Algebra and Discrete Mathematics
Volume 13 **(2012)**. Number 1. pp. 18 – 25

RESEARCH ARTICLE

On S-quasinormally embedded subgroups of finite groups

Kh. A. Al-Sharo, Olga Shemetkova and Xiaolan Yi[1]

Communicated by V. V. Kirichenko

Abstract. Let G be a finite group. A subgroup A is called: 1) S-quasinormal in G if A is permutable with all Sylow subgroups in G 2) S-quasinormally embedded in G if every Sylow subgroup of A is a Sylow subgroup of some S-quasinormal subgroup of G. Let B_{seG} be the subgroup generated by all the subgroups of B which are S-quasinormally embedded in G. A subgroup B is called SE-supplemented in G if there exists a subgroup T such that $G = BT$ and $B \cap T \leq B_{seG}$. The main result of the paper is the following.

Theorem. *Let H be a normal subgroup in G, and p a prime divisor of $|H|$ such that $(p-1, |H|) = 1$. Let P be a Sylow p-subgroup in H. Assume that all maximal subgroups in P are SE-supplemented in G. Then H is p-nilpotent and all its G-chief p-factors are cyclic.*

1. Introduction

All groups considered in this paper will be finite. A subgroup A of a group G is said to be S-quasinormal in G if it permutes with every Sylow subgroup of G. This concept was introduced by Kegel in [1] and has been studied in [2]–[15]. In 1998, Ballester-Bolinches and Pedraza-Aguilera [3] introduced the following definition: A subgroup B of a group G is said to be S-quasinormally embedded in G if for each prime p dividing the

[1]Research of the third author (corresponding author) was supported by NNSF of China (grant no. 11101369).
2010 Mathematics Subject Classification: *20D10, 20D20, 20D25.*
Key words and phrases: *Finite group, p-nilpotent, S-quasinormal subgroup.*

order of B, a Sylow p-subgroup of B is also a Sylow p-subgroup of some S-quasinormal subgroup of G. In 2007, Al-Sharo and Shemetkova proved the following.

Theorem 1. *Let H be a normal subgroup of a group G, and let p be the smallest prime dividing $|H|$. Let P be a Sylow p-subgroup of H. Assume that every maximal subgroup of P is S-quasinormally embedded in G. Then H is p-nilpotent and its non-Frattini G-chief p-factors are cyclic* (see [10, Theorem 1.2]).

In 2007, Skiba introduced [11] the concept of S-core as follows.

Definition 1. Let B be a subgroup of a group G. Let B_{sG} be the subgroup generated by all the subgroups of B which are S-quasinormal in G. The subgroup B_{sG} is called the S-core of H in G.

A subgroup B of G is called S-supplemented in G if there exists a subgroup T such that $G = BT$ and $B \cap T \le B_{sG}$.

By using the concept of S-supplemented subgroup, Skiba proved the following important result.

Theorem 2. *Let E be a normal subgroup of a group G. Suppose that for every non-cyclic Sylow subgroup P of E, all maximal subgroups of P are S-supplemented in G. Then each G-chief factor of E is cyclic* (see [13, Theorem A]).

Recently, based on the concept of S-quasinormally embedded subgroup, Skiba introduced [14] the following.

Definition 2. Let B be a subgroup of a group G. Let B_{seG} be the subgroup generated by all the subgroups of B which are S-quasinormally embedded in G. The subgroup B_{seG} is called the SE-core of B in G.

A subgroup B of G is called SE-supplemented in G if there exists a subgroup T such that $G = BT$ and $B \cap T \le B_{seG}$.

In the present paper, by using the concept of SE-supplemented subgroup, we will prove the following improvement of Theorem 1.

Theorem 3. *Let H be a normal subgroup in G, and p a prime divisor of $|H|$ such that $(p-1, |H|) = 1$. Let P be a Sylow p-subgroup in H. Assume that all maximal subgroups in P are SE-supplemented in G. Then H is p-nilpotent and all its G-chief p-factors are cyclic.*

Corollary 1. *Let H be a normal subgroup in G, and p a prime divisor of $|H|$ such that $(p-1,|H|)=1$. Let P be a Sylow p-subgroup in H. Assume that all maximal subgroups in P are S-supplemented in G. Then H is p-nilpotent and all its G-chief p-factors are cyclic.*

Theorem 2 can be easily deduced from Corollary 1 though we should notice that Theorem 2 is used in the proof of Theorem 3. The next corollary is a strengthened version of Theorem 1.

Corollary 2. *Let H be a normal subgroup in G, and p a prime divisor of $|H|$ such that $(p-1,|H|)=1$. Let P be a Sylow p-subgroup in H. Assume that all maximal subgroups in P are S-quasinormally embedded in G. Then H is p-nilpotent and all its G-chief p-factors are cyclic.*

2. Preliminaries

We use standard notations (see [16]). A subgroup T is called a supplement to a subgroup B in a group G if $G=BT$. We denote by H_G the core of H in G, the largest normal subgroup of G contained in H. A group (a subgroup) S is called a Schmidt group (a Schmidt subgroup) if every proper subgroup of S is nilpotent. We denote by $\pi(G)$ the set of all prime divisors of $|G|$. A group G is called p-supersoluble if every chief p-factor of G is cyclic.

Lemma 1. *Let G be a group and $H \leq K \leq G$.*

(1) If H is S-quasinormal in G, then H is S-quasinormal in K.

(2) If $H \trianglelefteq G$, then K/H is S-quasinormal in G/H if and only if K is S-quasinormal in G.

(3) If H is S-quasinormal in G, then H is subnormal in G.

(4) If A and B are S-quasinormal in G, then $A\cap B$ and $\langle A,B\rangle$ are S-quasinormal in G (see [1]).

Lemma 2. *Let A,B be some subgroups in G.*

(1) If A is S-quasinormal in G, then $A\cap B$ is S-quasinormal in B.

(2) If If A is S-quasinormal in G, then A/A_G is nilpotent (see [2]).

Lemma 3. *Suppose that a subgroup U is S-quasinormally embedded in a group G. Let $H \leq G$, and K be a normal subgroup of G. Then:*

(a) If $U \leq H$, then U is S-quasinormally embedded in H.

(b) UK is S-quasinormally embedded in G, and UK/K is S -quasinormally embedded in G/K (see [3]).

Lemma 4. *Let H be an SE-supplemented subgroup of G, and N a normal subgroup in G.*

(1) If $H \leq K \leq G$, then H is SE-supplemented in K.

(2) If $N \leq H$, then H/N is SE-supplemented in G/N.

(3) If $(|N|, |H|) = 1$, then HN/N is SE-supplemented in G/N (see [14, Lemma 2.8]).

The following result is well known.

Lemma 5. *Let p be a prime divisor of G such that $(p-1, |G|) = 1$.*

(1) If $M \leq G$ and $|G : M| = p$, then M is normal in G.

(2) If a Sylow p-subgroup of G is cyclic, then G is p-nilpotent.

(3) If G is p-supersoluble, then G is p-nilpotent.

Lemma 6. *If a p-subgroup H is S-quasinormal in G, then $H \leq O_p(G)$ and $O^p(G) \leq N_G(H)$* (see [15]).

Lemma 7. *If G is a Schmidt group, then:*

(1) G is a p-closed $\{p, q\}$-group for some primes p, q;

(2) if P is a Sylow p-subgroup of G, then $P/\Phi(P)$ is a chief factor of G and $|P/\Phi(P)| = p^n \equiv 1 \pmod q$ where n is the order of p modulo q (see [17, Theorem 26.1]) and [16, Theorem VII.6.18]*)*.

Lemma 8. *Let $R \trianglelefteq G$. Assume that $R/O_{p'}(G)$ is not contained in the hypercentre of $G/O_{p'}(G)$. Then G has a p-closed Schmidt subgroup S such that a Sylow p-subgroup $S_p \neq 1$ of S is contained in R* (see [18, Lemma 3]).

Lemma 9. *Let p be a prime divisor of G such that $(p-1, |G|) = 1$. Let G_p be a Sylow p-subgroup of G, $K \trianglelefteq G$, $P = G_p \cap K$. If G/K is a p-group and every maximal subgroup of G_p either contains P or has a p-nilpotent supplement in G, then K is p-nilpotent.*

Proof. Assume that K is not p-nilpotent. Then by [20, Theorem IV.4.7] we have $P \not\leq \Phi(G_p)$. Let M_1 be a maximal subgroup in G_p not containing P. It follows that there exists a p-nilpotent subgroup T_1 such that $G = M_1T_1$. Clearly, $G_p = M_1(G_p \cap T_1)$, and we can assume that $T_1 = N_G(H_1)$ where H_1 is a Hall p'-subgroup of K. We see that by [19] every two Hall p'-subgroup of K are conjugate in K (by assumption, either $p = 2$ or $|G|$ is odd). By Frattini argument, $G = KT_1 = PT_1$, hence $G_p = P(G_p \cap T_1)$ and $G_p \cap T_1 \not\leq P$. Let M_2 be a maximal subgroup in G_p containing $G_p \cap T_1$. Then $G = M_2T_2$ where T_2 is the normalizer in G of some Hall

p'-subgroup H_2 of K. Since $H_1^x = H_2$, $T_1^x = T_2$ for some $x \in G$, it follows that $G = M_2T_2 = M_2T_1^x = M_1T_1 = M_2T_1$. Therefore

$$G_p = M_1(G_p \cap T_1) = M_2(G_p \cap T_1) = M_2,$$

a contradiction. □

3. Proof of Theorem 3

Suppose that the theorem is not true and choose a counterexample (G, H) for which $|G| + |H|$ is minimal. We will prove several propositions and will get a contradiction. It follows from Lemma 5 that P is non-cyclic.

(1) $O_{p'}(H) = 1$.

Assume that $O_{p'}(H) \neq 1$. Applying Lemma 4 we see that the theorem is true for $(G/O_{p'}(H), H/O_{p'}(H))$, and then it is true for (G, H), a contradiction.

(2) $H = G$.

Assume that $H \neq G$. By Lemma 4 the theorem is true for the pair (H, H). Hence H is p-nilpotent. It follows by (1) that H is a p-group. By Theorem 2 every G-chief factor of H is cyclic, a contradiction.

From (1) and (2) we get the following.

(3) $O_{p'}(G) = 1$.

(4) $|P| > p^2$.

Assume that $|P| = p^2$. Applying Lemma 5 and Lemma 8 we see that P is contained in a p-closed Schmidt subgroup S of order p^2q^b where q is a prime and $p^2 \equiv 1 \pmod{q}$. Clearly, a Sylow q-subgroup of S is maximal in S. By Lemma 4 all subgroups of order p in P are SE-supplemented in S. Applying Lemmas 1 and 3 we see that all subgroups of order p in P are S-quasinormal in S. Therefore S has a subgroup of order pq^b, a contradiction.

(5) P is non-normal in G.

Assume that P is normal in G. Since the theorem is true for (G, P), G is p-supersoluble and so p-nilpotent by Lemma 5, a contradiction.

The following two propositions follow from Lemma 4 and the minimality of the counterexample G.

(6) If N is minimal normal subgroup in G contained in P, then G/N is p-supersoluble.

(7) If $P \leq M < G$, then M is p-nilpotent.

(8) G is p-soluble.

Assume that G is not p-soluble. By Lemma 6 the unit subgroup 1 is the only S-quasinormal subgroup contained in P. In particular, $P_G = 1$. Since $(p-1, |G|) = 1$, we have $p = 2$. By (7) there is a unique minimal normal subgroup K in G, and $PK = G$.

Let M be a maximal subgroup in P such that $M \not\geq P \cap K$. Since M is SE-supplemented in G, there is a subgroup T such that $G = MT$ and $M \cap T \leq M_{seG}$. If $M_{seG} = 1$, we have $|T|_2 = 2$, and therefore T is 2-nilpotent. Assume that $M_{seG} \neq 1$. Then there exists a non-identity subgroup L in M such that L is S-quasinormally embedded in G. Therefore L is a Sylow p-subgroup of some S-qusinormal subgroup D. If $D_G = 1$, it follows that D is nilpotent by Lemma 2. Then by Lemma 6 we have $F(G) \neq 1$, which contradicts (3) and $P_G = 1$. Therefore $K \leq D_G \neq 1$ and $L \geq P \cap K$. So we proved that every maximal subgroup in P not containing $P \cap K$ has a 2-nilpotent supplement. By Lemma 9 we have that K is 2-nilpotent, and (8) is proved.

The final contradiction.

From (1–8) it follows that G has a unique minimal normal subgroup K, and the following properties are valid: 1) K is a p-group and $K \neq P$; 2) G/K is p-nilpotent; 3) $K = C_G(K) = F(G)$.

Let M be a maximal subgroup in P such that $M \not\geq K$. Since M is SE-supplemented in G, there is a subgroup T such that $G = MT$ and $M \cap T \leq M_{seG}$. If $M_{seG} = 1$, we have $|T|_p = p$, and therefore T is p-nilpotent. Assume that $M_{seG} \neq 1$. Then there exists a non-identity subgroup L in M such that L is S-quasinormally embedded in G. Therefore L is a Sylow p-subgroup of some S-qusinormal subgroup D. If $D_G \neq 1$, then $K \leq D_G$ and $K \leq L \leq M$, a contradiction. Let $D_G = 1$. Then by Lemma 2 we have that D is nilpotent, and so $L = D$ is an S-qusinormal p-subgroup. By Lemma 6 we have that $O^p(G) \leq N_G(L)$. So, from $L \leq M \trianglelefteq P$ and $G = PO^p(G)$ it follows that

$$K \leq \langle L^x \mid x \in G\rangle = \langle L^x \mid x \in P\rangle \leq M,$$

a contradiction. We proved that every maximal subgroup in P not containing K has a p-nilpotent supplement in G. But then by Lemma 9 we have that KQ is p-nilpotent.

The proof of Theorem 3 is completed.

References

[1] O. Kegel, *Sylow-Gruppen and Subnormalteiler endlicher Gruppen*, Math. Z. **78** (1962), 205–221.

[2] W. E. Deskins, *On quasinormal subgroups of finite groups*, Math. Z. **82** (1963), 125–132.

[3] A. Ballester-Bolinches, M. C. Pedraza-Aguilera, *Sufficient conditions for supersolvability of finite groups*, J. Pure Appl. Algebra **127** (1998), 113–118.

[4] M. Asaad, *On maximal subgroups of finite groups*, Comm. Algebra **26** (1998), 3647–3652.

[5] M. Asaad, P. Csörgő, *Influence of minimal subgroups on the structure of finite groups*, Arch. Math. (Basel) **72** (1999), 401–404.

[6] A. Ballester-Bolinches, X. Y. Guo, *On complemented subgroups of finite groups*, Arch. Math. (Basel), **72** (1999), 161–166.

[7] Y. Li, Y. Wang, *The influence of minimal subgroups on the structure of a finite group*, Proc. Amer. Math. Soc. **131** (2002), 337–341.

[8] Y. Li, Y. Wang, *The influence of π-quasinormality of some subgroups of a finite group*, Arch. Math. (Basel) **81** (2003), 245–252.

[9] Y. Li, Y. Wang, *On π-quasinormally embedded subgroups of finite groups*, J. Algebra **281** (2004), 109–123.

[10] Kh. A. Al-Sharo, Olga Shemetkova, *An application of the concept of a generalized central element*, Algebra discrete Math. No. **4** (2007), 1–10.

[11] A. N. Skiba, *On weakly S-permutable subgroups of finite groups*, J. Algebra **315** (2007), 192–209.

[12] L. A. Shemetkov, Skiba, *On the $\mathfrak{X}\Phi$-hypercentre of finite groups*, J. Algebra **322** (2009), 2106–2117.

[13] A. N. Skiba, *On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups*, J. Group Theory **13** (2010), 841–850.

[14] A. N. Skiba, *On the SE-core of subgroups of a finite group*, Problems of Physics, Mathematics and Technics No. **4(5)** (2010), 39–45.

[15] P. Schmid, *Subgroups permutable with all Sylow subgroups*, J. Algebra **82** (1998), 285–293.

[16] K. Doerk, T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin–New York, 1992.

[17] L. A. Shemetkov, *Formations of finite groups*, Nauka, Moscow, 1978.

[18] O. L. Shemetkova, *On finite groups with Q-central elements of prime order*, Proc. of the Institute of Mathematics, Minsk, Republic of Belarus **16**, No. 1 (2008), 97–99.

[19] F. Gross, *Conjugacy of odd Hall subgroups*, Bull. London Math. Soc. **19** (1987), 311–319.

[20] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin–Heidelberg–New York, 1967.

Contact information

Kh. A. Al-Sharo Al al-Bayt University, St. Al-Zohoor 5–3, Mafraq 25113, Jordan
E-Mail: sharo_kh@yahoo.com

O. L. Shemetkova Russian Economic University named after G. V. Plekhanov, Stremyanny Per., 36, 117997 Moscow, Russia
E-Mail: ol-shem@mail.ru

Xiaolan Yi Zhejiang Sci-Tech University, Hangzhou 310018, P. R. China
E-Mail: yixiaolan2005@126.com

Received by the editors: 31.01.2012
and in final form 31.01.2012.