An Index for Intersecting Branes in Matrix Models[⋆]

Harold STEINACKER and Jochen ZAHN

Fakultät für Physik, Universität Wien, Boltzmanngasse 5, 1090 Wien, Austria
E-mail: *harold.steinacker@univie.ac.at, jochen.zahn@univie.ac.at*

Abstract. We introduce an index indicating the occurrence of chiral fermions at the intersection of branes in matrix models. This allows to discuss the stability of chiral fermions under perturbations of the branes.

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1 Introduction

The IKKT or IIB model [8] admits solutions which can be interpreted as branes embedded in a flat target space, cf. [11, 12] for recent reviews. Of particular interest is the case of intersecting branes, as these can give rise to chiral fermions living on the intersection [6], thus having the potential of yielding a physically viable model. The aim of this note is to show that the occurrence of chiral fermions can be rephrased as the non-vanishing of a certain index, which counts the number of zero modes of the Dirac operator, weighted with their chirality. Moreover, conditions ensuring the stability of the index under perturbations are given. This is also demonstrated in a concrete example.

Let us explain the relation of the index we propose with different indices discussed in the context of emergent geometries in matrix models. In [2], two-dimensional compact branes embedded in $\mathbb{R}^3$ are studied. In our language, the intersection of such branes with a point $y \in \mathbb{R}^3$ is considered, and the index counts the difference of the number of positive and negative modes of the corresponding Dirac operator. The set of y's where the index changes is then interpreted as the locus of the brane. Crucial differences to our setting are the odd dimension of the target space (so that there is no chirality operator and eigenvalues of the Dirac operator are not symmetric around 0), and the restriction to finite matrices.

Another index for noncommutative branes was considered in [1]. The difference to our definition is the usage of another Dirac operator, the so-called Ginsparg–Wilson Dirac operator, which does not coincide with the Dirac operator appearing in the IKKT action.

The article is structured as follows: In the next section, we recall aspects of the matrix model framework and its effective (brane) geometry. We introduce the notion of intersecting branes, and introduce the index indicating the occurrence of chiral fermions. In Section 3, we present conditions guaranteeing the stability of the index under deformations, and discuss a concrete example. We conclude with a brief summary and an outlook.

2 Matrix models, intersecting branes, and chiral fermions

We briefly collect the essential ingredients of the matrix model framework and its effective geometry, referring to the recent review [12] for more details. The starting point is the maximally

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supersymmetric IKKT or IIB model [8], whose action is given by

$$S = \mathrm{Tr}\left(\left[X^A, X^B\right]\left[X^C, X^D\right]\eta_{AC}\eta_{BD} + 2\bar{\Psi}D\Psi\right).$$

Here the X^A are Hermitian matrices, i.e., operators acting on a separable Hilbert space $\mathcal{H}$. The indices run from 0 to 9, and will be raised or lowered with the invariant tensor η_{AB} of $SO(9,1)$. Furthermore, Ψ is a matrix-valued Majorana Weyl spinor of $SO(9,1)$, and D is the Dirac operator, defined by

$$(D\Psi)^a = \Gamma_{Ab}^{\ \ a}\left[X^A, \Psi^b\right],$$

where Γ_A are the 10-dimensional γ matrices.

Even though this will not be used explicitly, the picture to have in mind is that the matrix configurations X^A describe embedded noncommutative branes. By this one means that the X^A can be interpreted as quantized embedding functions [12] of a $2n$-dimensional submanifold $\mathcal{M}^{2n} \hookrightarrow \mathbb{R}^{10}$. More precisely, there should be some quantization map $\mathcal{Q} : \mathcal{C}(\mathcal{M}^{2n}) \to L(\mathcal{H})$ which maps classical functions on $\mathcal{M}^{2n}$ to a noncommutative (matrix) algebra, such that commutators can be interpreted as quantized Poisson brackets. The X^A are then the image of classical embedding functions x^A under this map. For more details, we refer to [12].

If the matrices X^A are of block-diagonal form

$$X^A = \begin{pmatrix} X_L^A & 0 \\ 0 & X_R^A \end{pmatrix},$$

we speak of two intersecting branes. If we analogously split the fermions as

$$\Psi = \begin{pmatrix} \Psi_{LL} & \Psi_{LR} \\ \Psi_{RL} & \Psi_{RR} \end{pmatrix},$$

then the Dirac operator acts on the off-diagonal components as

$$D_{LR}\Psi_{LR} = \Gamma_A\left(X_L^A\Psi_{LR} - \Psi_{LR}X_R^A\right).$$

We will consider the case when, in the semiclassical picture, the two branes $\mathcal{M}_{L/R}$ are of the form

$$\mathcal{M}_{L/R} = \mathcal{M}^d \times \mathcal{M}'_{L/R}, \tag{1}$$

where d is even and $\mathcal{M}^d$ is embedded in the subspace of $\mathbb{R}^{10}$ generated by e^μ, $0 \le \mu \le d-1$ and $\mathcal{M}'_{L/R}$ are embedded in the directions spanned by e^i, $d \le i \le 9$. Furthermore, the symplectic form on $\mathcal{M}_{L/R}$ is required to respect the split (1), i.e., it should vanish for one vector in $T\mathcal{M}^d$ and one in $T\mathcal{M}'_{L/R}$. In formal terms, this means that

$$\mathcal{H}_{L/R} = \mathcal{H}_{L/R}^{(d)} \otimes \mathcal{H}_{L/R}^{(10-d)}, \qquad X_{L/R}^\mu = Y_{L/R}^\mu \otimes \mathbb{1}, \qquad X_{L/R}^i = \mathbb{1} \otimes Y_{L/R}^i. \tag{2}$$

Here μ labels the indices $0 \le \mu \le d-1$, whereas i labels $d \le i \le 9$. Furthermore,

$$\mathcal{H}_L^{(d)} \simeq \mathcal{H}_R^{(d)} \simeq \mathcal{H}^{(d)} \tag{3}$$

and, under this isomorphism, $X_L^\mu = X_R^\mu$. This encodes the requirement that the two branes share a common d-dimensional brane. Using the identification of $\mathcal{H}_L^{(d)}$ and $\mathcal{H}_R^{(d)}$, we may write the Dirac operator as

$$D_{LR}\Psi_{LR} = D_{LR}^{(d)}\Psi_{LR} + D_{LR}^{(10-d)}\Psi_{LR} = \Gamma_\mu[X^\mu, \Psi_{LR}] + \Gamma_i\left(X_L^i\Psi_{LR} - \Psi_{LR}X_R^i\right).$$

We also split the chirality operator (note the different signs in $\chi^{(d)}$ and $\chi^{(10-d)}$ stemming from the signature $(-, +, \ldots, +)$ of η),

$$\chi = \chi^{(d)}\chi^{(10-d)}, \qquad \chi^{(d)} = i^{-d/2+1}\Gamma_0 \cdots \Gamma_{d-1}, \qquad \chi^{(10-d)} = i^{-(10-d)/2}\Gamma_d \cdots \Gamma_9,$$

and remark that it fulfills

$$\left[\chi^{(d)}, \chi^{(10-d)}\right] = 0, \qquad \left(\chi^{(d)}\right)^2 = 1, \qquad \left(\chi^{(10-d)}\right)^2 = 1,$$

and

$$\left\{\chi^{(d)}, D_{LR}^{(d)}\right\} = 0, \qquad \left\{\chi^{(10-d)}, D_{LR}^{(10-d)}\right\} = 0,$$
$$\left[\chi^{(d)}, D_{LR}^{(10-d)}\right] = 0, \qquad \left[\chi^{(10-d)}, D_{LR}^{(d)}\right] = 0.$$

We also note that the Γ matrices may be represented as

$$\Gamma_\mu = \gamma_\mu \otimes \mathbb{1}_{2^{5-d/2}}, \qquad \Gamma_i = \gamma_{d+1} \otimes \delta_i, \tag{4}$$

where the γ_μ form the d-dimensional Lorentzian Clifford algebra, γ_{d+1} is the corresponding chirality operator, and the δ_i form the $(10-d)$-dimensional Euclidean Clifford algebra.

Given that the $X_{L/R}$ are represented on Hilbert spaces $\mathcal{H}_{L/R}$, the off-diagonal fermions are elements of $\mathcal{H}_{LR} = B(\mathcal{H}_R, \mathcal{H}_L) \otimes \mathbb{C}^{2^5}$. Due to the split (2), a general ansatz for solutions of $D_{LR}\Psi_{LR} = 0$ is[1]

$$\Psi_{LR} = \Psi_{LR}^{(d)} \otimes \Psi_{LR}^{(10-d)}, \qquad \Psi_{LR}^{(d)} \in \mathcal{H}_{LR}^{(d)}, \qquad \Psi_{LR}^{(10-d)} \in \mathcal{H}_{LR}^{(10-d)},$$

where we defined

$$\mathcal{H}_{LR}^{(d)} = B\left(\mathcal{H}^{(d)}\right) \otimes \mathbb{C}^{2^{d/2}}, \qquad \mathcal{H}_{LR}^{(10-d)} = B\left(\mathcal{H}_R^{(10-d)}, \mathcal{H}_L^{(10-d)}\right) \otimes \mathbb{C}^{2^{5-d/2}}.$$

Here we used (3) and the same factorization of the spinorial representation space as in (4). Using the operator norm, $\mathcal{H}_{LR}^{(10-d)}$ can be given the structure of a Banach space. Due to (4), have

$$D_{LR}^{(10-d)} = \gamma_{d+1} \otimes \Delta_{LR}^{(10-d)}, \qquad \chi^{(10-d)} = \mathbb{1}_{(d/2)^2} \otimes \theta^{(10-d)},$$

where $\Delta_{LR}^{(10-d)}$ and $\theta^{(10-d)}$ are anticommuting operators on $\mathcal{H}_{LR}^{(10-d)}$. In particular, non-zero eigenvalues of $\Delta_{LR}^{(10-d)}$ come in pairs $\pm m$, which are interchanged by $\theta^{(10-d)}$, and whose eigenvectors $v_{\pm m}$ may be combined to eigenvectors $v_m^\pm$ of $\theta^{(10-d)}$ of opposite chirality. It is then clear that given an eigenvector $\Psi_{LR}^{(10-d)}$ of $\Delta_{LR}^{(10-d)}$ with eigenvalue m, the Dirac equation for $\Psi_{LR}^{(d)}$ becomes, cf. (2),

$$\Gamma^\mu\left[Y_\mu, \Psi_{LR}^{(d)}\right] + m\gamma_{d+1}\Psi_{LR}^{(d)} = 0,$$

which does not admit chiral solutions unless $m = 0$. Furthermore, given a zero mode $\Psi_{LR}^{(10-d)}$, the chirality of $\Psi_{LR}^{(10-d)}$ w.r.t. $\theta^{(10-d)}$ determines $\chi^{(d)}\Psi_{LR}^{(d)}$, i.e., the d-dimensional chirality of $\Psi_{LR}^{(d)}$, by the total chirality constraint $\chi\Psi_{LR} = \Psi_{LR}$. Hence, a d-dimensional chiral fermion requires a zero eigenvector of $\Delta_{LR}^{(10-d)}$ with no corresponding eigenvector of opposite chirality[2]. Note that if we have a chiral fermion in the LR sector, then the Majorana condition ensures that the RL sector contains the conjugate fermion with opposite chirality.

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[1]Note that the condition (2) is crucial here. In [10], the same ansatz for Ψ_{LR} is used, but (2) is not fulfilled. Hence, in that work, the ansatz is not general enough to find all solutions of the Dirac equation.

[2]Otherwise, their combination will in general acquire a mass through quantum corrections.

By our assumptions, $\Delta_{LR}^{(10-d)}$ is a Dirac operator on the intersection of branes with Riemannian signature, so we may expect it to have discrete spectrum (in Section 3.1, this is shown to be the case in a concrete example). The above discussion then motivates the following definition of an index for the Dirac operator $\Delta_{LR}^{(10-d)}$:

$$\Xi\big(\Delta_{LR}^{(10-d)}\big) = \mathrm{Tr}_{\mathcal{H}_{LR}^{(10-d)}}\left(P_\Gamma\big((\Delta_{LR}^{(10-d)})^2\big)\theta^{(10-d)}\right).$$

Here Γ is some closed curve that encircles the origin and does not intersect an eigenvalue of $\big(\Delta_{LR}^{(10-d)}\big)^2$, and $P_\Gamma\big((\Delta_{LR}^{(10-d)})^2\big)$ is the orthogonal projector on the eigenspaces whose eigenvalues are encircled by Γ. As discussed above, nonzero eigenvalues of $\big(\Delta_{LR}^{(10-d)}\big)^2$ occur in pairs of opposite chirality, so the definition is independent of the choice of Γ. The index counts the number of 0 eigenmodes, weighted with their chirality. This index can also be written in the form

$$\Xi\big(\Delta_{LR}^{(10-d)}\big) = \mathrm{Tr}_{\mathcal{H}_{LR}^{(10-d)}}\left(e^{-t\left(\Delta_{LR}^{(10-d)}\right)^2}\theta^{(10-d)}\right)$$

for generic $t > 0$, which is analogous to the usual definition of the index on compact Riemannian spaces, cf. [4, Theorem 3.50].

The motivation for introducing an index to describe chirality is that it takes discrete values, so by continuity, one would expect it to be constant under deformations of the branes. In the next section, we will discuss criteria which indeed ensure this.

3   Deformation stability of chiral modes

Let us begin by recalling a notion from perturbation theory. Let A be a closed, in general unbounded operator on a Banach space. Then B is A-bounded, if $D(A) \subset D(B)$, and there are positive constants a, b such that

$$\|Bx\| \leq a\|Ax\| + b\|x\|$$

holds for all $x \in D(A)$. A straightforward consequence of [9, Theorem IV.3.18] is now the following:

Proposition 1. *Let A have discrete spectrum. Given a closed curve Γ in $\mathbb{C}$ that encircles a finite part of the spectrum, we define the projector $P_\Gamma(A)$ on the corresponding eigenspaces. Given an A-bounded operator B, the map $\lambda \mapsto P_\Gamma(A+\lambda B)$ is norm-continuous in a small enough neighborhood of 0.*

Now fix some $X_{L/R}^i$. By the above proposition and the fact that Ξ takes discrete values, one easily obtains precise conditions that ensure the invariance of the index under perturbations of the $X_{L/R}^i$:

Proposition 2. *Let $\tilde{X}_{L/R}^i \in L\big(\mathcal{H}_{L/R}^{(10-d)}\big)$ be self-adjoint and $\tilde{\Delta}_{LR}^{(10-d)}$ the corresponding Dirac operator. Assume that $\big(\tilde{\Delta}_{LR}^{(10-d)}\big)^2$ and $\big\{\tilde{\Delta}_{LR}^{(10-d)}, \Delta_{LR}^{(10-d)}\big\}$ are $\big(\Delta_{LR}^{(10-d)}\big)^2$ bounded. Then there is a neighborhood U of 0 such that*

$$\Xi\big(\Delta_{LR}^{(10-d)} + \lambda\tilde{\Delta}_{LR}^{(10-d)}\big) = \Xi\big(\Delta_{LR}^{(10-d)}\big)$$

for all $\lambda \in U$.

Remark 1. If $\mathcal{H}_{L/R}^{(10-d)}$ are finite-dimensional, then $\mathcal{H}_{LR}^{(10-d)}$ has finite even dimension. It is then no longer necessary to restrict the trace to a finite number of eigenvalues, so one can dispose of the projector in the definition of Ξ. It follows that for finite-dimensional representation spaces (corresponding to compact branes), the chirality index always vanishes.

3.1 An example

Up to now, the discussion was generic, in particular independent of the commutation relations of the X^i. Let us now consider the concrete example of intersecting Moyal planes (recall that a Moyal plane is defined by canonical commutation relations $[X^i, X^j] = i\Theta^{ij}$, with Θ a real antisymmetric matrix). Take $d = 6$, and let $X^i_{L/R}$ span two 2-dimensional orthogonal Moyal planes, i.e.,

$$X^6_L = x, \qquad X^7_L = p_x, \qquad X^8_L = 0, \qquad X^9_L = 0,$$
$$X^6_R = 0, \qquad X^7_R = 0, \qquad X^8_R = y, \qquad X^9_R = p_y,$$

where (x, p_x) and (y, p_y) are the canonical position and momentum operators on $\mathcal{H}^{(10-d)}_{L/R} = L^2(\mathbb{R})$. As shown in [6] (and also below), the index of this configuration is 1. It is easy to see that

$$\left(\Delta^{(10-d)}_{LR}\right)^2 = x^2 + y^2 + p_x^2 + p_y^2 + 2\Sigma_{67} + 2\Sigma_{89}, \tag{5}$$

where

$$\Sigma_{ij} = \tfrac{i}{4}[\Gamma^i, \Gamma^j].$$

This operator acts on $\mathcal{H}^{(10-d)}_{LR} \simeq L^2(\mathbb{R}^2) \otimes \mathbb{C}^4$, where we use that

$$B\left(L^2(\mathbb{R}), L^2(\mathbb{R})\right) \simeq L^2\left(\mathbb{R}^2\right).$$

As the first four terms on the r.h.s. of (5) form a positive definite quadratic form, it follows from the above proposition that the index is invariant under perturbations $X^i_{L/R} \to X^i_{L/R} + \lambda \tilde{X}^i_{L/R}$ for small enough λ, if the $\tilde{X}^i_{L/R}$ are bounded or linear (corresponding to intersections at angles[3]) in the $X^i_{L/R}$ (or a sum of such contributions). In order to see this explicitly, let us consider the case where the $\tilde{X}^i$ are linear in the X^i. As an example, consider

$$X^6 = x + cy, \qquad X^7 = p_x, \qquad X^8 = y, \qquad X^9 = p_y.$$

For the square of the Dirac operator, one obtains

$$\left(\Delta^{(10-d)}_{LR}\right)^2 = \underbrace{x^2 + \left(1 + c^2\right)y^2 - 2cxy + p_x^2 + p_y^2}_{\Delta_1} + \underbrace{2\Sigma_{67} - 2\Sigma_{89} + 2c\Sigma_{69}}_{\Delta_2}.$$

Here Δ_1 acts on $L^2(\mathbb{R}^2)$, while Δ_2 acts on the spinorial representation space $\mathbb{C}^4$. To have a zero eigenvector of $\left(\Delta^{(10-d)}_{LR}\right)^2$ requires a pair of eigenvectors of Δ_1 and Δ_2 which add up to zero. Let us compute the lowest eigenvalue of Δ_1. We use the ansatz

$$\Psi = e^{-\frac{1}{2}(Ax^2 + By^2 + 2Cxy)}.$$

The eigenvalue equation $\Delta_1 \Psi = \eta \Psi$ then leads to

$$-A^2 - C^2 + 1 = 0,$$
$$-C^2 - B^2 + 1 + c^2 = 0,$$
$$-AC - BC - c = 0,$$

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[3]For intersections at angles in the context of string compactifications, cf. [3, 5, 7].

the eigenvalue being given by $\eta = A + B$. It is straightforward to find the eigenvalue $\eta = \sqrt{4 + c^2}$. For the eigenvalues of the spinorial part Δ_2, one finds

$$\eta = \pm c, \qquad \eta = \pm\sqrt{4 + c^2}.$$

Hence, there is exactly one way to cancel the eigenvector of Δ_1, i.e., there is one eigenvector of $\left(\Delta_{LR}^{(10-d)}\right)^2$ with eigenvalue 0 (the higher eigenvalues of Δ_1 can obviously not lead to further zero eigenvalues). One can also explicitly check that it has positive chirality. Analogously, one can treat the $d = 4$ dimensional intersection of a 6- and an 8-dimensional brane, and similar configurations [6].

An example of intersecting branes with a vanishing index is provided by a degenerate intersection of two quantum planes, such as

$$X^6 = x + y, \qquad X^7 = p_x, \qquad X^8 = 0, \qquad X^9 = p_y.$$

In this case the part of $\left(\Delta_{LR}^{(10-d)}\right)^2$ that is quadratic in the coordinates of the quantum plane is given by

$$(x - y)^2 + p_x^2 + p_y^2,$$

which is not a positive definite quadratic form. In particular, the condition of being $\left(\Delta_{LR}^{(10-d)}\right)^2$ bounded is not fulfilled for rotations of this plane. One easily checks that the index for this configuration vanishes. This underlines the necessity of spanning the full $\mathbb{R}^{(10-d)}$ in order to get chiral fermions, as already pointed out in [6].

4   Summary and outlook

We presented a definition of an index describing the occurrence of chiral fermions on intersecting branes in matrix models and discussed the stability of this index under perturbations. In particular, this implies the existence of chiral fermions for branes intersecting at angles. The drawback of our approach is that it requires strong restrictions on the embedding, in particular (2). It is for example not applicable for situations in which (in the semiclassical picture) the brane $\mathcal{M}^d$ is not flat. One possibility to treat this case could be to work in the semiclassical limit, or to use a modified chirality operator, like[4]

$$\chi = \varepsilon^{A_1\ldots A_{2n}C_1\ldots C_{10-2n}}\varepsilon_{B_1\ldots B_{2n}C_1\ldots C_{10-2n}}X^{B_1}\cdots X^{B_{2n}}\Gamma_{A_1}\cdots\Gamma_{A_{2n}}$$

for a $2n$-dimensional brane. We plan to come back to this issue in future work.

As noted in Remark 1, the index always vanishes for intersections of compact fuzzy spaces $\mathcal{K}_i \subset \mathbb{R}^{(10-d)}$. This raises an apparent paradox, since the results on chiral fermions on intersections should apply at least approximately for each intersection. What happens is that pairs of "almost-localized" fermionic near-zero modes arise on the intersections $\mathcal{K}_i \cap \mathcal{K}_j$, such that for each "effectively" chiral fermion localized on some intersection, there is another fermion with opposite chirality at some other intersection[5]. This means that if, e.g., the chiral fermions of the standard model arise from some intersections such as in [6], there are additional sectors with fermions of opposite chirality localized at different intersections. The approximate localization on different intersections suggests that these unwanted sectors could be effectively hidden or removed in some way. A natural strategy to achieve this is to give up the product ansatz (2), as proposed in [10], and as realized, e.g., by solutions with split noncommutativity [13]. These are interesting directions for further research.

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[4]This particular operator has the disadvantage that it does in general not anticommute with the Dirac operator, but it may be useful nevertheless.

[5]This is verified in numerical simulations.

Acknowledgments

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