# A Centerless Virasoro Algebra of Master Symmetries for the Ablowitz–Ladik Hierarchy*

*Luc HAINE and Didier VANDERSTICHELEN*

*Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium*
E-mail: *luc.haine@uclouvain.be*, *didier.vanderstichelen@uclouvain.be*

Received July 31, 2013, in final form November 30, 2013; Published online December 12, 2013
http://dx.doi.org/10.3842/SIGMA.2013.079

**Abstract.** We show that the (semi-infinite) Ablowitz–Ladik (AL) hierarchy admits a centerless Virasoro algebra of master symmetries in the sense of Fuchssteiner [*Progr. Theoret. Phys.* **70** (1983), 1508–1522]. An explicit expression for these symmetries is given in terms of a slight generalization of the Cantero, Moral and Velázquez (CMV) matrices [*Linear Algebra Appl.* **362** (2003), 29–56] and their action on the tau-functions of the hierarchy is described. The use of the CMV matrices turns out to be crucial for obtaining a Lax pair representation of the master symmetries. The AL hierarchy seems to be the first example of an integrable hierarchy which admits a *full centerless* Virasoro algebra of master symmetries, in contrast with the Toda lattice and Korteweg–de Vries hierarchies which possess only "half of" a Virasoro algebra of master symmetries, as explained in Adler and van Moerbeke [*Duke Math. J.* **80** (1995), 863–911], Damianou [*Lett. Math. Phys.* **20** (1990), 101–112] and Magri and Zubelli [*Comm. Math. Phys.* **141** (1991), 329–351].

*Key words:* Ablowitz–Ladik hierarchy; master symmetries; Virasoro algebra

*2010 Mathematics Subject Classification:* 37K10; 17B68

## Contents



*This paper is a contribution to the Special Issue in honor of Anatol Kirillov and Tetsuji Miwa. The full collection is available at http://www.emis.de/journals/SIGMA/InfiniteAnalysis2013.html

# 1 Introduction

The group $U(n)$ of $n \times n$ unitary matrices, with Haar measure $\mathrm{d}U$ normalized as a probability measure, has eigenvalue probability distribution given by the Weyl formula

$$\frac{1}{n!}|\Delta_n(z)|^2 \prod_{k=1}^{n} \frac{\mathrm{d}z_k}{2\pi i z_k}, \qquad z_k = e^{i\varphi_k} \in S^1, \qquad \varphi_k \in \left]-\pi, \pi\right],$$

with $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle, and $\Delta_n(z)$ the Vandermonde determinant

$$\Delta_n(z) = \det\left(z_l^{k-1}\right)_{1\le k,l\le n} = \prod_{1\le k<l\le n}(z_l - z_k). \tag{1.1}$$

Thus, for $\eta, \theta \in \left]-\pi, \pi\right]$, with $\eta \le \theta$, the probability that a randomly chosen matrix from $U(n)$ has no eigenvalues within the arc of circle $\{z \in S^1 : \eta < \arg(z) < \theta\}$ is given by

$$\tau_n(\eta, \theta) = \frac{1}{(2\pi)^n n!}\int_\theta^{2\pi+\eta}\cdots\int_\theta^{2\pi+\eta}\prod_{1\le k<l\le n}|e^{i\varphi_k} - e^{i\varphi_l}|^2 \mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_n.$$

Obviously, this probability depends only on the length $\theta - \eta$.

The starting motivation for the present work was our attempt in [24] to understand a differential equation satisfied by the function $-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\tau_n(-\theta, \theta)$, obtained by Tracy and Widom in [36], from the point of view of the Adler–Shiota–van Moerbeke approach [3], in terms of Virasoro constraints. Introducing the 2-Toda time-dependent tau-functions

$$\tau_n(t, s; \eta, \theta) = \frac{1}{n!}\int_{[\theta, 2\pi+\eta]^n} \mathrm{d}I_n(t, s, z) \tag{1.2}$$

with $(t, s) = (t_1, t_2, \ldots, s_1, s_2, \ldots)$ and

$$\mathrm{d}I_n(t, s, z) = |\Delta_n(z)|^2 \prod_{k=1}^{n}\left(e^{\sum\limits_{j=1}^{\infty}(t_j z_k^j + s_j z_k^{-j})}\frac{\mathrm{d}z_k}{2\pi i z_k}\right),$$

deforming the probabilities $\tau_n(\eta, \theta) = \tau_n(0, 0; \eta, \theta)$, we discovered that they satisfy a set of Virasoro constraints indexed by *all* integers, decoupling into a boundary-part and a time-part

$$\frac{1}{i}\left(e^{ik\theta}\frac{\partial}{\partial\theta} + e^{ik\eta}\frac{\partial}{\partial\eta}\right)\tau_n(t, s; \eta, \theta) = L_k^{(n)}\tau_n(t, s; \eta, \theta), \qquad k \in \mathbb{Z}, \qquad i = \sqrt{-1}, \tag{1.3}$$

with the time-dependent operators $L_k^{(n)}$ providing a *centerless* representation of the *full* Virasoro algebra, that is

$$\left[L_k^{(n)}, L_l^{(n)}\right] = (k - l)L_{k+l}^{(n)}, \qquad \forall\, k, l \in \mathbb{Z}. \tag{1.4}$$

The basic trick for this result was to use the Lagrangian approach [31] for obtaining Virasoro constraints in matrix models, showing that the following variational formulas hold $\forall\, k \ge 0$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{d}I_n\left(z_\alpha \mapsto z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}\right)\Big|_{\varepsilon=0} = \left(L_k^{(n)} - L_{-k}^{(n)}\right)\mathrm{d}I_n,$$
$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathrm{d}I_n\left(z_\alpha \mapsto z_\alpha e^{i\varepsilon(z_\alpha^k + z_\alpha^{-k})}\right)\Big|_{\varepsilon=0} = i\left(L_k^{(n)} + L_{-k}^{(n)}\right)\mathrm{d}I_n,$$

with $L_k^{(n)}$ given by

$$L_k^{(n)} = \sum_{j=1}^{k-1} \frac{\partial^2}{\partial t_j \partial t_{k-j}} + n\frac{\partial}{\partial t_k} + \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{j+k}} - \sum_{j=k+1}^{\infty} j s_j \frac{\partial}{\partial s_{j-k}} - \sum_{j=1}^{k-1} j s_j \frac{\partial}{\partial t_{k-j}} - nks_k, \qquad k \geq 1, \tag{1.5}$$

$$L_0^{(n)} = \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_j} - \sum_{j=1}^{\infty} j s_j \frac{\partial}{\partial s_j}, \tag{1.6}$$

$$L_{-k}^{(n)} = -\sum_{j=1}^{k-1} \frac{\partial^2}{\partial s_j \partial s_{k-j}} - n\frac{\partial}{\partial s_k} - \sum_{j=1}^{\infty} j s_j \frac{\partial}{\partial s_{j+k}} + \sum_{j=k+1}^{\infty} j t_j \frac{\partial}{\partial t_{j-k}} + \sum_{j=1}^{k-1} j t_j \frac{\partial}{\partial s_{k-j}} + nkt_k, \qquad k \geq 1. \tag{1.7}$$

When $\eta = \theta$, the integral (1.2) is obviously independent of $\theta$, and the left-hand side of (1.3) is equal to zero. By using Weyl's integration formula, one can recognize it as the partition function of the unitary matrix model, introduced in [30]. After [24] was completed, we found out that our result in this case had already been obtained by Bowick, Morozov and Shevitz [8], though these authors didn't notice the commutation relations (1.4) of the centerless Virasoro algebra (see Corollary 4.2 and Remark 4.3). Kharchev and Mironov [27] first recognized that the partition function of the unitary matrix model is a special tau function of the two-dimensional Toda lattice (in short 2DTL) hierarchy of Ueno and Takasaki [37], by using bi-orthogonal polynomials on the circle. Then, Kharchev, Mironov and Zhedanov [28, 29] showed that the coefficients entering the Szegö type recursion relations satisfied by these bi-orthogonal polynomials solve the semi-infinite Ablowitz–Ladik (AL in short) hierarchy, a result which is already implicitly contained in [27]. We remind the reader that the first vector field of the AL hierarchy is the system of differential-difference equations introduced by Ablowitz and Ladik [1, 2] in the form

$$\begin{aligned} \dot{x}_n &= x_{n+1} - 2x_n + x_{n-1} - x_n y_n (x_{n+1} + x_{n-1}), \\ \dot{y}_n &= -y_{n+1} + 2y_n - y_{n-1} + x_n y_n (y_{n+1} + y_{n-1}). \end{aligned} \tag{1.8}$$

Upon making the change of variable $t \to it$, when $y_n = \mp \overline{x_n}$ the system reduces to the equation

$$-i\dot{x}_n = x_{n+1} - 2x_n + x_{n-1} \pm |x_n|^2 (x_{n+1} + x_{n-1}),$$

which is a discrete version of the focusing/defocusing nonlinear Schrödinger equation.

The functions $\tau_n(t, s; \eta, \theta)$ are thus *special* instances of tau-functions of the semi-infinite AL hierarchy. The Virasoro constraints they satisfy suggest that the semi-infinite AL hierarchy admits a *full centerless* Virasoro algebra of additional symmetries (so-called master symmetries), a notion which will be explained below. The goal of this paper is to identify the Virasoro algebra of master symmetries both on the variables $x_n$, $y_n$, $n \geq 0$, as well as on the *general* tau-functions of the AL hierarchy. Since the pioneering works [27, 28, 29] the fact that the semi-infinite AL hierarchy is related to (bi)-orthogonal polynomials on the circle in the same way as the semi-infinite Toda lattice hierarchy is related to orthogonal polynomials on the line, has been rediscovered several times, see for instance [5, 6, 9, 32]. We now introduce the necessary tools to explain this connection.

We denote by $\mathbb{C}[z, z^{-1}]$ the ring of Laurent polynomials over $\mathbb{C}$. A bilinear form

$$\mathcal{L} : \mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}] \to \mathbb{C}, \qquad (f, g) \mapsto \mathcal{L}[f, g], \tag{1.9}$$

will be called a bi-moment functional. The bi-moments associated to $\mathcal{L}$ are

$$\mu_{mn} = \mathcal{L}\big[z^m, z^n\big], \qquad \forall\, m, n \in \mathbb{Z}. \tag{1.10}$$

We assume that $\mathcal{L}$ satisfies the *Toeplitz condition*

$$\mathcal{L}\big[z^m, z^n\big] = \mathcal{L}\big[z^{m-n}, 1\big], \qquad \forall\, m, n \in \mathbb{Z}. \tag{1.11}$$

Because of the Toeplitz condition (1.11), the bi-moments depend only on the difference $m - n$ and we shall often write

$$\mu_{mn} := \mu_{m-n}. \tag{1.12}$$

In the rest of the paper, we shall freely use both notations for the bi-moments. An important example of a Toeplitz bi-moment functional is provided by

$$\mathcal{L}[f, g] = \oint_{S^1} f(z) g\big(z^{-1}\big) w(z) \frac{\mathrm{d}z}{2\pi i z}, \tag{1.13}$$

with $w(z)$ some weight function on the unit circle $S^1$ which is not necessarily positive or even real valued. We shall also assume $\mathcal{L}$ to be *quasi-definite*, that is

$$\det\big(\mu_{kl}\big)_{0\le k,l\le n-1} \neq 0, \qquad \forall\, n \ge 1. \tag{1.14}$$

This is a necessary and sufficient condition for the existence of a sequence of bi-orthogonal polynomials $\big\{p_n^{(1)}(z), p_n^{(2)}(z)\big\}_{n\ge 0}$ with respect to $\mathcal{L}$, that is $p_n^{(1)}(z)$ and $p_n^{(2)}(z)$ are polynomials of degree $n$, satisfying the orthogonality conditions

$$\mathcal{L}\big[p_m^{(1)}(z), p_n^{(2)}(z)\big] = h_n \delta_{m,n}, \qquad h_n \neq 0, \qquad \forall\, m, n \in \mathbb{N}.$$

Introducing the variables

$$x_n = p_n^{(1)}(0), \qquad y_n = p_n^{(2)}(0), \qquad n \ge 0, \tag{1.15}$$

the *monic* bi-orthogonal polynomials $\big\{p_n^{(1)}(z), p_n^{(2)}(z)\big\}_{n\ge 0}$ satisfy the Szegö type recurrence relations

$$p_{n+1}^{(1)}(z) - z p_n^{(1)}(z) = x_{n+1} z^n p_n^{(2)}\big(z^{-1}\big), \qquad p_{n+1}^{(2)}(z) - z p_n^{(2)}(z) = y_{n+1} z^n p_n^{(1)}\big(z^{-1}\big), \tag{1.16}$$

from which it easily follows that

$$\frac{h_{n+1}}{h_n} = 1 - x_{n+1} y_{n+1}, \qquad n \ge 0. \tag{1.17}$$

In [5, 6, 27, 28, 29] the AL hierarchy[1] is embedded in the 2DTL hierarchy by using a pair $(L_1, L_2)$ of Hessenberg matrices representing respectively the operator of multiplication $\mathbb{C}[z] \to \mathbb{C}[z] : f(z) \to z f(z)$ in the bases $p^{(1)}(z) = \big(p_n^{(1)}(z)\big)_{n\ge 0}$ and $p^{(2)}(z) = \big(p_n^{(2)}(z)\big)_{n\ge 0}$ of bi-orthogonal polynomials

$$z p^{(1)}(z) = L_1 p^{(1)}(z), \qquad z p^{(2)}(z) = L_2 p^{(2)}(z).$$

However, to represent the Virasoro algebra of master symmetries, what we shall need is a basis of the ring $\mathbb{C}\big[z, z^{-1}\big]$ of *Laurent polynomials* in which both the operators of multiplication by $z$ and $z^{-1}$ admit nice matrix representations. Thus, we shall adopt the more recent point of view

[1] In [5, 6] the terminology "Toeplitz hierarchy" instead of "AL hierarchy" is used.

of Nenciu [32] who used the celebrated Cantero, Moral and Velázquez matrices (CMV matrices in short) to obtain a Lax pair representation for the AL hierarchy in the special defocusing case, that is when $y_n = \overline{x_n}$. We can now describe the content of our paper.

To deal with the general AL hierarchy, in Section 2, we first develop a slight generalization of the CMV matrices as introduced in [10]. The generalized CMV matrices are *pentadiagonal* (semi-infinite) matrices $A_1$, $A_2$ which will represent multiplication by $z$ in bases of *bi-orthogonal Laurent polynomials*[2], which will be denoted by $f(z) = (f_n(z))_{n\geq 0}$ and $g(z) = (g_n(z))_{n\geq 0}$, satisfying $\mathcal{L}[f_m, g_n] = \delta_{m,n}h_n$ and the five-term recurrence relations

$$zf(z) = A_1 f(z), \qquad zg(z) = A_2 g(z). \tag{1.18}$$

In these bases, we shall have that

$$z^{-1}f(z) = A_1^* f(z), \qquad z^{-1}g(z) = A_2^* g(z),$$

with $A_1^* = hA_2^T h^{-1}$, $A_2^* = hA_1^T h^{-1}$ and $h$ the diagonal matrix $\operatorname{diag}(h_n)_{n\geq 0}$, so that $A_1^* = A_1^{-1}$ and $A_2^* = A_2^{-1}$. Putting $z_n = 1 - x_n y_n$, with $x_n$ and $y_n$ defined as in (1.15) (note that $x_0 = y_0 = 1$), the matrix $A_1$ reads

$$A_1 = \begin{pmatrix}
-x_1y_0 & y_0 & 0 & & & & & & \\
-x_2z_1 & -x_2y_1 & -x_3 & 1 & & & & & \\
z_1z_2 & y_1z_2 & -x_3y_2 & y_2 & 0 & & & O & \\
 & 0 & -x_4z_3 & -x_4y_3 & -x_5 & 1 & & & \\
 & & z_3z_4 & y_3z_4 & -x_5y_4 & y_4 & 0 & & \\
 & & & 0 & * & * & * & 1 & \\
 & O & & & * & * & * & * & 0 \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},$$

and $A_2$ is obtained from $A_1$ by exchanging the roles of the variables $x_n$ and $y_n$. This will be proven at the end of Section 2. To make contact with the work of Nenciu [32] as well as with the authoritative treatises on OPUC by Simon [34, 35], it suffices to specialize to the case $x_{n+1} = -\overline{\alpha_n}$, $y_{n+1} = -\alpha_n$, $n \geq 0$, where $\alpha_n$ are the so-called Verblunsky coefficients, remembering that $x_0 = y_0 = 1$.[3] We notice that Gesztesy, Holden, Michor and Teschl [20] have obtained a Lax pair representation for the doubly infinite AL hierarchy, involving a matrix similar to $A_1$ above (up to some conjugation). According to them, the proof is based on "fairly tedious computations". Our approach via bi-orthogonal Laurent polynomials and the "dressing method" explained below, is more conceptual.

In Section 3, we put this theory to use to obtain Lax pair representations both for the AL hierarchy and its Virasoro algebra of master symmetries. Our approach is based on a Favard like theorem which states that there is a one-to-one correspondence between pairs of CMV matrices $(A_1, A_2)$, with entries built in terms of $x_n$ and $y_n$ satisfying $x_0 = y_0 = 1$ and $x_n y_n \neq 1$, $n \geq 1$, and quasi-definite Toeplitz bi-moment functionals defined up to a multiplicative nonzero constant. This theorem can be proven as a generalization to bi-orthogonal Laurent polynomials of a similar result in [11], for orthogonal Laurent polynomials on the unit circle. For a complete and independent proof, see [38]. Thus to define the AL hierarchy vector fields $T_k$, $k \in \mathbb{Z}$, it is enough to define them on the bi-moments

$$T_k(\mu_j) \equiv \frac{\partial \mu_j}{\partial t_k} = \mu_{j+k}, \qquad T_{-k}(\mu_j) \equiv \frac{\partial \mu_j}{\partial s_k} = \mu_{j-k}, \qquad \forall\, k \geq 1, \tag{1.19}$$

[2]The paper [10] considers the case of a sesquilinear *hermitian* quasi-definite form on $\mathbb{C}[z, z^{-1}]$ satisfying the Toeplitz condition, dealing thus with orthogonal instead of bi-orthogonal Laurent polynomials.

[3]With these notations, the transpose $\mathcal{C}^T$ of the CMV matrix in [32, 34, 35] is given by $\mathcal{C}^T = (\sqrt{h})^{-1} A_1 \sqrt{h}$, with $\sqrt{h} = \operatorname{diag}(\sqrt{h_n})_{n\geq 0}$ and $h_{n+1}/h_n$ as in (1.17).

which, in the example of the bi-moment functional (1.13), corresponds to deform the weight $w(z)$ as follows

$$w(z;t,s) = w(z)\exp\left\{\sum_{j=1}^{\infty}\left(t_j z^j + s_j z^{-j}\right)\right\}. \tag{1.20}$$

Obviously $[T_k, T_l] = 0$, $\forall\, k,l\in\mathbb{Z}$, if we define $T_0\mu_j = \mu_j$. Then, all the objects introduced above become time dependent. In particular $x_n(t,s)$ and $y_n(t,s)$ depend on $t$, $s$. The Lax pair for the AL hierarchy is then obtained in Theorem 3.4 by "dressing up" the moment equations (1.19) written in matrix form (see (3.10)).

Following an idea introduced by Haine and Semengue [23] in the context of the semi-infinite Toda lattice, we define the following vector fields on the bi-moments

$$V_k(\mu_j) = (j+k)\mu_{j+k}, \qquad \forall\, k\in\mathbb{Z}. \tag{1.21}$$

These vector fields trivially satisfy the commutation relations

$$[V_k, V_l] = (l-k)V_{k+l}, \tag{1.22}$$

$$[V_k, T_l] = lT_{k+l}, \qquad \forall\, k,l\in\mathbb{Z}, \tag{1.23}$$

from which it follows that

$$[[V_k, T_l], T_l] = l[T_{k+l}, T_l] = 0, \qquad \forall\, k,l\in\mathbb{Z}. \tag{1.24}$$

Equations (1.22), (1.23) and (1.24) mean that the vector fields $V_k$, $k\in\mathbb{Z}$, form a centerless Virasoro algebra of master symmetries, in the sense of Fuchssteiner [18], for the AL hierarchy. We remind the reader that master symmetries are generators for time dependent symmetries of the hierarchy which are first degree polynomials in the time variables, that is

$$X_{k,l} = V_k + t[V_k, T_l], \qquad k\in\mathbb{Z},$$

are time dependent symmetries of the vector field $T_l$ (run with time $t$) as one immediately checks that

$$\frac{\partial X_{k,l}}{\partial t} + [T_l, X_{k,l}] = [V_k, T_l] + [T_l, V_k + t[V_k, T_l]] = 0,$$

from the commutation relations (1.24). Writing (1.21) in matrix form (see (3.18)) and "dressing up" these equations, leads then in Theorem 3.8 to the Lax pair representation of the master symmetries on the CMV matrices $(A_1, A_2)$, which was our first goal and is a new result.

In Section 4, we shall reach our second goal by translating the action of the master symmetries on the tau-functions of the AL hierarchy. One can show (see [5, 28, 29]) that the general solution of the AL hierarchy can be expressed in terms of the Toeplitz determinants

$$\tau_n(t,s) = \det\left(\mu_{k-l}(t,s)\right)_{0\le k,l\le n-1}, \tag{1.25}$$

as follows

$$x_n(t,s) = \frac{S_n(-\tilde{\partial}_t)\tau_n(t,s)}{\tau_n(t,s)}, \qquad y_n(t,s) = \frac{S_n(-\tilde{\partial}_s)\tau_n(t,s)}{\tau_n(t,s)}.$$

In this formula $S_n(t), t = (t_1, t_2, t_3, \ldots)$, are the so-called elementary Schur polynomials defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \sum_{n\in\mathbb{Z}} S_n(t_1, t_2, \ldots) z^n, \tag{1.26}$$

and $S_n(-\tilde{\partial}_t) = S_n\big(-\frac{\partial}{\partial t_1}, -\frac{1}{2}\frac{\partial}{\partial t_2}, -\frac{1}{3}\frac{\partial}{\partial t_3}, \dots\big)$, and similarly for $S_n(-\tilde{\partial}_s)$. The functions $\tau_n(t,s)$ are the tau-functions of the semi-infinite AL hierarchy. In the example of the bi-moment functional (1.13), a standard computation establishes that

$$\tau_n(t,s) = \frac{1}{n!}\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} w(z_k;t,s)\frac{\mathrm{d}z_k}{2\pi i z_k}, \tag{1.27}$$

with $w(z;t,s)$ the deformed weight introduced in (1.20), and $\Delta_n(z)$ the Vandermonde determinant (1.1). Such integrals appear in combinatorics as well as in random matrix theory, see [5, 6, 7, 17, 33, 36] and the references therein. The special case $\tau_n(t,s;\eta,\theta)$ (1.2) considered at the beginning of this Introduction corresponds to $w(z) = \chi_{]\eta,\theta[^c}(z)$, the characteristic function of the complement of an arc of circle $]\eta,\theta[= \{z \in S^1 : \eta < \arg z < \theta\}$.

By a simple computation, which will be recalled in Section 4, one obtains that the tau-functions (1.25) admit the expansion

$$\tau_n(t,s) = \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s), \tag{1.28}$$

where

$$p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} = \det\big(\mu_{i_k-j_l}(0,0)\big)_{0\le k,l\le n-1}, \tag{1.29}$$

are the so-called Plücker coordinates, and $S_{i_1,\dots,i_k}(t)$ denote the Schur polynomials

$$S_{i_1,\dots,i_k}(t) = \det\big(S_{i_r+s-r}(t)\big)_{1\le r,s\le k}. \tag{1.30}$$

In Theorem 4.1, we will show that the induced action of the master symmetries (1.21) on the Plücker coordinates of the tau-function $\tau_n(t,s)$ translates into the centerless Virasoro algebra of partial differential operators $L_k^{(n)}$, $k \in \mathbb{Z}$, in the $(t,s)$ variables, that was introduced at the beginning of the Introduction, a result we announced without proof in [24].

For the convenience of the reader, we summarize below our main results, which will be established respectively in Section 3 and Section 4 of the paper.

**Theorem 1.1.** *The centerless Virasoro algebra $\{V_k, k \in \mathbb{Z}\}$, of master symmetries of the Ablowitz–Ladik hierarchy which are defined on the bi-moments by (1.21), translates as follows on the CMV matrices and the tau-functions of the hierarchy:*

1) *On the CMV matrices $(A_1, A_2)$, the master symmetries admit the Lax pair representation*

$$V_k(A_1) = \Big[A_1, \big(D_1A_1^{k+1}\big)_{--} + \big(A_1^{k+1}D_1^*\big)_{--} + k\big(A_1^k\big)_{--}\Big], \qquad \forall\, k \in \mathbb{Z}, \tag{1.31}$$

$$V_k(A_2) = \Big[\big(D_2A_2^{1-k}\big)_{--} + \big(A_2^{1-k}D_2^*\big)_{--} - k\big(A_2^{-k}\big)_{--}, A_2\Big], \qquad \forall\, k \in \mathbb{Z}, \tag{1.32}$$

*where $A_{--}$ denotes the strictly lower triangular part of $A$, and $D_1$ and $(D_1^*)^T$ (respectively $D_2$ and $(D_2^*)^T$) represent the operator of derivation $\mathrm{d}/\mathrm{d}z$ in the bases $(f_n(z))_{n\ge0}$ and $\big(h_n^{-1}g_n(z^{-1})\big)_{n\ge0}$ (respectively $(g_n(z))_{n\ge0}$ and $\big(h_n^{-1}f_n(z^{-1})\big)_{n\ge0}$), with $f_n(z)$, $g_n(z)$ the bi-orthogonal Laurent polynomials satisfying (1.18) and $\mathcal{L}[f_m, g_n] = h_n\delta_{m,n}$.*

2) *On the tau-functions $\tau_n(t,s)$, the master symmetries are given by a centerless Virasoro algebra of partial differential operators in the $(t,s)$ variables*

$$V_k\tau_n(t,s) = L_k^{(n)}\tau_n(t,s), \qquad \forall\, k \in \mathbb{Z},$$

*with $L_k^{(n)}$ defined as in (1.5), (1.6) and (1.7).*

## 2 Bi-orthogonal Laurent polynomials and CMV matrices

In this section, given $\mathcal{L} : \mathbb{C}\big[z,z^{-1}\big] \times \mathbb{C}\big[z,z^{-1}\big] \to \mathbb{C}$, a bi-moment functional as in (1.9) which satisfies the Toeplitz condition (1.11) and is quasi-definite (1.14), we construct two sequences of bi-orthogonal Laurent polynomials (in short L-polynomials), which can be thought of as a Gram–Schmidt bi-orthogonalization process applied to the ordered bases $\big\{1, z, z^{-1}, z^2, z^{-2}, \dots\big\}$ and $\big\{1, z^{-1}, z, z^{-2}, z^2, \dots\big\}$ of $\mathbb{C}\big[z,z^{-1}\big]$. They will be called right and left bi-orthogonal L-polynomials respectively. This is a slight generalization of the Cantero, Moral and Velázquez [10] construction[4].

The two sequences of monic right and left bi-orthogonal L-polynomials we shall construct will be expressed in terms of the sequence of monic bi-orthogonal polynomials $\big\{p_n^{(1)}(z), p_n^{(2)}(z)\big\}_{n\geq 0}$, given by the well known formulae

$$p_n^{(1)}(z) = \frac{1}{\tau_n}\det\begin{pmatrix} \mu_{0,0} & \cdots & \mu_{0,n-1} & 1 \\ \mu_{1,0} & \cdots & \mu_{1,n-1} & z \\ \vdots & & \vdots & \vdots \\ \mu_{n,0} & \cdots & \mu_{n,n-1} & z^n \end{pmatrix},$$

$$p_n^{(2)}(z) = \frac{1}{\tau_n}\det\begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \cdots & \mu_{0,n} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1,0} & \mu_{n-1,1} & \cdots & \mu_{n-1,n} \\ 1 & z & \dots & z^n \end{pmatrix},$$

with $\tau_n = \det\big(\mu_{kl}\big)_{0\leq k,l\leq n-1}$. Denoting by $\{f_n, g_n\}_{n\geq 0}$ the sequence of monic right bi-orthogonal L-polynomials, multiplication by $z$ in the bases $(f_n)_{n\geq 0}$ and $(g_n)_{n\geq 0}$ of $\mathbb{C}\big[z,z^{-1}\big]$ will be represented by two pentadiagonal matrices $A_1$ and $A_2$, which we call the generalized CMV matrices (and similarly of course for the sequence of left bi-orthogonal L-polynomials). Moreover, the entries of $A_1$ and $A_2$ will have simple expressions in terms of the variables $x_n$ and $y_n$ entering the Szegö type recurrence relations (1.16).

### 2.1 Bi-orthogonal Laurent polynomials

The following definition is natural from our previous discussion. We define the vector subspaces

$$\mathbb{L}_{m,n} := \big\langle z^m, z^{m+1}, \dots, z^{n-1}, z^n\big\rangle, \qquad \forall\, m,n \in \mathbb{Z}, \qquad m \leq n,$$

and for $n \geq 0$

$$\mathbb{L}_{2n}^+ := \mathbb{L}_{-n,n}, \qquad \mathbb{L}_{2n+1}^+ := \mathbb{L}_{-n,n+1}, \qquad \mathbb{L}_{2n}^- := \mathbb{L}_{-n,n}, \qquad \mathbb{L}_{2n+1}^- := \mathbb{L}_{-n-1,n},$$

with the convention $\mathbb{L}_{-1}^+ = \mathbb{L}_{-1}^- = \{0\}$.

**Definition 2.1.** A sequence $\{f_n, g_n\}_{n\geq 0}$ in $\mathbb{C}[z,z^{-1}]$ is a sequence of right (left) bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if

1) $f_n, g_n \in \mathbb{L}_n^{+(-)} \setminus \mathbb{L}_{n-1}^{+(-)}$;
2) $\mathcal{L}[f_n, g_m] = h_n\delta_{n,m}$, with $h_n \neq 0$.

[4]The paper [10] deals with the case of a sesquilinear quasi-definite hermitian form on $\mathbb{C}[z,z^{-1}]$, satisfying the Toeplitz condition. Dropping the condition "hermitian" leads to bi-orthogonal L-polynomials, instead of orthogonal L-polynomials. For the applications we have in mind, see (1.13), it is better to assume $\mathcal{L}$ bilinear rather than sesquilinear.

**Remark 2.2.** Similarly to orthogonal polynomials, condition (2) in Definition 2.1 can be replaced equivalently by

$$(3r)\ \begin{cases} \mathcal{L}\big[f_{2n}, z^k\big] = 0, & \mathcal{L}\big[z^k, g_{2n}\big] = 0 \qquad \text{if } -n+1 \le k \le n, \\ \mathcal{L}\big[f_{2n}, z^{-n}\big] \neq 0, & \mathcal{L}\big[z^{-n}, g_{2n}\big] \neq 0, \\ \mathcal{L}\big[f_{2n+1}, z^k\big] = 0, & \mathcal{L}\big[z^k, g_{2n+1}\big] = 0 \qquad \text{if } -n \le k \le n, \\ \mathcal{L}\big[f_{2n+1}, z^{n+1}\big] \neq 0, & \mathcal{L}\big[z^{n+1}, g_{2n+1}\big] \neq 0, \end{cases}$$

in the case of right bi-orthogonal L-polynomials. For left bi-orthogonal L-polynomials the equivalent condition is

$$(3l)\ \begin{cases} \mathcal{L}\big[f_{2n}, z^k\big] = 0, & \mathcal{L}\big[z^k, g_{2n}\big] = 0 \qquad \text{if } -n \le k \le n-1, \\ \mathcal{L}\big[f_{2n}, z^{n}\big] \neq 0, & \mathcal{L}\big[z^{n}, g_{2n}\big] \neq 0, \\ \mathcal{L}\big[f_{2n+1}, z^k\big] = 0, & \mathcal{L}\big[z^k, g_{2n+1}\big] = 0 \qquad \text{if } -n \le k \le n, \\ \mathcal{L}\big[f_{2n+1}, z^{-n-1}\big] \neq 0, & \mathcal{L}\big[z^{-n-1}, g_{2n+1}\big] \neq 0. \end{cases}$$

We start by proving that sequences of right and left bi-orthogonal L-polynomials for a given Toeplitz bi-moment functional $\mathcal{L}$ are closely related to each other.

**Proposition 2.3.** *Let $f_n^*(z) = f_n\big(z^{-1}\big)$ and $g_n^*(z) = g_n\big(z^{-1}\big)$. Then $\{f_n, g_n\}_{n\ge 0}$ is a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\{g_n^*, f_n^*\}_{n\ge 0}$ is a sequence of left bi-orthogonal L-polynomials with respect to $\mathcal{L}$.*

**Proof.** We have $f_n^*, g_n^* \in \mathbb{L}_n^- \setminus \mathbb{L}_{n-1}^-$ if and only if $f_n, g_n \in \mathbb{L}_n^+ \setminus \mathbb{L}_{n-1}^+$. Using the Toeplitz condition (1.11), the result then follows from

$$\mathcal{L}\big[g_m^*(z), f_n^*(z)\big] = \mathcal{L}\big[g_m\big(z^{-1}\big), f_n\big(z^{-1}\big)\big] = \mathcal{L}\big[f_n(z), g_m(z)\big]. \qquad \blacksquare$$

Sequences of right or left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ are also very closely related to sequences of bi-orthogonal polynomials for $\mathcal{L}$. This is proven in the next theorem.

**Theorem 2.4.** *Let $\mathcal{L}$ be a Toeplitz bi-moment functional and let $\{f_n, g_n\}_{n\ge 0}$ be a sequence in $\mathbb{C}\big[z, z^{-1}\big]$. Let us define*

$$\begin{aligned} p_{2n}^{(1)}(z) &= z^n g_{2n}\big(z^{-1}\big), \qquad & p_{2n+1}^{(1)}(z) &= z^n f_{2n+1}(z), \\ p_{2n}^{(2)}(z) &= z^n f_{2n}\big(z^{-1}\big), \qquad & p_{2n+1}^{(2)}(z) &= z^n g_{2n+1}(z). \end{aligned} \tag{2.1}$$

*The sequence $\{f_n, g_n\}_{n\ge 0}$ is a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\big\{p_n^{(1)}, p_n^{(2)}\big\}_{n\ge 0}$ is a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$. Furthermore we have*

$$\mathcal{L}[f_n, g_n] = \mathcal{L}\big[p_n^{(1)}, p_n^{(2)}\big]. \tag{2.2}$$

*An analogous statement holds for sequences $\{f_n, g_n\}_{n\ge 0}$ of left bi-orthogonal L-polynomials, if we define*

$$\begin{aligned} \tilde{p}_{2n}^{(1)}(z) &= z^n f_{2n}(z), \qquad & \tilde{p}_{2n+1}^{(1)}(z) &= z^n g_{2n+1}\big(z^{-1}\big), \\ \tilde{p}_{2n}^{(2)}(z) &= z^n g_{2n}(z), \qquad & \tilde{p}_{2n+1}^{(2)}(z) &= z^n f_{2n+1}\big(z^{-1}\big). \end{aligned} \tag{2.3}$$

**Proof.** For $n \ge 0$, we define $\mathbb{P}_n = \langle 1, z, \ldots, z^n \rangle$ the vector subspace of polynomials with degree less than or equal to $n$, and $\mathbb{P}_{-1} := \{0\}$. For $\big\{p_n^{(1)}, p_n^{(2)}\big\}_{n\ge 0}$ defined as in (2.1) it is trivial that

$$\begin{aligned} p_{2n}^{(1)}, p_{2n}^{(2)} \in \mathbb{P}_{2n} \setminus \mathbb{P}_{2n-1} \quad &\Leftrightarrow \quad g_{2n}, f_{2n} \in \mathbb{L}_{2n}^+ \setminus \mathbb{L}_{2n-1}^+, \\ p_{2n+1}^{(1)}, p_{2n+1}^{(2)} \in \mathbb{P}_{2n+1} \setminus \mathbb{P}_{2n} \quad &\Leftrightarrow \quad f_{2n+1}, g_{2n+1} \in \mathbb{L}_{2n+1}^+ \setminus \mathbb{L}_{2n}^+. \end{aligned}$$

Furthermore we have using the Toeplitz condition (1.11)

$$\mathcal{L}\big[p^{(1)}_{2n+1}(z), z^k\big] = \mathcal{L}\big[z^n f_{2n+1}(z), z^k\big] = \mathcal{L}\big[f_{2n+1}(z), z^{k-n}\big],$$

and similarly

$$\begin{aligned}
&\mathcal{L}\big[p^{(1)}_{2n}(z), z^k\big] = \mathcal{L}\big[z^{n-k}, g_{2n}(z)\big], \qquad \mathcal{L}\big[z^k, p^{(2)}_{2n+1}(z)\big] = \mathcal{L}\big[z^{k-n}, g_{2n+1}(z)\big],\\
&\mathcal{L}\big[z^k, p^{(2)}_{2n}(z)\big] = \mathcal{L}\big[f_{2n}(z), z^{n-k}\big].
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&\mathcal{L}\big[p^{(1)}_{2n+1}(z), z^k\big] = 0, \quad 0 \le k \le 2n && \Leftrightarrow \quad \mathcal{L}\big[f_{2n+1}(z), z^k\big] = 0, && -n \le k \le n,\\
&\mathcal{L}\big[p^{(1)}_{2n}(z), z^k\big] = 0, \quad 0 \le k \le 2n-1 && \Leftrightarrow \quad \mathcal{L}\big[z^k, g_{2n}(z)\big] = 0, && -n+1 \le k \le n,\\
&\mathcal{L}\big[z^k, p^{(2)}_{2n+1}(z)\big] = 0, \quad 0 \le k \le 2n && \Leftrightarrow \quad \mathcal{L}\big[z^k, g_{2n+1}(z)\big] = 0, && -n \le k \le n,\\
&\mathcal{L}\big[z^k, p^{(2)}_{2n}(z)\big] = 0, \quad 0 \le k \le 2n-1 && \Leftrightarrow \quad \mathcal{L}\big[f_{2n}(z), z^k\big] = 0, && -n+1 \le k \le n,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}\big[p^{(1)}_{2n+1}(z), z^{2n+1}\big] \neq 0 && \Leftrightarrow \quad \mathcal{L}\big[f_{2n+1}(z), z^{n+1}\big] \neq 0,\\
&\mathcal{L}\big[p^{(1)}_{2n}(z), z^{2n}\big] \neq 0 && \Leftrightarrow \quad \mathcal{L}\big[z^{-n}, g_{2n}(z)\big] \neq 0,\\
&\mathcal{L}\big[z^{2n+1}, p^{(2)}_{2n+1}(z)\big] \neq 0 && \Leftrightarrow \quad \mathcal{L}\big[z^{n+1}, g_{2n+1}(z)\big] \neq 0,\\
&\mathcal{L}\big[z^{2n}, p^{(2)}_{2n}(z)\big] \neq 0 && \Leftrightarrow \quad \mathcal{L}\big[f_{2n}(z), z^{-n}\big] \neq 0.
\end{aligned}$$

Thus, according to Remark 2.2, $\{f_n, g_n\}_{n\ge 0}$ is a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\big\{p^{(1)}_n, p^{(2)}_n\big\}_{n\ge 0}$ is a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$. Equation (2.2) follows immediately from the definition (2.1) and the Toeplitz condition (1.11).

The statement (2.3) for sequences of left bi-orthogonal L-polynomials is an immediate consequence of the result for sequences of right bi-orthogonal L-polynomials and Proposition 2.3. This concludes the proof. ■

We are now able to prove the existence and the unicity of bi-orthogonal L-polynomials with respect to $\mathcal{L}$.

**Corollary 2.5.** *Consider a Toeplitz bi-moment functional $\mathcal{L}$. There exists a sequence of right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ and a sequence of left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ if and only if $\mathcal{L}$ is quasi-definite as defined in* (1.14)*. Each L-polynomial in these sequences is uniquely determined up to an arbitrary non-zero factor.*

**Proof.** By virtue of Theorem 2.4, the existence of a sequence of right or left bi-orthogonal L-polynomials with respect to $\mathcal{L}$ is equivalent to the existence of a sequence of bi-orthogonal polynomials with respect to $\mathcal{L}$, which are known to exist if and only $\mathcal{L}$ is quasi-definite. Since bi-orthogonal polynomials are uniquely determined up to an arbitrary non-zero factor, the same holds for right and left bi-orthogonal L-polynomials. ■

From now on we shall assume that $\{f_n, g_n\}_{n\ge 0}$ is a sequence of *monic* right bi-orthogonal L-polynomials with respect to $\mathcal{L}$, i.e. the coefficients of $z^{-n}$ in $f_{2n}$, $g_{2n}$ and $z^{n+1}$ in $f_{2n+1}$, $g_{2n+1}$ are equal to 1. We denote by $\big\{p^{(1)}_n, p^{(2)}_n\big\}_{n\ge 0}$ the associated sequence of monic bi-orthogonal polynomials with respect to $\mathcal{L}$, as defined by (2.1).

## 2.2 Five term recurrence relations

We now prove that bi-orthogonal L-polynomials with respect to a quasi-definite Toeplitz bimoment functional always satisfy five term recurrence relations. This generalizes the result obtained in [10] for orthogonal L-polynomials associated with a quasi-definite Toeplitz sesquilinear hermitian form. The essential ingredient in the proof in [10] is the Toeplitz condition. Consequently, it can immediately be translated to the case of bi-orthogonal L-polynomials.

**Theorem 2.6.** *Let $\{f_n, g_n\}_{n\geq 0}$ be a sequence of monic right bi-orthogonal L-polynomials with respect to $\mathcal{L}$, and $f_n^*(z) = f_n\big(z^{-1}\big)$, $g_n^*(z) = g_n\big(z^{-1}\big)$. Then for $n \geq 0$ there exist five-term recurrence relations*

$$zf_n(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i} f_i(z), \qquad zg_n(z) = \sum_{i=n-2}^{n+2} \beta_{n,i} g_i(z),$$
$$zf_n^*(z) = \sum_{i=n-2}^{n+2} \alpha^*_{n,i} f_i^*(z), \qquad zg_n^*(z) = \sum_{i=n-2}^{n+2} \beta^*_{n,i} g_i^*(z),$$

*where we use the convention $f_n(z) = g_n(z) = 0$ if $n < 0$, and*

$$\alpha^*_{n,i} = \frac{h_n}{h_i}\beta_{i,n}, \qquad \beta^*_{n,i} = \frac{h_n}{h_i}\alpha_{i,n},$$

*with $h_n = \mathcal{L}[f_n, g_n]$. Moreover, we have for all $n \geq 0$*

$$\alpha_{2n-1,2n-3} = 0, \qquad \alpha_{2n,2n+2} = 0, \qquad \beta_{2n-1,2n-3} = 0, \qquad \beta_{2n,2n+2} = 0.$$

**Proof.** As $f_n \in \mathbb{L}^+_n \setminus \mathbb{L}^+_{n-1}$, we have $zf_n(z) \in \mathbb{L}^+_{n+2}$. This implies that $zf_n$ admits an expansion in terms of $f_0, \dots, f_{n+2}$

$$zf_n(z) = \sum_{i=0}^{n+2} \alpha_{n,i} f_i(z),$$

with $\alpha_{n,i} \in \mathbb{C}$, $0 \leq i \leq n+2$. Consequently, by bi-orthogonality of the sequence $\{f_n, g_n\}_{n\geq 0}$ we have

$$\mathcal{L}[zf_n, g_m] = \sum_{i=0}^{n+2} h_i \alpha_{n,i} \delta_{i,m}.$$

But we also have

$$\mathcal{L}[zf_n, zg_k] = \mathcal{L}[f_n, g_k] = 0, \qquad 0 \leq k \leq n-1,$$

and $\langle g_0, \dots, g_{n-3}\rangle \subset \langle zg_0, \dots, zg_{n-1}\rangle$. It follows that

$$\mathcal{L}[zf_n, g_k] = 0, \qquad 0 \leq k \leq n-3.$$

Consequently we have $\alpha_{n,i} = 0$ if $i < n-2$, and thus

$$zf_n(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i} f_i(z).$$

We prove that $\alpha_{2n,2n+2} = \alpha_{2n-1,2n-3} = 0$. We first prove that $\alpha_{2n,2n+2} = 0$. Indeed, we have $zf_{2n}(z) \in \big\langle z^{1-n}, \dots, z^{1+n}\big\rangle$. Consequently, using condition $(3r)$ in Remark 2.2, we have

$\mathcal{L}[zf_{2n}, g_{2n+2}] = 0$ and thus $\alpha_{2n,2n+2} = 0$. We also have $\alpha_{2n-1,2n-3} = 0$. Indeed, we have $\mathcal{L}[zf_{2n-1}, g_{2n-3}] = \mathcal{L}[f_{2n-1}, z^{-1}g_{2n-3}]$, and $z^{-1}g_{2n-3}(z) \in \langle z^{1-n}, \dots, z^{n-2}\rangle$. From condition $(3r)$ in Remark 2.2, it follows that $\mathcal{L}[zf_{2n-1}, g_{2n-3}] = 0$. A similar argument gives $\beta_{2n,2n+2} = \beta_{2n-1,2n-3} = 0$. The proof of the other recurrence relations is similar.

The coefficients in the recurrence relations satisfy

$$\alpha_{n,i} = \frac{\mathcal{L}[zf_n, g_i]}{\mathcal{L}[f_i, g_i]}, \qquad \beta_{n,i} = \frac{\mathcal{L}[f_i, zg_n]}{\mathcal{L}[f_i, g_i]},$$
$$\alpha^*_{n,i} = \frac{\mathcal{L}[g^*_i, zf^*_n]}{\mathcal{L}[g^*_i, f^*_i]}, \qquad \beta^*_{n,i} = \frac{\mathcal{L}[zg^*_n, f^*_i]}{\mathcal{L}[g^*_i, f^*_i]}.$$

It follows from the definition of $\{g^*_n, f^*_n\}_{n\geq 0}$ that

$$\alpha^*_{n,i} = \frac{\mathcal{L}[g^*_i, zf^*_n]}{\mathcal{L}[g^*_i, f^*_i]} = \frac{\mathcal{L}[f_n, zg_i]}{\mathcal{L}[f_i, g_i]} = \frac{\mathcal{L}[f_n, zg_i]}{\mathcal{L}[f_n, g_n]}\frac{\mathcal{L}[f_n, g_n]}{\mathcal{L}[f_i, g_i]} = \beta_{i,n}\frac{h_n}{h_i}.$$

Similarly we have

$$\beta^*_{n,i} = \frac{\mathcal{L}[zg^*_n, f^*_i]}{\mathcal{L}[g^*_i, f^*_i]} = \frac{\mathcal{L}[zf_i, g_n]}{\mathcal{L}[f_i, g_i]} = \frac{\mathcal{L}[zf_i, g_n]}{\mathcal{L}[f_n, g_n]}\frac{\mathcal{L}[f_n, g_n]}{\mathcal{L}[f_i, g_i]} = \alpha_{i,n}\frac{h_n}{h_i}.$$

This concludes the proof. ■

**Corollary 2.7.** *With the same notations as in Theorem 2.6 we have*

$$z^{-1}f_n(z) = \sum_{i=n-2}^{n+2} \alpha^*_{n,i} f_i(z), \qquad z^{-1}g_n(z) = \sum_{i=n-2}^{n+2} \beta^*_{n,i} g_i(z),$$
$$z^{-1}f^*_n(z) = \sum_{i=n-2}^{n+2} \alpha_{n,i} f^*_i(z), \qquad z^{-1}g^*_n(z) = \sum_{i=n-2}^{n+2} \beta_{n,i} g^*_i(z).$$

Defining the vectors

$$f(z) = \big(f_n(z)\big)_{n\geq 0}, \qquad g(z) = \big(g_n(z)\big)_{n\geq 0}, \tag{2.4}$$
$$f^*(z) = f\big(z^{-1}\big) = \big(f^*_n(z)\big)_{n\geq 0}, \qquad g^*(z) = g\big(z^{-1}\big) = \big(g^*_n(z)\big)_{n\geq 0}, \tag{2.5}$$

the five term recurrence relations obtained in Theorem 2.6 and Corollary 2.7 can be written in vector form

$$\begin{cases} zf(z) = A_1 f(z), \\ zg(z) = A_2 g(z), \\ z^{-1}f(z) = A^*_1 f(z), \\ z^{-1}g(z) = A^*_2 g(z), \end{cases} \qquad \begin{cases} zf^*(z) = A^*_1 f^*(z), \\ zg^*(z) = A^*_2 g^*(z), \\ z^{-1}f^*(z) = A_1 f^*(z), \\ z^{-1}g^*(z) = A_2 g^*(z), \end{cases} \tag{2.6}$$

with

$$A_1 = \big(\alpha_{i,j}\big)_{i,j\geq 0}, \qquad A_2 = \big(\beta_{i,j}\big)_{i,j\geq 0},$$

where $\alpha_{i,j} = \beta_{i,j} = 0$ if $|i-j| > 2$, and

$$A^*_1 = hA_2^T h^{-1}, \qquad A^*_2 = hA_1^T h^{-1}, \tag{2.7}$$

where $h = \operatorname{diag}(h_n)_{n\geq 0}$. We call the matrices $A_1$, $A_2$ the (generalized) CMV matrices. Clearly, from (2.6), we have

$$A^*_1 = A_1^{-1}, \qquad A^*_2 = A_2^{-1}. \tag{2.8}$$

### 2.3 Explicit expression for the entries of the CMV matrices

Explicit expressions for the entries of the CMV matrices can be found in terms of the variables $x_n$, $y_n$ introduced in (1.15) entering the Szegö type recurrence relations (1.16).

**Theorem 2.8.** *The non-zero entries of the CMV matrices $A_1$ and $A_2$ are*

$$\begin{aligned}
&(A_1)_{2n-1,2n+1} = 1, && (A_1)_{2n-1,2n-1} = -x_{2n}y_{2n-1},\\
&(A_1)_{2n-1,2n} = -x_{2n+1}, && (A_1)_{2n-1,2n-2} = -x_{2n}(1-x_{2n-1}y_{2n-1}),\\
&(A_1)_{2n,2n+1} = y_{2n}, && (A_1)_{2n,2n-1} = y_{2n-1}(1-x_{2n}y_{2n}),\\
&(A_1)_{2n,2n} = -x_{2n+1}y_{2n}, && (A_1)_{2n,2n-2} = (1-x_{2n-1}y_{2n-1})(1-x_{2n}y_{2n}),
\end{aligned}$$

*and*

$$\begin{aligned}
&(A_2)_{2n-1,2n+1} = 1, && (A_2)_{2n-1,2n-1} = -x_{2n-1}y_{2n},\\
&(A_2)_{2n-1,2n} = -y_{2n+1}, && (A_2)_{2n-1,2n-2} = -y_{2n}(1-x_{2n-1}y_{2n-1}),\\
&(A_2)_{2n,2n+1} = x_{2n}, && (A_2)_{2n,2n-1} = x_{2n-1}(1-x_{2n}y_{2n}),\\
&(A_2)_{2n,2n} = -x_{2n}y_{2n+1}, && (A_2)_{2n,2n-2} = (1-x_{2n-1}y_{2n-1})(1-x_{2n}y_{2n}).
\end{aligned}$$

**Proof.** (1) We have

$$(A_1)_{2n-1,2n+1} = \frac{1}{h_{2n+1}}\mathcal{L}\big[zf_{2n-1}(z), g_{2n+1}(z)\big].$$

By virtue of Theorem 2.4 we obtain

$$(A_1)_{2n-1,2n+1} = \frac{1}{h_{2n+1}}\mathcal{L}\big[z^{2-n}p^{(1)}_{2n-1}(z), z^{-n}p^{(2)}_{2n+1}(z)\big] = \frac{1}{h_{2n+1}}\mathcal{L}\big[z^2p^{(1)}_{2n-1}(z), p^{(2)}_{2n+1}(z)\big].$$

As $z^2p^{(1)}_{2n-1}(z)$ is a monic polynomial of degree $2n+1$, using the bi-orthogonality of the polynomials, we have

$$(A_1)_{2n-1,2n+1} = \frac{1}{h_{2n+1}}\mathcal{L}\big[z^{2n+1}, p^{(2)}_{2n+1}(z)\big] = 1.$$

(2) We have

$$(A_1)_{2n-1,2n} = \frac{1}{h_{2n}}\mathcal{L}[zf_{2n-1}(z), g_{2n}(z)].$$

By virtue of Theorem 2.4 we obtain

$$(A_1)_{2n-1,2n} = \frac{1}{h_{2n}}\mathcal{L}\big[z^{2-n}p^{(1)}_{2n-1}(z), z^{n}p^{(1)}_{2n}\big(z^{-1}\big)\big] = \frac{1}{h_{2n}}\mathcal{L}\big[z^2p^{(1)}_{2n-1}(z), z^{2n}p^{(1)}_{2n}\big(z^{-1}\big)\big].$$

By using twice (1.16) we have

$$z^2p^{(1)}_{2n-1}(z) = p^{(1)}_{2n+1}(z) - x_{2n+1}z^{2n}p^{(2)}_{2n}\big(z^{-1}\big) - x_{2n}z^{2n}p^{(2)}_{2n-1}\big(z^{-1}\big),$$

and thus

$$\begin{aligned}
(A_1)_{2n-1,2n} = {}& \frac{1}{h_{2n}}\mathcal{L}\big[p^{(1)}_{2n+1}(z), z^{2n}p^{(1)}_{2n}\big(z^{-1}\big)\big] - \frac{x_{2n+1}}{h_{2n}}\mathcal{L}\big[p^{(2)}_{2n}\big(z^{-1}\big), p^{(1)}_{2n}\big(z^{-1}\big)\big]\\
& - \frac{x_{2n}}{h_{2n}}\mathcal{L}\big[p^{(2)}_{2n-1}\big(z^{-1}\big), p^{(1)}_{2n}\big(z^{-1}\big)\big].
\end{aligned}$$

As $z^{2n}p_{2n}^{(1)}(z^{-1})$ is a polynomial of degree $2n$, the first term is equal to 0 by bi-orthogonality. The remaining terms give

$$(A_1)_{2n-1,2n} = -\frac{x_{2n+1}}{h_{2n}}\mathcal{L}\big[p_{2n}^{(1)}(z), p_{2n}^{(2)}(z)\big] - \frac{x_{2n}}{h_{2n}}\mathcal{L}\big[p_{2n}^{(1)}(z), p_{2n-1}^{(2)}(z)\big] = -x_{2n+1}.$$

(3) We have

$$(A_1)_{2n-1,2n-1} = \frac{1}{h_{2n-1}}\mathcal{L}[zf_{2n-1}(z), g_{2n-1}(z)].$$

By virtue of Theorem 2.4 we obtain

$$(A_1)_{2n-1,2n-1} = \frac{1}{h_{2n-1}}\mathcal{L}\big[z^{2-n}p_{2n-1}^{(1)}(z), z^{1-n}p_{2n-1}^{(2)}(z)\big] = \frac{1}{h_{2n-1}}\mathcal{L}\big[zp_{2n-1}^{(1)}(z), p_{2n-1}^{(2)}(z)\big].$$

By using (1.16) and then (1.15) we have

$$\begin{aligned}(A_1)_{2n-1,2n-1} &= \frac{1}{h_{2n-1}}\mathcal{L}\big[p_{2n}^{(1)}(z) - x_{2n}z^{2n-1}p_{2n-1}^{(2)}\big(z^{-1}\big), p_{2n-1}^{(2)}(z)\big]\\ &= -\frac{x_{2n}}{h_{2n-1}}\mathcal{L}\big[z^{2n-1}p_{2n-1}^{(2)}\big(z^{-1}\big), p_{2n-1}^{(2)}(z)\big]\\ &= -\frac{x_{2n}}{h_{2n-1}}\mathcal{L}\big[y_{2n-1}z^{2n-1}, p_{2n-1}^{(2)}(z)\big] = -x_{2n}y_{2n-1}.\end{aligned}$$

(4) We have

$$(A_1)_{2n-1,2n-2} = \frac{1}{h_{2n-2}}\mathcal{L}[zf_{2n-1}(z), g_{2n-2}(z)].$$

By virtue of Theorem 2.4 we obtain

$$\begin{aligned}(A_1)_{2n-1,2n-2} &= \frac{1}{h_{2n-2}}\mathcal{L}\big[z^{2-n}p_{2n-1}^{(1)}(z), z^{n-1}p_{2n-2}^{(1)}\big(z^{-1}\big)\big]\\ &= \frac{1}{h_{2n-2}}\mathcal{L}\big[zp_{2n-1}^{(1)}(z), z^{2n-2}p_{2n-2}^{(1)}\big(z^{-1}\big)\big].\end{aligned}$$

Using (1.16) we obtain

$$\begin{aligned}(A_1)_{2n-1,2n-2} &= \frac{1}{h_{2n-2}}\mathcal{L}\big[p_{2n}^{(1)}(z) - x_{2n}z^{2n-1}p_{2n-1}^{(2)}\big(z^{-1}\big), z^{2n-2}p_{2n-2}^{(1)}\big(z^{-1}\big)\big]\\ &= \frac{1}{h_{2n-2}}\mathcal{L}\big[p_{2n}^{(1)}(z), z^{2n-2}p_{2n-2}^{(1)}\big(z^{-1}\big)\big] - \frac{x_{2n}}{h_{2n-2}}\mathcal{L}\big[zp_{2n-1}^{(2)}\big(z^{-1}\big), p_{2n-2}^{(1)}\big(z^{-1}\big)\big].\end{aligned}$$

The first term is equal to 0 as $z^{2n-2}p_{2n-2}^{(1)}\big(z^{-1}\big)$ is a polynomial of degree $2n-2$. Consequently, using (1.17), we have

$$\begin{aligned}(A_1)_{2n-1,2n-2} &= -\frac{x_{2n}}{h_{2n-2}}\mathcal{L}\big[zp_{2n-2}^{(1)}(z), p_{2n-1}^{(2)}(z)\big] = -\frac{x_{2n}}{h_{2n-2}}\mathcal{L}\big[z^{2n-1}, p_{2n-1}^{(2)}(z)\big]\\ &= -\frac{h_{2n-1}}{h_{2n-2}}x_{2n} = -(1 - x_{2n-1}y_{2n-1})x_{2n}.\end{aligned}$$

(5) The other relations are proven in a similar way. This finishes the proof. ■

# 3 The AL hierarchy and a Lax pair for its master symmetries

In this section we "dress up" the equations defining the Ablowitz–Ladik hierarchy (1.19) and its master symmetries (1.21) on the bi-moments. This leads to Lax pair representations both for the hierarchy and its master symmetries on the CMV matrices. In all this section we shall denote the time variables $(t,s) = (t_1, t_2, \dots, s_1, s_2, \dots)$ of the AL hierarchy by $(t_k)_{k\in\mathbb{Z}}$, with $t_{-k} = s_k$, $k \geq 1$, and $T_0$ defined as in the Introduction (see below (1.20)). It is only in the next section that the notation $(t,s)$ will be more convenient.

## 3.1 The Ablowitz–Ladik hierarchy

Let

$$\chi(z) = \left(1, z, z^{-1}, z^2, z^{-2}, \dots\right)^T, \tag{3.1}$$

and let $\mathcal{L}$ be a quasi-definite bi-moment functional satisfying the Toeplitz condition. We introduce two matrices $S_1$ and $S_2$ by writing the vectors $f(z)$, $g(z)$ (2.4) of monic right bi-orthogonal L-polynomials with respect to $\mathcal{L}$ as follows

$$f(z) = S_1\chi(z), \qquad g(z) = h\left(S_2^T\right)^{-1}\chi(z), \tag{3.2}$$

with $h = \operatorname{diag}(h_n)_{n\geq 0}$ and $h_n = \mathcal{L}[f_n, g_n]$. With this definition, $S_1$ is a lower triangular matrix with all diagonal elements equal to 1, and $S_2$ is an upper triangular matrix such that $h^{-1}S_2$ has all diagonal elements equal to 1.

Associated to $\mathcal{L}$ we also define the semi-infinite bi-moment matrix

$$M = \begin{pmatrix} \mu_{0,0} & \mu_{0,1} & \mu_{0,-1} & \cdots \\ \mu_{1,0} & \mu_{1,1} & \mu_{1,-1} & \cdots \\ \mu_{-1,0} & \mu_{-1,1} & \mu_{-1,-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.3}$$

with $\mu_{m,n}$ as in (1.10), (1.12). The bi-moment matrix $M$ can be written in terms of the vector $\chi(z)$ in (3.1)

$$M = \left(\mathcal{L}\left[\left(\chi(z)\right)_m, \left(\chi(z)\right)_n\right]\right)_{0\leq m,n<\infty}.$$

The existence of a sequence of right bi-orthogonal L-polynomials for $\mathcal{L}$ is equivalent to the existence of a factorisation of the bi-moment matrix $M$ in a product of a lower triangular matrix and an upper triangular matrix with non-zero diagonal elements.

**Proposition 3.1.** *The bi-moment matrix $M$ factorizes in a product of a lower triangular matrix and an upper triangular matrix*

$$M = S_1^{-1}S_2.$$

**Proof.** By bi-orthogonality of the sequence $\{f_n, g_n\}_{n\geq 0}$, we have

$$\mathcal{L}[f_m, g_n] = h_m\delta_{m,n}.$$

This can be written in matrix form

$$h = \left(\mathcal{L}[f_m, g_n]\right)_{0\leq m,n<\infty}.$$

Using the expressions (3.2) we obtain

$$h = \left(\mathcal{L}\left[\left(S_1\chi(z)\right)_m, \left(h\left(S_2^T\right)^{-1}\chi(z)\right)_n\right]\right)_{0\leq m,n\leq\infty} = S_1 M S_2^{-1} h.$$

Consequently we have

$$M = S_1^{-1} S_2,$$

which establishes the result. ■

We define the semi-infinite shift matrix $\Lambda$ by

$$\Lambda\chi(z) = z\chi(z). \tag{3.4}$$

We have

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.5}$$

and $\Lambda^{-1} = \Lambda^T$. We leave to the reader to check that, because of the Toeplitz property satisfied by the bi-moments in (3.3), we have the commutation relation

$$[\Lambda, M] = 0. \tag{3.6}$$

The CMV matrices can be obtained by "dressing up" the shift $\Lambda$.

**Proposition 3.2.** *We have*

$$A_1 = S_1\Lambda S_1^{-1}, \qquad A_2 = h\left(S_2^T\right)^{-1}\Lambda S_2^T h^{-1}, \tag{3.7}$$

$$A_1^{-1} = S_2\Lambda^T S_2^{-1}, \qquad A_2^{-1} = h\left(S_1^{-1}\right)^T\Lambda^T S_1^T h^{-1}, \tag{3.8}$$

*with $S_1$ and $S_2$ defined in (3.2).*

**Proof.** We have

$$A_1 f(z) = zf(z) = zS_1\chi(z) = S_1\Lambda\chi(z) = S_1\Lambda S_1^{-1} f(z).$$

It follows that

$$A_1 = S_1\Lambda S_1^{-1}.$$

The proof for $A_2$ is similar. The factorisations in (3.8) follow from (2.7), (2.8) and (3.7). ■

Remember that because $\mathcal{L} : \mathbb{C}[z, z^{-1}] \times \mathbb{C}[z, z^{-1}] \to \mathbb{C}$ is a Toeplitz bi-moment functional, the bi-moments $\mu_{m,n} = \mathcal{L}[z^m, z^n]$ only depend on the difference $m - n$ and can be written as in (1.12) $\mu_{m,n} := \mu_{m-n}$.

The Ablowitz–Ladik hierarchy is defined on the space of bi-moments by the vector fields

$$\boxed{T_k\mu_j \equiv \frac{\partial\mu_j}{\partial t_k} = \mu_{j+k}, \qquad \forall\, k \in \mathbb{Z},} \tag{3.9}$$

where we have put $s_k = t_{-k}$ in (1.19). Obviously, these vector fields satisfy the commutation relations

$$[T_k, T_l] = 0, \qquad \forall\, k, l \in \mathbb{Z}.$$

It follows from the definition of $\Lambda$ in (3.4) and (3.9) that the time evolution of the bi-moment matrix $M$ is given by the equations

$$\boxed{\frac{\partial M}{\partial t_k} = \Lambda^k M, \qquad \forall\, k \in \mathbb{Z}.} \tag{3.10}$$

Equations (3.9) and (3.10) are two equivalent formulations of the Ablowitz–Ladik vector fields at the level of the bi-moments.

For a square matrix $A$, we define

- $A_0$ the diagonal part of $A$;
- $A_-$ (resp. $A_+$) the lower (resp. upper) triangular part of $A$;
- $A_{--}$ (resp. $A_{++}$) the strictly lower (resp. strictly upper) triangular part of $A$.

We establish the following lemma, based on the factorisation of the moment matrix $M$ in Proposition 3.1 in a product of a lower triangular and an upper triangular matrix.

**Lemma 3.3.** *We have for $k \in \mathbb{Z}$*

$$\frac{\partial S_1}{\partial t_k} S_1^{-1} = -\big(A_1^k\big)_{--}, \tag{3.11}$$

$$\big(S_2^T h^{-1}\big)^{-1} \frac{\partial\big(S_2^T h^{-1}\big)}{\partial t_k} = \big(A_2^{-k}\big)_{--}. \tag{3.12}$$

**Proof.** On the one hand, we have using Proposition 3.1

$$\frac{\partial M}{\partial t_k} = -S_1^{-1}\frac{\partial S_1}{\partial t_k} S_1^{-1} S_2 + S_1^{-1}\frac{\partial S_2}{\partial t_k}.$$

On the other hand, from equation (3.10) we have

$$\frac{\partial M}{\partial t_k} = \Lambda^k M = \Lambda^k S_1^{-1} S_2.$$

As $A_1 = S_1 \Lambda S_1^{-1}$, we obtain

$$A_1^k = -\frac{\partial S_1}{\partial t_k} S_1^{-1} + \frac{\partial S_2}{\partial t_k} S_2^{-1}.$$

Since $\frac{\partial S_1}{\partial t_k}$ is strictly lower triangular, the first term in the right hand side of this equation is strictly lower triangular. The second term is upper triangular. Consequently, taking the strictly lower triangular part of both sides of the equation yields

$$\frac{\partial S_1}{\partial t_k} S_1^{-1} = -\big(A_1^k\big)_{--},$$

which establishes (3.11).

To establish the other formula, we write $M = \big(S_1^{-1} h\big)\big(h^{-1} S_2\big)$ which gives

$$\frac{\partial M}{\partial t_k} = \frac{\partial\big(S_1^{-1} h\big)}{\partial t_k}\big(h^{-1} S_2\big) + \big(S_1^{-1} h\big)\frac{\partial\big(h^{-1} S_2\big)}{\partial t_k}.$$

Using the commutation relation (3.6) and (3.10), we also have

$$\frac{\partial M}{\partial t_k} = M\Lambda^k = \left(S_1^{-1}h\right)\left(h^{-1}S_2\right)\Lambda^k.$$

As $A_2 = \left(S_2^T h^{-1}\right)^{-1}\Lambda\left(S_2^T h^{-1}\right)$, we obtain after some algebra

$$A_2^{-k} = \frac{\partial\left(S_1^{-1}h\right)^T}{\partial t_k}\left(\left(S_1^{-1}h\right)^T\right)^{-1} + \left(S_2^T h^{-1}\right)^{-1}\frac{\partial\left(S_2^T h^{-1}\right)}{\partial t_k}.$$

Since $\left(S_1^{-1}h\right)^T$ is upper triangular, the first term in the right hand side of this equation is upper triangular. As $S_2^T h^{-1}$ is lower triangular with all diagonal entries equal to 1, the second term is strictly lower triangular. Consequently, taking the strictly lower triangular part of both sides of the equation yields

$$\left(S_2^T h^{-1}\right)^{-1}\frac{\partial\left(S_2^T h^{-1}\right)}{\partial t_k} = \left(A_2^{-k}\right)_{--},$$

which establishes (3.12), completing the proof. ■

We are now able to obtain a Lax pair representation for the Ablowitz–Ladik hierarchy.

**Theorem 3.4.** *The "dressed up" form of the moment equation* (3.10) *gives the following Lax pair representation for the Ablowitz–Ladik hierarchy on the semi-infinite CMV matrices* $(A_1, A_2)$

$$\boxed{\frac{\partial A_1}{\partial t_k} = \left[A_1, \left(A_1^k\right)_{--}\right], \qquad \frac{\partial A_2}{\partial t_k} = \left[A_2, \left(A_2^{-k}\right)_{--}\right], \qquad \forall\, k \in \mathbb{Z}.} \tag{3.13}$$

**Proof.** As $A_1 = S_1\Lambda S_1^{-1}$ and $A_2 = \left(S_2^T h^{-1}\right)^{-1}\Lambda\left(S_2^T h^{-1}\right)$, we have

$$\frac{\partial A_1}{\partial t_k} = \left[\frac{\partial S_1}{\partial t_k}S_1^{-1}, A_1\right] \qquad \text{and} \qquad \frac{\partial A_2}{\partial t_k} = \left[A_2, \left(S_2^T h^{-1}\right)^{-1}\frac{\partial\left(S_2^T h^{-1}\right)}{\partial t_k}\right].$$

By Lemma 3.3 we obtain

$$\frac{\partial A_1}{\partial t_k} = \left[-\left(A_1^k\right)_{--}, A_1\right] \qquad \text{and} \qquad \frac{\partial A_2}{\partial t_k} = \left[A_2, \left(A_2^{-k}\right)_{--}\right],$$

which establishes (3.13), concluding the proof. ■

**Remark 3.5.** Looking back at the explicit expressions for the entries of the CMV matrices in Theorem 2.8, the reader will observe that the entries of $A_2$ are obtained from those of $A_1$ by exchanging the roles of the variables $x_n$ and $y_n$. Also $A_1$ contains as entries $-x_{2n+1}$ and $y_{2n}$ and thus $A_2$ contains as entries $x_{2n}$ and $-y_{2n+1}$, $n \geq 0$ (remember that $x_0 = y_0 = 1$). Thus the pair of Lax equations in (3.13) completely determines the Ablowitz–Ladik hierarchy in terms of the variables $x_n$ and $y_n$.

Using the explicit expressions in terms of the variables $x_n$ and $y_n$ for the entries of the CMV matrices obtained in Theorem 2.8, and Theorem 3.4, one easily computes the equations for the vector fields $T_1$ and $T_{-1}$

$$\begin{aligned} \frac{\partial x_n}{\partial t_1} &= (1 - x_n y_n)x_{n+1}, & \frac{\partial x_n}{\partial t_{-1}} &= -(1 - x_n y_n)x_{n-1}, \\ \frac{\partial y_n}{\partial t_1} &= -(1 - x_n y_n)y_{n-1}, & \frac{\partial y_n}{\partial t_{-1}} &= (1 - x_n y_n)y_{n+1}. \end{aligned}$$

After the rescaling $x_n \to e^{-2t}x_n$, $y_n \to e^{2t}y_n$, the vector field $T_1 - T_{-1}$ reduces to the Ablowitz–Ladik equations as written in (1.8). In this paper, we won't discuss the Hamiltonian structure of the AL hierarchy in terms of the CMV matrices $A_1$ and $A_2$. One can show that for $k \geq 1$

$$\frac{\partial x_n}{\partial t_k} = (1 - x_n y_n)\frac{\partial H_k^{(1)}}{\partial y_n}, \qquad \frac{\partial x_n}{\partial t_{-k}} = (1 - x_n y_n)\frac{\partial H_k^{(2)}}{\partial y_n},$$
$$\frac{\partial y_n}{\partial t_k} = -(1 - x_n y_n)\frac{\partial H_k^{(1)}}{\partial x_n}, \qquad \frac{\partial y_n}{\partial t_{-k}} = -(1 - x_n y_n)\frac{\partial H_k^{(2)}}{\partial x_n},$$

where $H_k^{(1)} = -\frac{1}{k}\operatorname{Tr} A_1^k$, $H_k^{(2)} = \frac{1}{k}\operatorname{Tr} A_2^k$ and Tr denotes the formal trace, see [38] for a proof inspired by [5] in the context of Hessenberg matrices.

### 3.2 A Lax pair for the master symmetries

In this section we translate the action of the master symmetries vector fields $V_k$, $k \in \mathbb{Z}$, defined on the bi-moments by (1.21), on the CMV matrices $(A_1, A_2)$.

We first decompose the vector fields $V_k$ as follows

$$V_k = kT_k + \mathcal{V}_k, \tag{3.14}$$

where $T_k$ are the Ablowitz–Ladik vector fields (3.9). At the level of the bi-moments, the vector fields $\mathcal{V}_k$ are given by

$$\boxed{\mathcal{V}_k\mu_j \equiv \frac{\mathrm{d}}{\mathrm{d}u_k}\mu_j = j\mu_{j+k}, \qquad j, k \in \mathbb{Z}.} \tag{3.15}$$

These vector fields satisfy the following commutation relations

$$[\mathcal{V}_k, \mathcal{V}_l] = (l - k)\mathcal{V}_{k+l}, \qquad [\mathcal{V}_k, T_l] = lT_{k+l}.$$

It follows that

$$[[\mathcal{V}_k, T_l], T_l] = 0, \qquad \forall\, k, l \in \mathbb{Z}.$$

Consequently, like the vector fields $V_k$, the vector fields $\mathcal{V}_k$, $k \in \mathbb{Z}$, form a Virasoro algebra of master symmetries for the Ablowitz–Ladik hierarchy.

The differentiation of $\chi(z)$ with respect to $z$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}z}\chi(z) = \delta\chi(z), \tag{3.16}$$

where

$$\delta = \Delta\Lambda^T, \qquad \text{with} \qquad \Delta = \operatorname{diag}(0, 1, -1, 2, -2, \ldots), \tag{3.17}$$

and $\Lambda$ is as in (3.5).

Remembering the notation (1.12), (3.15) writes

$$\frac{\mathrm{d}}{\mathrm{d}u_k}\mu_{m,n} = (m - n)\mu_{m+k,n},$$

which is equivalent to the following equation on the bi-moment matrix $M$

$$\boxed{\frac{\mathrm{d}M}{\mathrm{d}u_k} = \Delta\Lambda^k M - \Lambda^k M\Delta = \big[\Delta, \Lambda^k M\big].} \tag{3.18}$$

Remember from (3.2) that

$$f(z)=S_1\chi(z),\qquad g(z)=h\big(S_2^T\big)^{-1}\chi(z),\tag{3.19}$$

and, according to (2.6) and (2.7), these vectors satisfy

$$A_1f(z)=zf(z),\qquad A_1^T\big(h^{-1}g^*(z)\big)=z\big(h^{-1}g^*(z)\big),\tag{3.20}$$

$$A_2g(z)=zg(z),\qquad A_2^T\big(h^{-1}f^*(z)\big)=z\big(h^{-1}f^*(z)\big).\tag{3.21}$$

We define the semi-infinite matrices $D_1$, $D_1^*$ and $D_2$, $D_2^*$ by the relations

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z)=D_1f(z),\qquad \frac{\mathrm{d}}{\mathrm{d}z}\big(h^{-1}g^*(z)\big)=(D_1^*)^T\big(h^{-1}g^*(z)\big),\tag{3.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}g(z)=D_2g(z),\qquad \frac{\mathrm{d}}{\mathrm{d}z}\big(h^{-1}f^*(z)\big)=(D_2^*)^T\big(h^{-1}f^*(z)\big).\tag{3.23}$$

These matrices can be "dressed up" as explained in the next lemma.

**Lemma 3.6.** *We have*

$$D_1=S_1\Delta\Lambda^TS_1^{-1},\qquad D_1^*=-S_2\Lambda^T\Delta S_2^{-1},\tag{3.24}$$

$$D_2=\big(S_2^Th^{-1}\big)^{-1}\Delta\Lambda^T\big(S_2^Th^{-1}\big),\qquad D_2^*=-\big(S_1^Th^{-1}\big)^{-1}\Lambda^T\Delta\big(S_1^Th^{-1}\big),\tag{3.25}$$

*with $\Delta$ as in* (3.17)*.*

**Proof.** Using (3.19) and (3.22), we have

$$D_1f(z)=\frac{\mathrm{d}}{\mathrm{d}z}f(z)=S_1\frac{\mathrm{d}}{\mathrm{d}z}\chi(z).$$

By definition of $\delta$ in (3.16) and (3.17), we get

$$D_1f(z)=S_1\delta S_1^{-1}f(z)=S_1\Delta\Lambda^TS_1^{-1}f(z).$$

This proves the first formula in (3.24).

Using (3.19) and remembering from (2.5) that $g^*(z)=g\big(z^{-1}\big)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}g^*(z)=h\big(S_2^T\big)^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\chi(z^{-1})=-h\big(S_2^T\big)^{-1}z^{-2}\left(\frac{\mathrm{d}}{\mathrm{d}u}\chi(u)\right)\bigg|_{u=z^{-1}},$$

which gives, using (3.16), (3.17), (3.19) and remembering the definition (3.4) of the shift matrix $\Lambda$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}g^*(z)&=-h\big(S_2^T\big)^{-1}\delta z^{-2}\chi(z^{-1})=-h\big(S_2^T\big)^{-1}\Delta\Lambda^T\Lambda^2\chi(z^{-1})\\&=-h\big(S_2^T\big)^{-1}\Delta\Lambda\big(h\big(S_2^T\big)^{-1}\big)^{-1}g^*(z).\end{aligned}$$

Consequently, using the definition (3.22) of $D_1^*$

$$(D_1^*)^T\big(h^{-1}g^*(z)\big)=\frac{\mathrm{d}}{\mathrm{d}z}\big(h^{-1}g^*(z)\big)=-\big(S_2^T\big)^{-1}\Delta\Lambda S_2^T\big(h^{-1}g^*(z)\big).$$

This proves the second formula in (3.24).

The proof of (3.25) is identical to the proof of (3.24) using (3.19) and the definitions of $D_2$ and $D_2^*$ in (3.23). This establishes the lemma. ■

**Lemma 3.7.** *We have for $k \in \mathbb{Z}$*

$$\frac{\mathrm{d}S_1}{\mathrm{d}u_k}S_1^{-1} = -\big(D_1A_1^{k+1}\big)_{--} - \big(A_1^{k+1}D_1^*\big)_{--}, \tag{3.26}$$

$$\big(S_2^Th^{-1}\big)^{-1}\frac{\mathrm{d}\big(S_2^Th^{-1}\big)}{\mathrm{d}u_k} = -\big(D_2A_2^{1-k}\big)_{--} - \big(A_2^{1-k}D_2^*\big)_{--}. \tag{3.27}$$

**Proof.** By substituting the factorisation $M = S_1^{-1}S_2$ of the moment matrix into (3.18), we obtain

$$-S_1^{-1}\frac{\mathrm{d}S_1}{\mathrm{d}u_k}S_1^{-1}S_2 + S_1^{-1}\frac{\mathrm{d}S_2}{\mathrm{d}u_k} = \Delta\Lambda^kS_1^{-1}S_2 - \Lambda^kS_1^{-1}S_2\Delta.$$

Multiplying this equation on the left by $S_1$ and on the right by $S_2^{-1}$, we get

$$-\frac{\mathrm{d}S_1}{\mathrm{d}u_k}S_1^{-1} + \frac{\mathrm{d}S_2}{\mathrm{d}u_k}S_2^{-1} = \underbrace{S_1\Delta\Lambda^kS_1^{-1}}_{\text{Term1}} - \underbrace{S_1\Lambda^kS_1^{-1}S_2\Delta S_2^{-1}}_{\text{Term2}}. \tag{3.28}$$

Using the factorisation of $A_1$ given in (3.7) and the factorisation of $D_1$ in (3.24), Term1 gives

$$\text{Term1} = S_1\Delta\Lambda^T\Lambda^{k+1}S_1^{-1} = \big(S_1\Delta\Lambda^TS_1^{-1}\big)\big(S_1\Lambda^{k+1}S_1^{-1}\big) = D_1A_1^{k+1}.$$

Similarly, Term2 gives

$$\text{Term2} = A_1^kS_2\Delta S_2^{-1} = A_1^{k+1}A_1^{-1}S_2\Delta S_2^{-1}.$$

Using the factorisation of $A_1^{-1}$ in (3.8) we get

$$\text{Term2} = A_1^{k+1}\big(S_2\Lambda^TS_2^{-1}\big)S_2\Delta S_2^{-1} = A_1^{k+1}\big(S_2\Lambda^T\Delta S_2^{-1}\big) = -A_1^{k+1}D_1^*,$$

where we have used the expression of $D_1^*$ in Lemma 3.6. Substituting these results in (3.28), we obtain

$$-\frac{\mathrm{d}S_1}{\mathrm{d}u_k}S_1^{-1} + \frac{\mathrm{d}S_2}{\mathrm{d}u_k}S_2^{-1} = D_1A_1^{k+1} + A_1^{k+1}D_1^*.$$

The first term in the left-hand side is strictly lower triangular, while the second term in the left-hand side is upper triangular. Consequently, taking the strictly lower triangular part in both sides, we obtain

$$\frac{\mathrm{d}S_1}{\mathrm{d}u_k}S_1^{-1} = -\big(D_1A_1^{k+1}\big)_{--} - \big(A_1^{k+1}D_1^*\big)_{--},$$

which establishes (3.26).

To establish the other formula, we substitute the factorisation $M = \big(S_1^{-1}h\big)\big(h^{-1}S_2\big)$ into equation (3.18) rewritten as

$$\frac{\mathrm{d}M}{\mathrm{d}u_k} = \big[\Delta, M\Lambda^k\big],$$

which follows from the commutation relation (3.6). This gives

$$\frac{\mathrm{d}\big(S_1^{-1}h\big)}{\mathrm{d}u_k}\big(h^{-1}S_2\big) + \big(S_1^{-1}h\big)\frac{\mathrm{d}\big(h^{-1}S_2\big)}{\mathrm{d}u_k} = \Delta\big(S_1^{-1}h\big)\big(h^{-1}S_2\big)\Lambda^k - \big(S_1^{-1}h\big)\big(h^{-1}S_2\big)\Lambda^k\Delta.$$

Multiplying this equation on the left by $\left(S_1^{-1}h\right)^{-1}$ and on the right by $\left(h^{-1}S_2\right)^{-1}$, we get

$$\begin{aligned}
&\left(S_1^{-1}h\right)^{-1}\frac{\mathrm{d}\left(S_1^{-1}h\right)}{\mathrm{d}u_k}+\frac{\mathrm{d}\left(h^{-1}S_2\right)}{\mathrm{d}u_k}\left(h^{-1}S_2\right)^{-1}\\
&\qquad=\underbrace{\left(S_1^{-1}h\right)^{-1}\Delta\left(S_1^{-1}h\right)\left(h^{-1}S_2\right)\Lambda^k\left(h^{-1}S_2\right)^{-1}}_{\text{Term1}}-\underbrace{\left(h^{-1}S_2\right)\Lambda^k\Delta\left(h^{-1}S_2\right)^{-1}}_{\text{Term2}}.
\end{aligned}\tag{3.29}$$

Using the factorisation of $A_2$ in (3.7) and the factorisation of $D_2$ in (3.25), Term2 gives

$$\begin{aligned}
\text{Term2}&=\left(h^{-1}S_2\right)\Lambda^{k-1}\Lambda\Delta\left(h^{-1}S_2\right)^{-1}\\
&=\left(h^{-1}S_2\right)\Lambda^{k-1}\left(h^{-1}S_2\right)^{-1}\left(h^{-1}S_2\right)\Lambda\Delta\left(h^{-1}S_2\right)^{-1}=\left(A_2^T\right)^{1-k}D_2^T.
\end{aligned}$$

Similarly, using the factorisation of $A_2$ in (3.7), gives

$$\text{Term1}=\left(S_1^{-1}h\right)^{-1}\Delta\left(S_1^{-1}h\right)\left(A_2^T\right)^{-k}=\left(S_1^{-1}h\right)^{-1}\Delta\left(S_1^{-1}h\right)\left(A_2^T\right)^{-1}\left(A_2^T\right)^{1-k}.$$

Using the factorisation of $A_2^{-1}$ in (3.8) and the factorisation of $D_2^*$ in (3.25), we get

$$\text{Term1}=\left(h^{-1}S_1\right)\Delta\Lambda\left(h^{-1}S_1\right)^{-1}\left(A_2^T\right)^{1-k}=-(D_2^*)^T\left(A_2^T\right)^{1-k}.$$

Substituting these results in the transpose of (3.29), we obtain

$$\frac{\mathrm{d}\left(S_1^{-1}h\right)^T}{du_k}\left(\left(S_1^{-1}h\right)^T\right)^{-1}+\left(S_2^Th^{-1}\right)^{-1}\frac{\mathrm{d}\left(S_2^Th^{-1}\right)}{\mathrm{d}u_k}=-D_2A_2^{1-k}-A_2^{1-k}D_2^*.$$

Since $\left(S_1^{-1}h\right)^T$ is upper triangular and $S_2^Th^{-1}$ is lower triangular with diagonal elements equal to 1, by taking the strictly lower part of both sides of this equation, we obtain (3.27). This concludes the proof of the lemma. ∎

We are now able to obtain a Lax pair representation for the master symmetries vector fields $\mathcal{V}_k$, $k\in\mathbb{Z}$.

**Theorem 3.8.** *The "dressed up" form of the moment equation* (3.18) *gives the following Lax pair representation for the master symmetries vector fields $\mathcal{V}_k$ on the semi-infinite CMV matrices* $(A_1,A_2)$

$$\boxed{\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u_k}A_1&=\left[A_1,\left(D_1A_1^{k+1}\right)_{--}+\left(A_1^{k+1}D_1^*\right)_{--}\right], &&\forall\,k\in\mathbb{Z},\\
\frac{\mathrm{d}}{\mathrm{d}u_k}A_2&=\left[\left(D_2A_2^{1-k}\right)_{--}+\left(A_2^{1-k}D_2^*\right)_{--},A_2\right], &&\forall\,k\in\mathbb{Z},
\end{aligned}}\tag{3.30}$$

*or equivalently*

$$\boxed{\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u_k}A_1&=A_1^{k+1}+\left[\left(D_1A_1^{k+1}\right)_+-\left(A_1^{k+1}D_1^*\right)_{--},A_1\right], &&\forall\,k\in\mathbb{Z},\\
\frac{\mathrm{d}}{\mathrm{d}u_k}A_2&=A_2^{1-k}+\left[A_2,\left(A_2^{1-k}D_2^*\right)_+-\left(D_2A_2^{1-k}\right)_{--}\right], &&\forall\,k\in\mathbb{Z}.
\end{aligned}}$$

**Proof.** As $A_1=S_1\Lambda S_1^{-1}$ and $A_2=\left(S_2^Th^{-1}\right)^{-1}\Lambda\left(S_2^Th^{-1}\right)$, we have

$$\frac{\mathrm{d}A_1}{\mathrm{d}u_k}=\left[\frac{\mathrm{d}S_1}{\mathrm{d}u_k}S_1^{-1},A_1\right]\qquad\text{and}\qquad\frac{\mathrm{d}A_2}{\mathrm{d}u_k}=\left[A_2,\left(S_2^Th^{-1}\right)^{-1}\frac{\mathrm{d}\left(S_2^Th^{-1}\right)}{\mathrm{d}u_k}\right].$$

Using (3.26) and (3.27) in Lemma 3.7, we obtain (3.30).

From (3.20), (3.22) and from (3.21), (3.23), we deduce that $[A_1, D_1] = 1$ and $[D_2^*, A_2] = 1$. From these commutation relations, one readily obtains that

$$\begin{aligned}
&\big[A_1, \big(D_1 A_1^{k+1}\big)_+\big] + \big[A_1, \big(D_1 A_1^{k+1}\big)_{--}\big] = A_1^{k+1},\\
&\big[\big(A_2^{1-k} D_2^*\big)_+, A_2\big] + \big[\big(A_2^{1-k} D_2^*\big)_{--}, A_2\big] = A_2^{1-k},
\end{aligned}$$

which gives the equivalent formulation for the representation of the master symmetries on the CMV matrices $(A_1, A_2)$. This concludes the proof. ■

We notice that as a consequence of the Lax pair representation (3.13) for the AL hierarchy in Theorem 3.4, the relation between the vector fields $V_k$ and $\mathcal{V}_k$ in (3.14) and the Lax pair representation (3.30) of $\mathcal{V}_k$ in Theorem 3.8, we have established the Lax pair representation (1.31), (1.32) of the vector fields $V_k$ as announced in Theorem 1.1 in the Introduction.

We emphasize that Theorem 3.8 exhibits a *full* centerless Virasoro algebra of master symmetries for the AL hierarchy. This result stands in contrast with the Toda lattice and Korteweg–de Vries hierarchies which possess only half of a Virasoro algebra of master symmetries $\mathcal{V}_k$, $k \geq -1$, satisfying $[\mathcal{V}_k, \mathcal{V}_l] = (l-k)\mathcal{V}_{k+l}$, $k, l \geq -1$, see [4, 12, 15, 16, 21, 39].

Using the explicit form of the CMV matrices $(A_1, A_2)$ in Theorem 2.8, and Theorem 3.8, remembering Remark 3.5, one can compute the first few master symmetries vector fields $\mathcal{V}_{-2}$, $\mathcal{V}_{-1}$, $\mathcal{V}_0$, $\mathcal{V}_1$ in terms of the variables $x_n$, $y_n$:

$$\begin{aligned}
\mathcal{V}_{-2}(x_n) &= (n-4)x_{n-2}(1-x_{n-1}y_{n-1})(1-x_n y_n)\\
&\quad - x_{n-1}(1-x_n y_n)\big((n-4)x_{n-1}y_n + (n-1)x_n y_{n+1}\big)\\
&\quad - 2x_{n-1}(1-x_n y_n)\sum_{k=1}^{n} y_k x_{k-1} + x_n \sum_{k=1}^{n} y_k^2 x_{k-1}^2\\
&\quad - 2x_n \sum_{k=2}^{n} y_k x_{k-2} + 2x_n \sum_{k=2}^{n} y_k y_{k-1} x_{k-1} x_{k-2},\\
\mathcal{V}_{-2}(y_n) &= -n y_{n+2}(1-x_n y_n)(1-x_{n+1}y_{n+1})\\
&\quad + y_{n+1}(1-x_n y_n)\big(n x_n y_{n+1} + (n-1)x_{n-1}y_n\big)\\
&\quad + 2y_{n+1}(1-x_n y_n)\sum_{k=1}^{n} y_k x_{k-1} - y_n \sum_{k=1}^{n} y_k^2 x_{k-1}^2\\
&\quad + 2y_n \sum_{k=2}^{n} y_k x_{k-2} - 2y_n \sum_{k=2}^{n} y_k y_{k-1} x_{k-1} x_{k-2},\\
\mathcal{V}_{-1}(x_n) &= (n-2)x_{n-1}(1-x_n y_n) - x_n \sum_{k=1}^{n} y_k x_{k-1},\\
\mathcal{V}_{-1}(y_n) &= -n y_{n+1}(1-x_n y_n) + y_n \sum_{k=1}^{n} y_k x_{k-1},\\
\mathcal{V}_0(x_n) &= n x_n,\\
\mathcal{V}_0(y_n) &= -n y_n,\\
\mathcal{V}_1(x_n) &= n x_{n+1}(1-x_n y_n) - x_n \sum_{k=1}^{n} x_k y_{k-1},\\
\mathcal{V}_1(y_n) &= -(n-2)y_{n-1}(1-x_n y_n) + y_n \sum_{k=1}^{n} x_k y_{k-1}.
\end{aligned}$$

## 4 The action of the master symmetries on the tau-functions

As we recalled in the Introduction in formula (1.25), the tau-functions of the semi-infinite AL hierarchy are given by

$$\tau_n(t,s) = \det\big(\mu_{k-l}(t,s)\big)_{0\le k,l<n}. \tag{4.1}$$

It immediately follows from the generating function of the elementary Schur polynomials (1.26) that

$$\frac{\partial}{\partial t_k} S_n(t) = S_{n-k}(t), \tag{4.2}$$

which shows that the formal solution of the AL hierarchy (1.19) on the moments is

$$\mu_j(t,s) = \sum_{m,n=0}^{\infty} S_m(t) S_n(s) \mu_{j+m-n}(0,0), \qquad \forall\, j\in\mathbb{Z}. \tag{4.3}$$

The expansion (1.28) of the tau-functions in terms of the Plücker coordinates (1.29) and the Schur polynomials (1.30) easily follows. Indeed, by substituting (4.3) into (4.1), we have

$$\tau_n(t,s) = \sum_{\substack{0\le i_0,i_1,\dots,i_{n-1}\\ 0\le j_0,j_1,\dots,j_{n-1}}} \det\big[\mu_{k-l+i_k-j_l}(0,0)\big]_{0\le k,l<n} S_{i_0}(t)\cdots S_{i_{n-1}}(t) S_{j_0}(s)\cdots S_{j_{n-1}}(s).$$

Relabeling the indices as follows $i_k \mapsto i_k - k$, $j_l \mapsto j_l - l$, we get

$$\begin{aligned}
\tau_n(t,s) &= \sum_{\substack{0\le i_0,\dots,i_{n-1}\\ 0\le j_0,\dots,j_{n-1}}} \det\big[\mu_{i_k-j_l}(0,0)\big]_{0\le k,l<n} S_{i_0}(t) S_{i_1-1}(t)\cdots S_{i_{n-1}-(n-1)}(t)\\
&\quad\times S_{j_0}(s) S_{j_1-1}(s)\cdots S_{j_{n-1}-(n-1)}(s)\\
&= \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} \sum_{\sigma_1,\sigma_2\in S_n} (-1)^{\sigma_1}(-1)^{\sigma_2} \det\big[\mu_{i_k-j_l}(0,0)\big]_{0\le k,l<n} S_{i_{\sigma_1(0)}}(t)\\
&\quad\times S_{i_{\sigma_1(1)}-1}(t)\cdots S_{i_{\sigma_1(n-1)}-(n-1)}(t) S_{j_{\sigma_2(0)}}(s) S_{j_{\sigma_2(1)}-1}(s)\cdots S_{j_{\sigma_2(n-1)}-(n-1)}(s)\\
&= \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s),
\end{aligned} \tag{4.4}$$

with $(-1)^\sigma$ the sign of the permutation $\sigma$.

The aim of this section is to establish the second part of Theorem 1.1.

**Theorem 4.1.** *For all $k\in\mathbb{Z}$, we have*

$$L_k^{(n)}\tau_n(t,s) = \sum_{\substack{0\le i_0<\dots<i_{n-1}\\ 0\le j_0<\dots<j_{n-1}}} V_k\Big(p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\Big) S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s), \tag{4.5}$$

*with $L_k^{(n)}$, $k\in\mathbb{Z}$, defined as in (1.5), (1.6), (1.7), and $V_k\Big(p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\Big)$ the Lie derivative of the Plücker coordinates (1.29) in the direction of the master symmetries $V_k$ of the AL hierarchy, as defined in (1.21).*

This theorem is the key to the quick derivation of the various "Virasoro-type" constraints satisfied by *special* tau-functions of the AL hierarchy. As an illustration we establish the following result.

**Corollary 4.2.** *The partition function of the unitary matrix model*

$$\tau_n(t,s) = \int_{U(n)} \exp\left\{ \sum_{j=1}^{\infty} \left( t_j \operatorname{Tr} U^j + s_j \operatorname{Tr} U^{-j} \right) \right\} \mathrm{d}U,$$

*where $U(n)$ is the group of unitary $n \times n$ matrices and $\mathrm{d}U$ is the standard Haar measure, normalized so that the total volume is 1, satisfies the Virasoro constraints*

$$L_k^{(n)} \tau_n(t,s) = 0, \qquad \forall\, k \in \mathbb{Z},$$

*with $L_k^{(n)}$ defined as in (1.5), (1.6) and (1.7).*

**Proof.** By using Weyl's integral formula, one has that

$$\tau_n(t,s) = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \exp\left\{ \sum_{j=1}^{\infty} \left( t_j z_k^j + s_j z_k^{-j} \right) \right\} \frac{\mathrm{d}z_k}{2\pi i z_k},$$

is a tau-function of the AL hierarchy as in (1.27), with $w(z) = 1$ in the deformed weight (1.20). Thus the initial moments (at time $(t,s) = (0,0)$) are given by

$$\mu_j(0,0) = \oint_{S^1} z^j \frac{\mathrm{d}z}{2\pi i z} = \delta_{j,0},$$

with $\delta_{j,k}$ the usual Kronecker symbol. By the definition (1.21) of the master symmetries $V_k$, it follows that

$$V_k(\mu_j)_{|(t,s)=(0,0)} = (j+k)\mu_{j+k}(0,0) = (j+k)\delta_{j+k,0} = 0,$$

which, using the definition of the Plücker coordinates (1.29) and formula (4.5), establishes the result. ∎

**Remark 4.3.** After [24] was completed, we found out that Corollary 4.2, which can be seen as a particular case of our result recalled in (1.3), had already been obtained by Bowick, Morozov and Shevitz [8], using the Lagrangian approach [31] to derive Virasoro constraints. However, these authors didn't notice the commutation relations (1.4) of the centerless Virasoro algebra. In contrast with Corollary 4.2, the partition function of the Hermitian matrix model (which is a tau-function of the Toda lattice hierarchy) and the partition function of 2d-quantum gravity (which is a tau-function of the KdV hierarchy) are characterized by Virasoro constraints $L_k\tau(t) = 0$, $k \geq -1$, corresponding to "half of" a Virasoro algebra, see [4, 14, 19, 22, 25, 30, 31] for the explicit form of the operators $L_k$ in those cases.

Actually, in the proof of Theorem 4.1, we shall need to know that the operators $L_k^{(n)}$, $k \in \mathbb{Z}$, satisfy the commutation relations of the centerless Virasoro algebra. For the convenience of the reader we repeat the proof given in [24]. Consider the complex Lie algebra $\mathcal{A}$ given by the direct sum of two commuting copies of the Heisenberg algebra with bases $\{\hbar_a, a_j \,|\, j \in \mathbb{Z}\}$ and $\{\hbar_b, b_j \,|\, j \in \mathbb{Z}\}$ and defining commutation relations

$$\begin{aligned}
&[\hbar_a, a_j] = 0, \qquad [a_j, a_k] = j\delta_{j,-k}\hbar_a, \\
&[\hbar_b, b_j] = 0, \qquad [b_j, b_k] = j\delta_{j,-k}\hbar_b, \\
&[\hbar_a, \hbar_b] = 0, \qquad [a_j, b_k] = 0, \qquad [\hbar_a, b_j] = 0, \qquad [\hbar_b, a_j] = 0,
\end{aligned} \tag{4.6}$$

with $j,k\in\mathbb{Z}$. Let $\mathcal{B}$ be the space of formal power series in the variables $t_1,t_2,\dots$ and $s_1,s_2,\dots$, and consider the following representation of $\mathcal{A}$ in $\mathcal{B}$

$$\begin{aligned} & a_j=\frac{\partial}{\partial t_j}, \qquad a_{-j}=jt_j, \qquad b_j=\frac{\partial}{\partial s_j}, \qquad b_{-j}=js_j,\\ & a_0=b_0=\mu, \qquad \hbar_a=\hbar_b=1, \end{aligned} \tag{4.7}$$

for $j>0$, and $\mu\in\mathbb{C}$. Define the operators

$$A_k^{(n)}=\frac{1}{2}\sum_{j\in\mathbb{Z}} :a_{-j}a_{j+k}:, \qquad B_k^{(n)}=\frac{1}{2}\sum_{j\in\mathbb{Z}} :b_{-j}b_{j+k}:, \tag{4.8}$$

where $k\in\mathbb{Z}$, $a_j$, $b_j$ are as in (4.7) with $\mu=n$, and where the colons indicate normal ordering, defined by

$$:a_ja_k:=\begin{cases} a_ja_k & \text{if } j\le k,\\ a_ka_j & \text{if } j>k, \end{cases}$$

and a similar definition for $:b_jb_k:$, obtained by changing the $a$'s in $b$'s in the former. Expanding the expressions in (4.8) we obtain for $k>0$

$$\begin{aligned} A_0^{(n)}&=\sum_{j>0} jt_j\frac{\partial}{\partial t_j}+\frac{n^2}{2},\\ A_k^{(n)}&=\frac{1}{2}\sum_{0<j<k}\frac{\partial^2}{\partial t_j\partial t_{k-j}}+\sum_{j>k}(j-k)t_{j-k}\frac{\partial}{\partial t_j}+n\frac{\partial}{\partial t_k},\\ A_{-k}^{(n)}&=\frac{1}{2}\sum_{0<j<k} j(k-j)t_jt_{k-j}+\sum_{j>k} jt_j\frac{\partial}{\partial t_{j-k}}+nkt_k, \end{aligned}$$

and similar expressions for $B_k^{(n)}$, by changing the $t$-variables in $s$-variables. Using these notations, we can rewrite (1.5), (1.6) and (1.7) as follows

$$\begin{aligned} L_k^{(n)}&=A_k^{(n)}-B_{-k}^{(n)}+\frac{1}{2}\sum_{j=1}^{k-1}(a_j-b_{-j})(a_{k-j}-b_{j-k}), \qquad k\ge 1,\\ L_0^{(n)}&=A_0^{(n)}-B_0^{(n)},\\ L_{-k}^{(n)}&=A_{-k}^{(n)}-B_k^{(n)}-\frac{1}{2}\sum_{j=1}^{k-1}(a_{-j}-b_j)(a_{j-k}-b_{k-j}), \qquad k\ge 1. \end{aligned} \tag{4.9}$$

As shown in [26] (see Lecture 2) the operators $A_k^{(n)}$, $k\in\mathbb{Z}$, provide a representation of the Virasoro algebra in $\mathcal{B}$ with central charge $c=1$, that is

$$\big[A_k^{(n)},A_l^{(n)}\big]=(k-l)A_{k+l}^{(n)}+\delta_{k,-l}\frac{k^3-k}{12}, \tag{4.10}$$

for $k,l\in\mathbb{Z}$. Similarly, the operators $B_k^{(n)}$ satisfy the commutation relations

$$\big[B_k^{(n)},B_l^{(n)}\big]=(k-l)B_{k+l}^{(n)}+\delta_{k,-l}\frac{k^3-k}{12}, \tag{4.11}$$

for $k,l\in\mathbb{Z}$. Furthermore we have for $k,l\in\mathbb{Z}$

$$\big[a_k,A_l^{(n)}\big]=ka_{k+l}, \qquad \big[b_k,B_l^{(n)}\big]=kb_{k+l}, \qquad \big[a_k,B_l^{(n)}\big]=0, \qquad \big[b_k,A_l^{(n)}\big]=0. \tag{4.12}$$

**Proposition 4.4.** *The operators $L_k^{(n)}$ defined as in (1.5), (1.6), (1.7) satisfy the commutation relations of the centerless Virasoro algebra*

$$\big[L_k^{(n)}, L_l^{(n)}\big] = (k-l)L_{k+l}^{(n)}, \qquad \forall\, k,l \in \mathbb{Z}. \tag{4.13}$$

**Proof.** We give the proof for $k,l \geq 0$, the other cases being similar. As $\big[A_i^{(n)}, B_j^{(n)}\big] = 0$, $i,j \in \mathbb{Z}$, we have using (4.6), (4.10), (4.11) and (4.12)

$$\begin{aligned}\big[L_k^{(n)}, L_l^{(n)}\big] &= (k-l)\big(A_{k+l}^{(n)} - B_{-k-l}^{(n)}\big) - \frac{1}{2}\sum_{j=1}^{l-1} j(a_{j+k} - b_{-j-k})(a_{l-j} - b_{j-l})\\ &\quad - \frac{1}{2}\sum_{j=1}^{l-1}(l-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) + \frac{1}{2}\sum_{j=1}^{k-1} j(a_{j+l} - b_{-j-l})(a_{k-j} - b_{j-k})\\ &\quad + \frac{1}{2}\sum_{j=1}^{k-1}(k-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}).\end{aligned}$$

Relabeling the indices in the sums, we have

$$\begin{aligned}\big[L_k^{(n)}, L_l^{(n)}\big] &= (k-l)\big(A_{k+l}^{(n)} - B_{-k-l}^{(n)}\big) - \frac{1}{2}\sum_{j=k+1}^{k+l-1} (j-k)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l})\\ &\quad - \frac{1}{2}\sum_{j=1}^{l-1}(l-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) + \frac{1}{2}\sum_{j=l+1}^{k+l-1} (j-l)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l})\\ &\quad + \frac{1}{2}\sum_{j=1}^{k-1}(k-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) = (k-l)L_{k+l}^{(n)}.\end{aligned}$$

This concludes the proof. ■

The plan of the rest of the section is as follows. After some algebraic preliminaries, we shall translate the master symmetries on the Plücker coordinates $p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}$. Next we shall compute the action of the Virasoro operators on the products $S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)$ of Schur polynomials. Finally we shall end with the proof of Theorem 4.1.

## 4.1 Some algebraic lemmas

We shall need the following lemmas. In order to formulate them, we introduce some notations. Given $n$ vectors $x_1,\dots,x_n \in \mathbb{R}^n$, we shall denote by $|x_1x_2\dots x_n|$ the determinant of the $n\times n$ matrix formed with the columns $x_i$. Also, given two vectors $x$ and $y$, $x\wedge y$ denotes the usual wedge product, with components $(x\wedge y)_{rs} = x_ry_s - x_sy_r$. Finally, for an $n\times n$ matrix $A$, $A_r$ will denote the $r$th column of A, and $A_r^T$ the $r$th column of the transposed matrix, and $\operatorname{tr}(A)$ will mean the trace of $A$. With these conventions, we have the following lemma.

**Lemma 4.5** (Haine–Semengue [23])**.** *Let $A$ and $B$ be $n\times n$ matrices, with $A$ invertible. Then*

*(i)* $$\sum_{r=1}^{n} |A_1\dots A_{r-1}B_rA_{r+1}\dots A_n| = (\det A)\operatorname{tr}(BA^{-1}),$$

*(ii)* $$\sum_{1\leq r<s\leq n} |A_1\dots A_{r-1}B_rA_{r+1}\dots A_{s-1}B_sA_{s+1}\dots A_n| = (\det A)\sum_{1\leq r<s\leq n}\big(\big(BA^{-1}\big)_r \wedge \big(BA^{-1}\big)_s\big)_{rs}.$$

**Proof.** $(i)$ Let $A$, $B$ be $n \times n$ matrices, with $A$ invertible. As $A$ is invertible, its columns form a basis of $\mathbb{C}^n$ and thus we have

$$B_r = Ac^{(r)} = \sum_j c_j^{(r)} A_j, \tag{4.14}$$

for a certain $c^{(r)} \in \mathbb{C}^n$, whose components are $c_j^{(r)} = \left(A^{-1}B\right)_{jr}$. It then follows that

$$\begin{aligned}\sum_{r=1}^{n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_n| &= \sum_{r=1}^{n} \Big|A_1 \dots A_{r-1} \Big(\sum_j c_j^{(r)} A_j\Big) A_{r+1} \dots A_n\Big| \\ &= \det A \sum_{r=1}^{n} c_r^{(r)} = (\det A) \operatorname{tr}\left(BA^{-1}\right).\end{aligned}$$

$(ii)$ Using (4.14), we have

$$\begin{aligned}&\sum_{1\le r<s\le n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_{s-1} B_s A_{s+1} \dots A_n| \\ &\quad= \sum_{1\le r<s\le n} \Big|A_1 \dots A_{r-1} \Big(\sum_j c_j^{(r)} A_j\Big) A_{r+1} \dots A_{s-1} \Big(\sum_j c_j^{(s)} A_j\Big) A_{s+1} \dots A_n\Big| \\ &\quad= \sum_{1\le r<s\le n} \Big|A_1 \dots A_{r-1} \big(c_r^{(r)} A_r + c_s^{(r)} A_s\big) A_{r+1} \dots A_{s-1} \big(c_r^{(s)} A_r + c_s^{(s)} A_s\big) A_{s+1} \dots A_n\Big| \\ &\quad= \det A \sum_{1\le r<s\le n} \big(c_r^{(r)} c_s^{(s)} - c_s^{(r)} c_r^{(s)}\big) = \det A \sum_{1\le r<s\le n} \big((A^{-1}B)_r \wedge (A^{-1}B)_s\big)_{rs}.\end{aligned}$$

We thus obtain

$$\begin{aligned}&\sum_{1\le r<s\le n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_{s-1} B_s A_{s+1} \dots A_n| \\ &\quad= \det A \sum_{1\le r<s\le n} \big(\left(BA^{-1}\right)_r \wedge \left(BA^{-1}\right)_s\big)_{rs},\end{aligned}$$

where we have used the fact that for $X$, $Y$ two $n \times n$ matrices, we have

$$\sum_{1\le r<s\le n} \big((XY)_r \wedge (XY)_s\big)_{rs} = \sum_{1\le r<s\le n} \big((YX)_r \wedge (YX)_s\big)_{rs}. \tag{4.15}$$

This concludes the proof of the lemma. ∎

We will also need a transposed version of this lemma.

**Lemma 4.6.** *With the same conditions as in Lemma 4.5, we have*

$$\begin{aligned}&(i) \quad \sum_{r=1}^{n} \big|A_1^T \dots A_{r-1}^T (B)_r^T A_{r+1}^T \dots A_n^T\big| = \sum_{r=1}^{n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_n|, \\ &(ii) \quad \sum_{1\le r<s\le n} \big|A_1^T \dots A_{r-1}^T B_r^T A_{r+1}^T \dots A_{s-1}^T B_s^T A_{s+1}^T \dots A_n^T\big| \\ &\qquad\quad = \sum_{1\le r<s\le n} |A_1 \dots A_{r-1} B_r A_{r+1} \dots A_{s-1} B_s A_{s+1} \dots A_n|.\end{aligned}$$

**Proof.** Both formulas are direct consequences of Lemma 4.5, by observing that for $X$, $Y$ two $n \times n$ matrices, we have (4.15) and

$$\left(X_r^T \wedge X_s^T\right)_{rs} = (X_r \wedge X_s)_{rs}.$$ ∎

We give two consequences of this lemma. First we particularize the preceding lemma to the Plücker coordinates, and then we particularize it to the Schur polynomials.

**Lemma 4.7.** *For $m \in \mathbb{Z}$ we have*

$$(i) \quad \sum_{l=1}^{n} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-l}-m,\dots,j_{n-1}}} = \sum_{l=1}^{n} p_{\substack{i_0,\dots,i_{n-l}+m,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}},$$

$$(ii) \quad \sum_{1\le r<s\le n} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-s}-m,\dots,j_{n-r}-m,\dots,j_{n-1}}} = \sum_{1\le r<s\le n} p_{\substack{i_0,\dots,i_{n-s}+m,\dots,i_{n-r}+m,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}}.$$

**Proof.** Define the $n \times n$ matrices

$$A = (\mu_{i_k - j_l})_{0\le k,l\le n-1} \qquad \text{and} \qquad B(m) = (\mu_{i_k - j_l + m})_{0\le k,l\le n-1}.$$

We then have

$$\sum_{l=1}^{n} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-l}-m,\dots,j_{n-1}}} = \sum_{l=1}^{n} \left| A_1 \dots A_{n-l-1} \big(B(m)\big)_{n-l} A_{n-l+1} \dots A_{n-1} \right|$$
$$= \sum_{l=1}^{n} \left| A_1^T \dots A_{n-l-1}^T \big(B(m)\big)_{n-l}^T A_{n-l+1}^T \dots A_{n-1}^T \right| = \sum_{l=1}^{n} p_{\substack{i_0,\dots,i_{n-l}+m,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}},$$

where we have used Lemma 4.6$(i)$ in the second equality. This proves $(i)$. The proof of $(ii)$ is similar. ■

**Lemma 4.8.** *The following holds*

$$(i) \quad \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)$$
$$= \det \begin{pmatrix} S_{i_{n-1}-n}(t) & S_{i_{n-1}-(n-2)}(t) & \cdots & S_{i_{n-1}}(t) \\ S_{i_{n-2}-n}(t) & S_{i_{n-2}-(n-2)}(t) & \cdots & S_{i_{n-2}}(t) \\ \vdots & \vdots & & \vdots \\ S_{i_0-n}(t) & S_{i_0-(n-2)}(t) & \cdots & S_{i_0}(t) \end{pmatrix},$$

$$(ii) \quad \sum_{l=1}^{n-1} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)+1,\dots,i_1-1}(t)$$
$$= \det \begin{pmatrix} S_{i_{n-1}-(n-1)}(t) & \cdots & S_{i_{n-1}-2}(t) & S_{i_{n-1}}(t) \\ S_{i_{n-2}-(n-1)}(t) & \cdots & S_{i_{n-2}-2}(t) & S_{i_{n-2}}(t) \\ \vdots & & \vdots & \vdots \\ S_{i_1-(n-1)}(t) & \cdots & S_{i_1-2}(t) & S_{i_1}(t) \end{pmatrix}.$$

**Proof.** We prove $(i)$. Define the $n \times n$ matrices

$$A = \big(S_{i_{n-k}-(n-k)+l-k}(t)\big)_{1\le k,l\le n}, \qquad B(m) = \big(S_{i_{n-k}-(n-k)+l-k+m}(t)\big)_{1\le k,l\le n}.$$

We have $S_{i_{n-1}-(n-1),\dots,i_0}(t) = \det A$. It then follows that

$$\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) = \sum_{l=1}^{n} \left| A_1^T \dots A_{n-l-1}^T \big(B(-1)\big)_{n-l}^T A_{n-l+1}^T \dots A_{n-1}^T \right|.$$

Using Lemma 4.6($i$) we get

$$\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) = \sum_{l=1}^{n} \left|A_1 \dots A_{n-l-1}\big(B(-1)\big)_{n-l} A_{n-l+1} \dots A_{n-1}\right|.$$

In the right-hand side, in the $l^{\text{th}}$ term, the $l^{\text{th}}$ and $(l-1)^{\text{th}}$ columns coincide in the determinant, provided that $l \neq 1$. Consequently, only the first term of the right-hand side gives a non zero contribution. This proves ($i$). The proof of ($ii$) is similar. ■

### 4.2 Expression of the master symmetries on the Plücker coordinates

We now translate the master symmetries on Plücker coordinates.

**Lemma 4.9.** *Let $V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}$ denote the Lie derivative of the Plücker coordinates in the direction of the vector fields $V_k$. Then for $k \in \mathbb{Z}$,*

$$\begin{aligned} V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} &= \sum_{l=0}^{n-1} (i_l + k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=0}^{n-1} j_l p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{l-1},j_l-k,j_{l+1},\dots,j_{n-1}}} \\ &= \sum_{l=0}^{n-1} i_l p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=0}^{n-1} (j_l - k) p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{l-1},j_l-k,j_{l+1},\dots,j_{n-1}}}. \end{aligned} \tag{4.16}$$

**Proof.** Fix $0 \le i_0 < i_1 < \dots < i_{n-1}$ and $0 \le j_0 < j_1 < \dots < j_{n-1}$. We introduce the $n \times n$ matrices

$$A = \big(\mu_{i_k - j_l}(0,0)\big)_{0 \le k,l \le n-1}, \qquad B(m) = \big(\mu_{i_k - j_l + m}(0,0)\big)_{0 \le k,l \le n-1},$$

as well as the diagonal matrix $D = \operatorname{diag}(j_0, \dots, j_{n-1})$. We notice that $p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} = \det A$, by definition of the Plücker coordinates. From the definition of $V_k$ and using Leibniz's rule we find for $k \in \mathbb{Z}$

$$V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} = \sum_{l=0}^{n-1} \begin{vmatrix} \mu_{i_0-j_0} & \mu_{i_0-j_1} & \cdots & \mu_{i_0-j_{n-1}} \\ \vdots & \vdots & & \vdots \\ \mu_{i_{l-1}-j_0} & \mu_{i_{l-1}-j_1} & \cdots & \mu_{i_{l-1}-j_{n-1}} \\ (i_l - j_0 + k)\mu_{i_l-j_0+k} & (i_l - j_1 + k)\mu_{i_l-j_1+k} & \cdots & (i_l - j_{n-1} + k)\mu_{i_l-j_{n-1}+k} \\ \mu_{i_{l-1}-j_0} & \mu_{i_{l-1}-j_1} & \cdots & \mu_{i_{l-1}-j_{n-1}} \\ \vdots & \vdots & & \vdots \\ \mu_{i_{n-1}-j_0} & \mu_{i_{n-1}-j_1} & \cdots & \mu_{i_{n-1}-j_{n-1}} \end{vmatrix},$$

or equivalently,

$$V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} = \sum_{l=0}^{n-1} (i_l + k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=1}^{n} \left|A_1^T \dots A_{l-1}^T \big(B(k)D\big)_l^T A_{l+1}^T \dots A_n^T\right|.$$

Using Lemma 4.6($i$) we obtain

$$\begin{aligned} V_k p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} &= \sum_{l=0}^{n-1} (i_l + k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=1}^{n} \left|A_1 \dots A_{l-1} \big(B(k)D\big)_l A_{l+1} \dots A_n\right| \\ &= \sum_{l=0}^{n-1} (i_l + k) p_{\substack{i_0,\dots,i_{l-1},i_l+k,i_{l+1},\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} - \sum_{l=1}^{n} j_{l-1} \left|A_1 \dots A_{l-1} \big(B(k)\big)_l A_{l+1} \dots A_n\right|. \end{aligned}$$

This gives the first equality in (4.16). The second equality in (4.16) can be derived from the first one by using Lemma 4.7($i$). ■

### 4.3 Action of the Virasoro operators $L_k^{(n)}$ on the Schur polynomials

Next we shall compute the action of the Virasoro operators on the products of Schur polynomials $S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)$. We have the following lemma.

**Lemma 4.10.**

$$
\begin{aligned}
(i)\quad & L_0^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&= \sum_{l=0}^{n-1}(i_l - j_l) S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s),\\
(ii)\quad & L_1^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&= \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&\quad - \sum_{l=1}^{n}(j_{n-l}+1) S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s),\\
(iii)\quad & L_2^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&= \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&\quad + \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s)\\
&\quad - \sum_{l=1}^{n}(j_{n-l}+2) S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s)\\
&\quad - \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_{n-k}-(n-k)+1,\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s)\\
&\quad + s_1 \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s)\\
&\quad - s_1 \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)S_{j_{n-1}-(n-1),\dots,j_0}(s).
\end{aligned}
$$

**Proof.** By using Leibniz's rule and (4.2) we have for $j \ge 1$,

$$
\frac{\partial}{\partial t_j} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-j,\dots,i_0}(t), \tag{4.17}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial t_1^2} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)\\
&\quad + 2\sum_{1\le r<s\le n} S_{i_{n-1}-(n-1),\dots,i_{n-r}-(n-r)-1,\dots,i_{n-s}-(n-s)-1,\dots,i_0}(t).
\end{aligned} \tag{4.18}
$$

Define the following $n\times n$ matrices

$$A(t) := \begin{pmatrix} S_{i_{n-1}-(n-1)}(t) & \dots & S_{i_{n-1}}(t) \\ \vdots & & \vdots \\ S_{i_0-(n-1)}(t) & \dots & S_{i_0}(t) \end{pmatrix},$$
$$B(j,t) := \begin{pmatrix} S_{i_{n-1}-(n-1)-j}(t) & \dots & S_{i_{n-1}-j}(t) \\ \vdots & & \vdots \\ S_{i_0-(n-1)-j}(t) & \dots & S_{i_0-j}(t) \end{pmatrix},$$

and $D = \operatorname{diag}(n-1, n-2, \dots, 0)$. We shall denote $\hat{A}(s)$ and $\hat{B}(j,s)$ the same matrices with $t \to s$ and $(i_0,\dots,i_{n-1}) \to (j_0,\dots,j_{n-1})$. From the definition (1.26) of the elementary Schur polynomials it follows easily that for $j \geq 0$,

$$\sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+j}} S_i(t) = (i-j)S_{i-j}(t),$$
$$\sum_{k=j+1}^{\infty} kt_k \frac{\partial}{\partial t_{k-j}} S_i(t) = (i+j)S_{i+j}(t) - \sum_{1\leq l\leq j} lt_l S_{i+j-l}(t).$$

Consequently, by first using Leibniz's rule and then Lemma 4.5($i$) we have for $j\geq 0$

$$\begin{aligned} \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+j}} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \sum_{l=1}^{n} (i_{n-l}-j) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-j,\dots,i_0}(t) \\ &\quad - \big(\det A(t)\big) \operatorname{tr}\big(A(t)^{-1}B(j,t)D\big), \end{aligned} \tag{4.19}$$

$$\begin{aligned} \sum_{k=j+1}^{\infty} kt_k \frac{\partial}{\partial t_{k-j}} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \sum_{l=1}^{n} (i_{n-l}+j) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)+j,\dots,i_0}(t) \\ &\quad - \big(\det A(t)\big) \operatorname{tr}\big(A(t)^{-1}B(-j,t)D\big) \\ &\quad - \sum_{m=1}^{j} mt_m \big(\det A(t)\big) \operatorname{tr}\big(A(t)^{-1}B(m-j,t)\big). \end{aligned} \tag{4.20}$$

We are now ready to prove the lemma.

($i$) From (4.9), we have $L_0^{(n)} = A_0^{(n)} - B_0^{(n)}$. Using (4.19) with $j=0$, we obtain

$$\begin{aligned} A_0^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_k} S_{i_{n-1}-(n-1),\dots,i_0}(t) + \frac{n^2}{2} S_{i_{n-1}-(n-1),\dots,i_0}(t) \\ &= \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_0}(t) - \big(\det A(t)\big)\operatorname{tr}\big(A(t)^{-1}B(0,t)D\big) + \frac{n^2}{2} S_{i_{n-1}-(n-1),\dots,i_0}(t). \end{aligned}$$

We have $B(0,t) = A(t)$, and thus

$$\big(\det A(t)\big)\operatorname{tr}\big(A(t)^{-1}B(0,t)D\big) = \big(\det A(t)\big)\operatorname{tr}(D) = \frac{n(n-1)}{2} S_{i_{n-1}-(n-1),\dots,i_0}(t).$$

Consequently, we get

$$A_0^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \left[\sum_{l=1}^{n} i_{n-l} + \frac{n}{2}\right] S_{i_{n-1}-(n-1),\dots,i_0}(t).$$

Similarly, we get

$$B_0^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) = \left[\sum_{l=1}^{n} j_{n-l} + \frac{n}{2}\right] S_{j_{n-1}-(n-1),\dots,j_0}(s).$$

Combining both equations, we obtain $(i)$.

$(ii)$ From (4.9), we have $L_1^{(n)} = A_1^{(n)} - B_{-1}^{(n)}$. We compute, using (4.17) and (4.19)

$$\begin{aligned} A_1^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \left[\sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{j+1}} + n \frac{\partial}{\partial t_1}\right] S_{i_{n-1}-(n-1),\dots,i_0}(t) \\ &= \sum_{l=1}^{n} \big(i_{n-l} + n - 1\big) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) - \big(\det A(t)\big) \operatorname{tr}\big(A(t)^{-1} B(1,t) D\big). \end{aligned}$$

By virtue of Lemma 4.5$(i)$, we have

$$\big(\det A(t)\big) \operatorname{tr}\big(A(t)^{-1} B(1,t) D\big) = (n-1)\big|\big(B(1,t)\big)_1 A_2(t) \dots A_n(t)\big|.$$

But by virtue of Lemma 4.8$(i)$, this gives

$$\big(\det A(t)\big) \operatorname{tr}\big(A(t)^{-1} B(1,t) D\big) = (n-1) \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t).$$

Hence, we obtain

$$A_1^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t). \tag{4.21}$$

Similarly, we have using (4.20)

$$\begin{aligned} B_{-1}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) &= \left[\sum_{j=2}^{\infty} j s_j \frac{\partial}{\partial s_{j-1}} + n s_1\right] S_{j_{n-1}-(n-1),\dots,j_0}(s) \\ &= \sum_{l=1}^{n} (j_{n-l} + 1) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) - \big(\det \hat{A}(s)\big) \operatorname{tr}\big(\hat{A}(s)^{-1} \hat{B}(-1,s) D\big) \\ &\quad - s_1 \big(\det \hat{A}(s)\big) \operatorname{tr}\big(\hat{A}(s)^{-1} \hat{B}(0,s)\big) + n s_1 S_{j_{n-1}-(n-1),\dots,j_0}(s). \end{aligned}$$

We have using Lemma 4.5$(i)$

$$\big(\det \hat{A}(s)\big) \operatorname{tr}\big(\hat{A}(s)^{-1} \hat{B}(-1,s) D\big) = 0,$$

and, obviously, we also have

$$\big(\det \hat{A}(s)\big) \operatorname{tr}\big(\hat{A}(s)^{-1} \hat{B}(0,s)\big) = n S_{j_{n-1}-(n-1),\dots,j_0}(s).$$

Consequently we obtain

$$B_{-1}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) = \sum_{l=1}^{n} (j_{n-l} + 1) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s). \tag{4.22}$$

Subtracting (4.21) and (4.22) gives $(ii)$.

$(iii)$ From (4.9), we have

$$L_2^{(n)} = A_2^{(n)} - B_{-2}^{(n)} + \frac{1}{2}\left(\frac{\partial}{\partial t_1} - s_1\right)^2.$$

We study separately the contributions of the three terms in the operator $L_2^{(n)}$ on the product of Schur functions. We start with the contribution of $A_2^{(n)}$. We compute, using (4.17), (4.18) and (4.19)

$$\begin{aligned}
A_2^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \left[\frac{1}{2}\frac{\partial^2}{\partial t_1^2} + \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{j+2}} + n\frac{\partial}{\partial t_2}\right] S_{i_{n-1}-(n-1),\dots,i_0}(t)\\
&= \sum_{l=1}^{n}\left(i_{n-l} + n - \frac{3}{2}\right) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)\\
&\quad + \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)\\
&\quad - \big(\det A(t)\big)\,\mathrm{tr}\,\big(A(t)^{-1}B(2,t)D\big).
\end{aligned}$$

The last term in this equation gives by developing the trace

$$\begin{aligned}
&\big(\det A(t)\big)\,\mathrm{tr}\,\big(A(t)^{-1}B(2,t)D\big)\\
&\qquad = \big(\det A(t)\big)\Big[(n-1)\big(A(t)^{-1}B(2,t)\big)_{11} + (n-2)\big(A(t)^{-1}B(2,t)\big)_{22}\Big].
\end{aligned}$$

We have

$$\big(\det A(t)\big)\,\mathrm{tr}\,\big(A(t)^{-1}B(2,t)\big) = \big(\det A(t)\big)\big[\big(A(t)^{-1}B(2,t)\big)_{11} + \big(A(t)^{-1}B(2,t)\big)_{22}\big],$$

and by a short computation

$$\big(A(t)^{-1}B(2,t)\big)_{22} = -\sum_{1\le k<l\le n}\big(\big(A(t)^{-1}B(1,t)\big)_k \wedge \big(A(t)^{-1}B(1,t)\big)_l\big)_{kl}.$$

Consequently we have

$$\begin{aligned}
\big(\det A(t)\big)\,\mathrm{tr}\,\big(A(t)^{-1}B(2,t)D\big) &= (n-1)\big(\det A(t)\big)\,\mathrm{tr}\,\big(A(t)^{-1}B(2,t)\big)\\
&\quad + \big(\det A(t)\big)\sum_{1\le k<l\le n}\Big(\big(A(t)^{-1}B(1,t)\big)_k \wedge \big(A(t)^{-1}B(1,t)\big)_l\Big)_{kl}.
\end{aligned}$$

Using Lemma 4.5, we obtain

$$\begin{aligned}
\big(\det A(t)\big)\,\mathrm{tr}\,\big(A(t)^{-1}B(2,t)D\big) &= (n-1)\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)\\
&\quad + \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t).
\end{aligned}$$

Hence, we get

$$A_2^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) = \sum_{l=1}^{n}\left(i_{n-l} - \frac{1}{2}\right) S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t). \tag{4.23}$$

We now turn to the contribution of $B_{-2}^{(n)}$. We have using (4.20)

$$
\begin{aligned}
B_{-2}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) &= \left[\frac{1}{2}s_1^2 + \sum_{j=3}^{\infty} j s_j \frac{\partial}{\partial s_{j-2}} + 2ns_2\right] S_{j_{n-1}-(n-1),\dots,j_0}(s) \\
&= \left[\frac{1}{2}s_1^2 + 2ns_2\right] S_{j_{n-1}-(n-1),\dots,j_0}(s) + \sum_{l=1}^{n} \big(j_{n-l}+2\big) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s) \\
&\quad - \big(\det \hat{A}(s)\big)\,\mathrm{tr}\,\big(\hat{A}(s)^{-1}\hat{B}(-2,s)D\big) - \sum_{m=1}^{2} m s_m \big(\det \hat{A}(s)\big)\,\mathrm{tr}\,\big(\hat{A}(s)^{-1}\hat{B}(m-2,s)\big).
\end{aligned}
$$

By a similar argument as above, we have

$$
\begin{aligned}
&\big(\det \hat{A}(s)\big)\,\mathrm{tr}\,\big(\hat{A}(s)^{-1}\hat{B}(-2,s)D\big) \\
&\quad = -\big(\det \hat{A}(s)\big) \sum_{1\le k<l\le n} \Big(\big(\hat{A}(s)^{-1}\hat{B}(-1,s)\big)_k \wedge \big(\hat{A}(s)^{-1}\hat{B}(-1,s)\big)_l\Big)_{kl},
\end{aligned}
$$

and thus using Lemma 4.5$(ii)$, we obtain

$$
\begin{aligned}
&\big(\det \hat{A}(s)\big)\,\mathrm{tr}\,\big(\hat{A}(s)^{-1}\hat{B}(-2,s)D\big) \\
&\quad = - \sum_{1\le k<l\le n} S_{j_{n-1}-(n-1),\dots,j_{n-k}-(n-k)+1,\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s).
\end{aligned}
$$

We also have, using Lemma 4.5$(i)$,

$$
\begin{aligned}
&\sum_{m=1}^{2} m s_m \big(\det \hat{A}(s)\big)\,\mathrm{tr}\,\big(\hat{A}(s)^{-1}\hat{B}(m-2,s)\big) \\
&\quad = s_1 \sum_{l=1}^{n} S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) + 2ns_2 S_{j_{n-1}-(n-1),\dots,j_0}(s).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
B_{-2}^{(n)} S_{j_{n-1}-(n-1),\dots,j_0}(s) &= \sum_{l=1}^{n} \big(j_{n-l}+2\big) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s) \\
&\quad + \sum_{1\le k<l\le n} S_{j_{n-1}-(n-1),\dots,j_{n-k}-(n-k)+1,\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\
&\quad - s_1 \sum_{l=1}^{n} S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) + \frac{1}{2}s_1^2 S_{j_{n-1}-(n-1),\dots,j_0}(s). \qquad (4.24)
\end{aligned}
$$

Finally, we turn to the contribution of the term $\frac{1}{2}\big(\frac{\partial}{\partial t_1} - s_1\big)^2$. We have using (4.17) and (4.18)

$$
\begin{aligned}
\frac{1}{2}\left[\frac{\partial}{\partial t_1} - s_1\right]^2 S_{i_{n-1}-(n-1),\dots,i_0}(t) &= \frac{1}{2}\left[\frac{\partial^2}{\partial t_1^2} - 2s_1\frac{\partial}{\partial t_1} + s_1^2\right] S_{i_{n-1}-(n-1),\dots,i_0}(t) \\
&= \frac{1}{2}\sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t) \\
&\quad + \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) \\
&\quad - s_1 \sum_{l=1}^{n} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) + \frac{1}{2}s_1^2 S_{i_{n-1}-(n-1),\dots,i_0}(t). \qquad (4.25)
\end{aligned}
$$

Combining (4.23), (4.24) and (4.25), we obtain $(iii)$. ■

**Remark 4.11.** We observe that by definition of the operators $L_k^{(n)}$ we have

$$
\begin{aligned}
&L_{-k}^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) \\
&\quad = -L_k^{(n)} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s) \Big|_{\substack{t \leftrightarrow s \\ (i_0,\dots,i_{n-1}) \leftrightarrow (j_0,\dots,j_{n-1})}}.
\end{aligned}
$$

## 4.4 Proof of the main theorem

We now turn to the last part of this section. We will prove Theorem 4.1. We first prove the following lemma.

**Lemma 4.12.**

$$
\begin{aligned}
&\sum_{l=1}^{n} \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\
&\qquad + \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 < j_1 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ -1,j_1,\dots,j_{n-1}}} S_{i_{n-1}-(n-1),\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s) \\
&\quad = \sum_{l=1}^{n} \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t) S_{j_{n-1}-(n-1),\dots,j_0}(s).
\end{aligned} \tag{4.26}
$$

**Proof.** For simplicity, we will use the notations

$$
\mathcal{S}_i(t) = S_{i_{n-1}-(n-1),\dots,i_0}(t), \qquad \mathcal{S}_j(s) = S_{j_{n-1}-(n-1),\dots,j_0}(s), \tag{4.27}
$$

when no 'special' shift on the indices of the Schur functions occur. Relabeling each term in the first sum of the left-hand side of (4.26) in the following way $j_{n-l} \mapsto j_{n-l} - 1$ gives

$$
\begin{aligned}
&\sum_{l=1}^{n} \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\
&\quad = \sum_{l=1}^{n} \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-l}-1 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-l}-1,\dots,j_{n-1}}} \mathcal{S}_i(t) \mathcal{S}_j(s).
\end{aligned}
$$

On the one hand, for a fixed $1 \le l \le n-1$, if $j_{n-l} = j_{n-l-1} + 1$, then $p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-l}-1,\dots,j_{n-1}}} = 0$. On the other hand, for a fixed $2 \le l \le n$, if $j_{n-l} = j_{n-l+1}$, then $S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l),\dots,j_0}(s) = 0$. Therefore

$$
\begin{aligned}
&\sum_{l=1}^{n} \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s) \\
&\quad = \sum_{l=1}^{n} \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ j_0,\dots,j_{n-l}-1,\dots,j_{n-1}}} \mathcal{S}_i(t) \mathcal{S}_j(s) \\
&\qquad - \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 < j_1 < \cdots < j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1} \\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s).
\end{aligned}
$$

Consequently, the left-hand side of (4.26) is equal to

$$\sum_{l=1}^{n}\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s)$$
$$+\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\-1,j_1,\dots,j_{n-1}}}\mathcal{S}_i(t)S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)$$
$$=\sum_{l=1}^{n}\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-l}-1,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s). \tag{4.28}$$

Similarly, one can show that the right-hand side of (4.26) is equal to

$$\sum_{l=1}^{n}\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)\mathcal{S}_j(s)$$
$$=\sum_{l=1}^{n}\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-l}+1,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s). \tag{4.29}$$

By virtue of Lemma 4.7($i$), (4.28) and (4.29) are equal. ■

**Proof of Theorem 4.1.** We will prove the theorem for $k\ge 0$. The case $k<0$ is similar. Using the Plücker expansion (4.4) of $\tau_n(t)$, and Lemmas 4.9 and 4.10 we have for $k=0,1$, using the notations (4.27),

$$V_k\tau_n(s,t)=\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} V_k p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s)$$
$$=\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}L_k^{(n)}\mathcal{S}_i(t)\mathcal{S}_j(s)=L_k^{(n)}\tau_n(s,t),$$

where, in the second equality, we have performed some relabeling of the indices as in the proof of Lemma 4.12. We will finish the proof with the case $k=2$, for which we provide some more details, but first we prove the theorem for general $k\ge 3$. We proceed by induction. Assume the theorem holds for some $k\ge 2$. We will establish it for $k+1$. The argument follows from the commutation relations (4.13) and (1.22). We have

$$(k-1)V_{k+1}\tau_n(s,t)=\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}}[V_1,V_k]p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s)$$
$$=\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\j_0,\dots,j_{n-1}}}\big[L_k^{(n)},L_1^{(n)}\big]\mathcal{S}_i(t)\mathcal{S}_j(s)=(k-1)L_{k+1}^{(n)}\tau_n(s,t),$$

where in the second equality we have used the induction hypothesis.

We now provide some details for the case $k=2$. Using Lemmas 4.10 and 4.12 we have

$$L_2^{(n)}\tau_n(s,t)=T_1+T_2+T_3+T_4$$
$$-s_1\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\-1,j_1,\dots,j_{n-1}}}\mathcal{S}_i(t)S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s), \tag{4.30}$$

with

$$\begin{aligned}
T_1 &:= \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \sum_{l=1}^{n} i_{n-l} S_{i_{n-1}-(n-1),\dots,i_{n-l}-(n-l)-2,\dots,i_0}(t)\mathcal{S}_j(s),\\
T_2 &:= -\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \sum_{l=1}^{n} (j_{n-l}+2)\mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+2,\dots,j_0}(s),\\
T_3 &:= \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \sum_{1\le k<l\le n} S_{i_{n-1}-(n-1),\dots,i_{n-k}-(n-k)-1,\dots,i_{n-l}-(n-l)-1,\dots,i_0}(t)\mathcal{S}_j(s),\\
T_4 &:= -\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\\
&\quad\times \sum_{1\le k<l\le n} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_{n-k}-(n-k)+1,\dots,j_{n-l}-(n-l)+1,\dots,j_0}(s).
\end{aligned}$$

We will consider separately the four terms $T_1$, $T_2$, $T_3$, $T_4$. By arguments similar to those used in the proof of Lemma 4.12, and using the fact that $S_{i_{n-1}-(n-1),\dots,i_0}(t)=0$ if $i_k<0$ for some $0\le k\le n-1$, we get for $T_1$

$$\begin{aligned}
T_1 &= \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s)\\
&\quad+\sum_{l=1}^{n-1} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ =i_{n-l}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s)\\
&\quad-\sum_{l=2}^{n} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ <i_{n-l}+1=i_{n-l+1}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s).
\end{aligned}$$

The two last terms in this expression annihilate, i.e.

$$\begin{aligned}
0 &= \sum_{l=1}^{n-1} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ =i_{n-l}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s)\\
&\quad-\sum_{l=2}^{n} \sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ <i_{n-l}+1=i_{n-l+1}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s).
\end{aligned} \tag{4.31}$$

Indeed, we have for $1\le l\le n-1$

$$\begin{aligned}
&\sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1\\ =i_{n-l}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2) p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s)\\
&\quad= \sum_{\substack{-1\le k_0-1<\cdots<k_{n-l-1}\\ =k_{n-l}-1<\cdots<k_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (k_{n-l-1}+2) p_{\substack{k_0,\dots,k_{n-l-2},k_{n-l},k_{n-l-1}+2,k_{n-l+1},\dots,k_{n-1}\\ j_0,\dots,j_{n-1}}}\\
&\qquad\times S_{k_{n-1}-(n-1),\dots,k_{n-l+1}-(n-l+1),k_{n-l-1}-(n-l),k_{n-l}-(n-l-1),k_{n-l-2}-(n-l-2),\dots,k_0}(t)\mathcal{S}_j(s),
\end{aligned}$$

where we have made the relabeling $i_{n-l-1} \mapsto k_{n-l}$, $i_{n-l} \mapsto k_{n-l-1}$, and $i_m \mapsto k_m$ if $m \neq n-l-1, n-l$. As the Plücker coordinates and the Schur functions are determinants, we have, permuting lines in the determinants,

$$p_{\substack{k_0,\dots,k_{n-l-2},k_{n-l},k_{n-l-1}+2,k_{n-l+1},\dots,k_{n-1}\\ j_0,\dots,j_{n-1}}} = -p_{\substack{k_0,\dots,k_{n-l-1}+2,\dots,k_{n-1}\\ j_0,\dots,j_{n-1}}},$$

and

$$S_{k_{n-1}-(n-1),\dots,k_{n-l+1}-(n-l+1),k_{n-l-1}-(n-l),k_{n-l}-(n-l-1),k_{n-l-2}-(n-l-2),\dots,k_0}(t) = -\mathcal{S}_k(t),$$

and hence

$$\begin{aligned}&\sum_{\substack{-1\le i_0-1<\cdots<i_{n-l-1}-1=i_{n-l}<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2)p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s)\\ &\quad= \sum_{\substack{-1\le k_0-1<\cdots<k_{n-l-1}=k_{n-l}-1<\cdots<k_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (k_{n-l-1}+2)p_{\substack{k_0,\dots,k_{n-l-1}+2,\dots,k_{n-1}\\ j_0,\dots,j_{n-1}}}\mathcal{S}_k(t)\mathcal{S}_j(s).\end{aligned}$$

Summing this expression for $1 \le l \le n-1$, and relabeling $l \mapsto l-1$ we get (4.31). Consequently we obtain

$$T_1 = \sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} (i_{n-l}+2)p_{\substack{i_0,\dots,i_{n-l}+2,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s). \tag{4.32}$$

By similar arguments, we have

$$\begin{aligned}T_2 = &-\sum_{l=1}^{n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} j_{n-l}p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-l}-2,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s)\\ &+\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}}\mathcal{S}_i(t)S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s),\end{aligned} \tag{4.33}$$

$$T_3 = \sum_{1\le k<l\le n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-l}+1,\dots,i_{n-k}+1,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s), \tag{4.34}$$

$$\begin{aligned}T_4 = &-\sum_{1\le k<l\le n} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-l}-1,\dots,j_{n-k}-1,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s)\\ &+\sum_{1\le k\le n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-k}-1,\dots,j_{n-1}}}\mathcal{S}_i(t)S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s).\end{aligned} \tag{4.35}$$

Substituting (4.32), (4.33), (4.34) and (4.35) in (4.30), using Lemma 4.7$(ii)$ and Lemma 4.9 we obtain

$$\begin{aligned}L_2^{(n)}\tau_n(s,t) = &\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} V_2 p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}}\mathcal{S}_i(t)\mathcal{S}_j(s)\\ &+\sum_{k=1}^{n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-k}-1,\dots,j_{n-1}}}\mathcal{S}_i(t)S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)\end{aligned}$$

$$
\begin{aligned}
&- s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)\\
&+ \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s).
\end{aligned}
$$

We prove that the last three terms in this expression annihilate

$$
\begin{aligned}
0 = {}& \sum_{k=1}^{n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-k}-1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)\\
&- s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0<j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s)\\
&+ \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s),
\end{aligned}
\tag{4.36}
$$

and hence

$$
L_2^{(n)} \tau_n(s,t) = \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_0<\cdots<j_{n-1}}} V_2 p_{\substack{i_0,\dots,i_{n-1}\\ j_0,\dots,j_{n-1}}} \mathcal{S}_i(t)\mathcal{S}_j(s). \tag{4.37}
$$

Indeed, developing the determinant $S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s)$ with respect to the last line, using the fact that the first elementary Schur polynomials are $S_0(s) = 1$ and $S_1(s) = s_1$, and Lemma 4.8$(ii)$, we have

$$
\begin{aligned}
&\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s)\\
&\quad = s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1}(s)\\
&\qquad - \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) \sum_{l=1}^{n-1} S_{j_{n-1}-(n-1),\dots,j_{n-l}-(n-l)+1,\dots,j_1-1}(s).
\end{aligned}
$$

By an argument similar to that of the proof of Lemma 4.12, we get

$$
\begin{aligned}
&\sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1,1}(s)\\
&\quad = s_1 \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1}(s)\\
&\qquad - \sum_{l=1}^{n-1} \sum_{\substack{0\le i_0<\cdots<i_{n-1}\\ 0\le j_1<\cdots<j_{n-1}}} p_{\substack{i_0,\dots,i_{n-1}\\ -1,j_1,\dots,j_{n-l}-1,\dots,j_{n-1}}} \mathcal{S}_i(t) S_{j_{n-1}-(n-1),\dots,j_1-1}(s).
\end{aligned}
$$

Noticing that $S_{j_{n-1}-(n-1),\dots,j_1-1}(s) = 0$ when $j_1 = 0$, and

$$
S_{j_{n-1}-(n-1),\dots,j_1-1}(s) = S_{j_{n-1}-(n-1),\dots,j_1-1,0}(s),
$$

when $j_1 > 0$, we get (4.36), and hence (4.37). This proves the case $k = 2$ and finishes the proof. ■

It would be nice to have a proof of Theorem 4.1 using the vertex operators techniques developed by the Kyoto school [13], but this remains a challenge for the future!

### Acknowledgements

The authors thank the referees for their useful comments on this work and for drawing attention to the references [27, 28, 29]. The first author acknowledges the partial support of the Belgian Interuniversity Attraction Poles P06/02 and P07/18. During part of this research, the second author was a Research Fellow of the Belgian National Science Foundation (FNRS), whose support is also gratefully acknowledged.

## References

[1] Ablowitz M.J., Ladik J.F., Nonlinear differential-difference equations, *J. Math. Phys.* **16** (1975), 598–603.

[2] Ablowitz M.J., Ladik J.F., Nonlinear differential-difference equations and Fourier analysis, *J. Math. Phys.* **17** (1976), 1011–1018.

[3] Adler M., Shiota T., van Moerbeke P., Random matrices, Virasoro algebras, and noncommutative KP, *Duke Math. J.* **94** (1998), 379–431, solv-int/9812006.

[4] Adler M., van Moerbeke P., Matrix integrals, Toda symmetries, Virasoro constraints, and orthogonal polynomials, *Duke Math. J.* **80** (1995), 863–911, solv-int/9706010.

[5] Adler M., van Moerbeke P., Integrals over classical groups, random permutations, Toda and Toeplitz lattices, *Comm. Pure Appl. Math.* **54** (2001), 153–205, math.CO/9912143.

[6] Adler M., van Moerbeke P., Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice, *Comm. Math. Phys.* **237** (2003), 397–440, math-ph/0201063.

[7] Aldous D., Diaconis P., Longest increasing subsequences: from patience sorting to the Baik–Deift–Johansson theorem, *Bull. Amer. Math. Soc.* **36** (1999), 413–432.

[8] Bowick M.J., Morozov A., Shevitz D., Reduced unitary matrix models and the hierarchy of $\tau$-functions, *Nuclear Phys. B* **354** (1991), 496–530.

[9] Cafasso M., Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz–Ladik hierarchy, *J. Phys. A: Math. Theor.* **42** (2009), 365211, 20 pages, arXiv:0804.3572.

[10] Cantero M.J., Moral L., Velázquez L., Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle, *Linear Algebra Appl.* **362** (2003), 29–56, math.CA/0204300.

[11] Cruz-Barroso R., González-Vera P., A Christoffel–Darboux formula and a Favard's theorem for orthogonal Laurent polynomials on the unit circle, *J. Comput. Appl. Math.* **179** (2005), 157–173.

[12] Damianou P.A., Master symmetries and $R$-matrices for the Toda lattice, *Lett. Math. Phys.* **20** (1990), 101–112.

[13] Date E., Kashiwara M., Jimbo M., Miwa T., Transformation groups for soliton equations, in Nonlinear Integrable Systems – Classical Theory and Quantum Theory (Kyoto, 1981), World Sci. Publishing, Singapore, 1983, 39–119.

[14] Dijkgraaf R., Verlinde H., Verlinde E., Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity, *Nuclear Phys. B* **348** (1991), 435–456.

[15] Faybusovich L., Gekhtman M., Poisson brackets on rational functions and multi-Hamiltonian structure for integrable lattices, *Phys. Lett. A* **272** (2000), 236–244, nlin.SI/0006045.

[16] Fernandes R.L., On the master symmetries and bi-Hamiltonian structure of the Toda lattice, *J. Phys. A: Math. Gen.* **26** (1993), 3797–3803.

[17] Forrester P.J., Witte N.S., Bi-orthogonal polynomials on the unit circle, regular semi-classical weights and integrable systems, *Constr. Approx.* **24** (2006), 201–237, math.CA/0412394.

[18] Fuchssteiner B., Mastersymmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, *Progr. Theoret. Phys.* **70** (1983), 1508–1522.

[19] Fukuma M., Kawai H., Nakayama R., Continuum Schwinger–Dyson equations and universal structures in two-dimensional quantum gravity, *Internat. J. Modern Phys. A* **6** (1991), 1385–1406.

[20] Gesztesy F., Holden H., Michor J., Teschl G., Local conservation laws and the Hamiltonian formalism for the Ablowitz–Ladik hierarchy, *Stud. Appl. Math.* **120** (2008), 361–423, arXiv:0711.1644.

[21] Grünbaum F.A., Haine L., A theorem of Bochner, revisited, in Algebraic Aspects of Integrable Systems, *Progr. Nonlinear Differential Equations Appl.*, Vol. 26, Birkhäuser Boston, Boston, MA, 1997, 143–172.

[22] Haine L., Horozov E., Toda orbits of Laguerre polynomials and representations of the Virasoro algebra, *Bull. Sci. Math.* **117** (1993), 485–518.

[23] Haine L., Semengue J.P., The Jacobi polynomial ensemble and the Painlevé VI equation, *J. Math. Phys.* **40** (1999), 2117–2134.

[24] Haine L., Vanderstichelen D., A centerless representation of the Virasoro algebra associated with the unitary circular ensemble, *J. Comput. Appl. Math.* **236** (2011), 19–27, arXiv:1001.4244.

[25] Kac V., Schwarz A., Geometric interpretation of the partition function of 2D gravity, *Phys. Lett. B* **257** (1991), 329–334.

[26] Kac V.G., Raina A.K., Bombay lectures on highest weight representations of infinite-dimensional Lie algebras, *Advanced Series in Mathematical Physics*, Vol. 2, World Scientific Publishing Co. Inc., Teaneck, NJ, 1987.

[27] Kharchev S., Mironov A., Integrable structures of unitary matrix models, *Internat. J. Modern Phys. A* **7** (1992), 4803–4824.

[28] Kharchev S., Mironov A., Zhedanov A., Faces of relativistic Toda chain, *Internat. J. Modern Phys. A* **12** (1997), 2675–2724, hep-th/9606144.

[29] Kharchev S., Mironov A., Zhedanov A., Different aspects of relativistic Toda chain, in Symmetries and Integrability of Difference Equations (Canterbury, 1996), *London Math. Soc. Lecture Note Ser.*, Vol. 255, Editors P.A. Clarkson, F.W. Nijhoff, Cambridge Univ. Press, Cambridge, 1999, 23–40, hep-th/9612094.

[30] Martinec E.J., On the origin of integrability in matrix models, *Comm. Math. Phys.* **138** (1991), 437–449.

[31] Mironov A., Morozov A., On the origin of Virasoro constraints in matrix models: Lagrangian approach, *Phys. Lett. B* **252** (1990), 47–52.

[32] Nenciu I., Lax pairs for the Ablowitz–Ladik system via orthogonal polynomials on the unit circle, *Int. Math. Res. Not.* **2005** (2005), 647–686, math-ph/0412047.

[33] Rains E.M., Increasing subsequences and the classical groups, *Electron. J. Combin.* **5** (1998), R12, 9 pages.

[34] Simon B., Orthogonal polynomials on the unit circle. Part 1. Classical theory, *American Mathematical Society Colloquium Publications*, Vol. 54, American Mathematical Society, Providence, RI, 2005.

[35] Simon B., Orthogonal polynomials on the unit circle. Part 2. Spectral theory, *American Mathematical Society Colloquium Publications*, Vol. 54, American Mathematical Society, Providence, RI, 2005.

[36] Tracy C.A., Widom H., Fredholm determinants, differential equations and matrix models, *Comm. Math. Phys.* **163** (1994), 33–72, hep-th/9306042.

[37] Ueno K., Takasaki K., Toda lattice hierarchy, in Group Representations and Systems of Differential Equations (Tokyo, 1982), *Adv. Stud. Pure Math.*, Vol. 4, North-Holland, Amsterdam, 1984, 1–95.

[38] Vanderstichelen D., Virasoro symmetries for the Ablowitz–Ladik hierarchy and non-intersecting Brownian motion models, Ph.D. Thesis, Université Catholique de Louvain, 2011.

[39] Zubelli J.P., Magri F., Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV, *Comm. Math. Phys.* **141** (1991), 329–351.