Symmetry, Integrability and Geometry: Methods and Applications SIGMA 9 (2013), 066, 21 pages

Symmetry and Intertwining Operators for the Nonlocal Gross–Pitaevskii Equation

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Received February 15, 2013, in final form October 26, 2013; Published online November 06, 2013
http://dx.doi.org/10.3842/SIGMA.2013.066

Abstract. We consider the symmetry properties of an integro-differential multidimensional Gross–Pitaevskii equation with a nonlocal nonlinear (cubic) term in the context of symmetry analysis using the formalism of semiclassical asymptotics. This yields a semiclassically reduced nonlocal Gross–Pitaevskii equation, which can be treated as a nearly linear equation, to determine the principal term of the semiclassical asymptotic solution. Our main result is an approach which allows one to construct a class of symmetry operators for the reduced Gross–Pitaevskii equation. These symmetry operators are determined by linear relations including intertwining operators and additional algebraic conditions. The basic ideas are illustrated with a 1D reduced Gross–Pitaevskii equation. The symmetry operators are found explicitly, and the corresponding families of exact solutions are obtained.

Key words: symmetry operators; intertwining operators; nonlocal Gross–Pitaevskii equation; semiclassical asymptotics; exact solutions

2010 Mathematics Subject Classification: 35Q55; 45K05; 76M60; 81Q20

1 Introduction

Symmetry operators, which, by definition, leave the set of solutions of an equation invariant, are of essential importance in the symmetry analysis of nonlinear partial differential equations (PDEs). The obvious use of symmetry operators of an equation is to generate new solutions from a known one. A modern symmetry analysis of differential equations (DEs) is based on Lie group theory. For example, if for an ordinary differential equation (ODE) there exists a Lie group of point transformations (point symmetries) which act on the space of independent and dependent variables, then they map any solution to another solution of the equation. In a more general case, an ODE can admit contact transformations (contact symmetries) acting on the independent and dependent variables, and also on the first derivatives of the dependent variables. In other words, point symmetries and contact symmetries provide examples of Lie groups of symmetry operators. The Lie group methods, as well as their applications to ODEs and PDEs, are described in many books and review articles (see, e.g., [7]). The prolongation of the action of a Lie group on the space of independent variables, dependent variables, and partial derivatives of the dependent variables up to any finite order allows to apply the Lie group theory to studying symmetries of PDEs [30]. The fundamental property of a Lie group is that it is completely characterized by its infinitesimal operator (generator). Given a system of DEs, finding the Lie symmetry group is reduced to solving a system of equations that determine the Lie group generators. The principal point is that the determining equations for the generator are linear and homogeneous. For a nonlinear PDE the determining equations take the form of an overdetermined system of linear homogeneous PDEs, which can be solved step-by-step to obtain infinitesimal operators in explicit form (see, e.g., [8, 29, 30]).

Solving the determining equations for given system of PDEs we can find the generators of point or contact symmetries for the system. Following the Lie theory, we can recover the Lie group of finite (i.e., not infinitesimal) symmetry transformations for given system of PDEs. Solutions of the determining equations, however, may contain not only independent variables, dependent variables, and first order derivatives (as with point and contact symmetries), but also higher-order derivatives. Generators of this type are called higher symmetries and they do not yield finite Lie groups. Higher symmetries are related to the so-called Lie–Bäcklund transformations that are widely used in symmetry analysis (see [2] and also, e.g., [29] and [8]). Note that higher-order symmetries do not generate symmetry operators. However, no general approaches to direct calculation of symmetry operators for nonlinear equations are known other than the use of the Lie group formalism. This is due to that the determining equations for symmetry operators are nonlinear operator equations. Solving them is a complicated mathematical problem which requires special techniques not developed yet. In addition, in order to solve determining equations for symmetry operators, we have to specify the structure of symmetry operators consistent with the determining equations, but there are no recipes for choosing such a structure. Therefore, finding the symmetry operators for nonlinear equations is in general an unrealistic task.

Note that for linear PDEs, symmetry operators which are widely used in quantum mechanics applications can be effectively found from linear determining equations, (see, e.g., [14, 21, 31] and references therein). This inspired us to seek a special class of nonlinear equations for which symmetry operators could be calculated using the methods applicable to linear equations. As an example of such a class of nonlinear equations we consider nonlinear integro-differential equations (IDEs) with partial derivatives. We call the equations of this class *nearly linear equations*. Symmetry operators for them can be found by solving linear operator equations (similarly to those for linear PDEs) and additional algebraic equations. We consider a generalized multidimensional integro-differential Gross–Pitaevskii equation (GPE) with partial derivatives and a nonlocal cubic nonlinear interaction term of general form. The WKB–Maslov method of semiclassical asymptotics [4, 24] is used to obtain a reduced GPE from the original GPE. The reduced GPE is quadratic in spatial coordinates and derivatives, and it contains a nonlocal cubic nonlinear interaction term of special form. This equation belongs to the class of nearly linear equations and determines the principal term of semiclassical asymptotic solution.

The main result of our work is an approach developed for finding symmetry operators for a reduced GPE by solving linear operator equations. This approach is illustrated by an example of a one-dimensional reduced GPE for which symmetry operators can be found explicitly. Using symmetry operators obtained two families of exact solutions can be generated for the reduced GPE. In Section 2 the integro-differential Gross–Pitaevskii equation is considered and its semiclassical reduction is presented. A method for integrating the reduced GPE is described and the essential idea of the method is realized; namely, the consistent system and the linear equation associated with the reduced GPE are found. In Section 3 we propose an approach to finding the class of symmetry operators of the reduced GPE by constructing intertwining operators. The general ideas are illustrated in Section 4 by the example of a one-dimensional GPE of special type. The symmetry operators for this equation are found explicitly, and two families of exact solutions are generated making use of the operators obtained.

2 The nonlocal Gross–Pitaevskii equation and the Cauchy problem

We consider here the Gross–Pitaevskii equation with a nonlocal interaction term of general form. Using the concepts of the semiclassical WKB–Maslov method, we arrive at a reduced nonlocal GPE and briefly explain an algorithm for solving the Cauchy problem.

The Gross–Pitaevskii equation and its modifications are widely used in study of coherent matter waves in Bose–Einstein condensates (BECs) [10]. Recent extensions to BEC studies involve long-range effects in the condensates described by a generalized GPE containing integral terms responsible for nonlocal interactions. We refer to equations of this class as nonlocal GPE (which are also known as Hartree-type equations). The nonlocal BEC models may keep the condensate wave function from collapse and stabilize the solutions in higher dimensions (see, e.g., [20], the review [13] and references therein). Nonlocal GPEs also serve as basic equations of models describing many-particle quantum systems, nonlinear optics phenomena [1], collective soliton excitations in atomic chains [28], etc.

Let us write the nonlocal Gross–Pitaevskii equation as

$$\hat{F}(\Psi)(\vec{x},t) = \{-i\hbar\partial_t + \hat{H}(t) + \varkappa\hat{V}(\Psi)(t)\}\Psi(\vec{x},t) = 0, \tag{2.1}$$

$$\hat{V}(\Psi)(t) = V(\Psi)(\hat{z},t) = \int_{\mathbb{R}^n} \mathrm{d}\,\vec{y}\,\Psi^*(\vec{y},t)V(\hat{z},\hat{w},t)\Psi(\vec{y},t), \tag{2.2}$$

where $\partial_t = \partial/\partial t$, $\Psi(\vec{x},t)$ is a smooth complex scalar function that belongs to a complex Schwartz space $\mathbb{S}$ in the space variable $\vec{x}\in\mathbb{R}^n$ at each time t.

The linear operators $\hat{H}(t) = H(\hat{z},t)$ and $V(\hat{z},\hat{w},t)$ in (2.1) are Hermitian Weyl-ordered functions [16] of time t and of noncommuting operators

$$\hat{z} = (\hat{\vec{p}},\vec{x}) = (-i\hbar\partial/\partial\vec{x},\vec{x}), \qquad \hat{w} = (-i\hbar\partial/\partial\vec{y},\vec{y}), \qquad \vec{x},\vec{y}\in\mathbb{R}^n,$$

with the commutators

$$[\hat{z}_k,\hat{z}_j]_- = [\hat{w}_k,\hat{w}_j]_- = i\hbar J_{kj}, \qquad [\hat{z}_k,\hat{w}_j]_- = 0, \qquad k,j=\overline{1,2n},$$

where $[\hat{A},\hat{B}]_- = \hat{A}\hat{B}-\hat{B}\hat{A}$, $J=\|J_{kj}\|_{2n\times 2n}$ is the unit symplectic matrix: $J=\begin{pmatrix}0 & -\mathbb{I}\\ \mathbb{I} & 0\end{pmatrix}_{2n\times 2n}$, and $\mathbb{I}=\mathbb{I}_{n\times n}$ is the $n\times n$ identity matrix. We use the space $\mathbb{S}$ to provide existence of the moments of $\Psi(\vec{x},t)$ and convergence of the integral in (2.2). In what follows, we use the norm $\|\Psi\|$, $\Psi\in\mathbb{S}$, of the space $L_2(\mathbb{R}^n_x)$, i.e., $\|\Psi\| = \sqrt{(\Psi,\Psi)}$, where $(\Phi,\Psi) = \int_{\mathbb{R}^n}\mathrm{d}\,\vec{x}\,\Phi^*(\vec{x})\Psi(\vec{x})$ denotes the Hermitian inner product of the functions $\Phi,\Psi\in\mathbb{S}$, and Φ^* denotes the complex conjugate to Φ.

From equation (2.1) it follows immediately that the squared norm of a solution $\Psi(\vec{x},t)$ is conserved, $\|\Psi(t)\|^2 = \|\Psi(0)\|^2 = \mathrm{const}$.

A specific and attractive feature of the nonlocal GPE (2.1) is that in the semiclassical approximation the input GPE is reduced to an equation containing nonlocal terms which can be expressed as a finite number of moments of the unknown function $\Psi(\vec{x},t)$. The reduced equation can be considered as nearly linear. The concept of the nearly linear equations implies that among the solutions of a nonlinear equation there exists a subset of solutions that regularly depend on the nonlinearity parameter [16]. In the multidimensional case, the GPE (2.1) with variable coefficients of general form cannot be integrated by well-known methods, such as the inverse scattering transform [27]. Therefore, analytical solutions to this equation can be constructed only approximately. An effective approach to constructing asymptotic solutions in this case is to find semiclassical asymptotics as $\hbar\to 0$.

Note that semiclassical asymptotic expansions can be assigned to the following basic classes. The semiclassical asymptotic solutions of the equation under consideration are constructed in a chosen class of functions $K_\hbar$. The functions of the class $K_\hbar$ are determined by specific features of the problem and singularly depend on the small parameter $\hbar$. In the general case, such a class of functions is constructed as follows: In the phase space of a dynamic system of equations corresponding to the equation with partial derivatives under consideration (the classical equations of motion in the case of a linear quantum mechanics Schrödinger equation), a Lagrangian manifold Λ^k, $k \leq n$, is defined. Here k is the dimension of Λ^k and n is the dimension of the configuration space of the phase space. The manifold Λ^k evolves in time for the Cauchy problem and is invariant for the spectral problem, i.e. Λ^k is not deformed and does not move in space. On the manifold Λ^k a set of functions is defined. The Maslov's canonical operator projects a function defined in the phase space onto a function given in the configuration space. If $k = n$, then the canonical operator should be a real phase operator [25], whereas if $k < n$, then the canonical operator should be a complex phase one [4, 24]. In constructing projections of Λ^k onto the configuration space, caustics can appear.

The solutions of the first class ($k = n$) are given by the WKB ansatz with a real phase [25], where the leading term of the asymptotics outside the neighborhoods of the focal points can be written as

$$\Psi(\vec{x}, t, \hbar) = \sum_{j=1}^{M} f_j(\vec{x}, t) \exp\Big\{\frac{i}{\hbar} S_j(\vec{x}, t)\Big\} e^{i\pi\mu_j/2}, \tag{2.3}$$
$$\mu_j \in \mathbb{Z}, \qquad \operatorname{Im} S_j(\vec{x}, t) = 0, \qquad f_j, S_j \in \mathbb{C}^\infty(\mathbb{R}^n).$$

Semiclassical asymptotic solutions of the form (2.3) for the Gross–Pitaevskii equation were constructed in [15, 22, 23] (see also [17]).

The solutions of the second class ($k = 0$) are constructed using a complex WKB–Maslov ansatz [4, 24]. For the Gross–Pitaevskii equation, asymptotic solutions of this type are considered in [6, 9, 34].

Note that constructing of semiclassical asymptotic solutions for nonlinear equations engender a number of problems: In general, the evolution law for a manifold Λ^k is unknown. In other words, the "classical dynamics" related to the nonlinear equation under consideration depends on the initial conditions for the equation. Moreover, the relevant "classical dynamics equations" are unknown a priori for the nonlinear equation and to deduce them is a real problem. For the Gross–Pitaevskii equation (2.1), this problem was solved for the class of functions concentrated on a zero-dimensional manifold Λ^0 [6] and for the class of functions concentrated on an n-dimensional manifold Λ^n [15, 22, 23].

Following [6], we denote the second class of functions by $\mathcal{P}^t_\hbar(Z(t,\hbar), S(t,\hbar))$ and define it as

$$\mathcal{P}^t_\hbar = \mathcal{P}^t_\hbar\big(Z(t,\hbar), S(t,\hbar)\big) = \Big\{\Phi : \Phi(\vec{x}, t, \hbar) = \varphi\Big(\frac{\Delta\vec{x}}{\sqrt{\hbar}}, t, \hbar\Big) \exp\Big[\frac{i}{\hbar}(S(t,\hbar) + \langle\vec{P}(t,\hbar), \Delta\vec{x}\rangle)\Big]\Big\},$$

where the function $\varphi(\vec{\xi}, t, \hbar)$ belongs to the Schwarz space $\mathbb{S}$ in the variable $\vec{\xi} \in \mathbb{R}^n$, smoothly depends on t, and regularly depends on $\sqrt{\hbar}$ as $\hbar \to 0$. Here $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$, and the real function $S(t,\hbar)$ and the $2n$-dimensional vector function $Z(t,\hbar) = (\vec{P}(t,\hbar), \vec{X}(t,\hbar))$, which characterize the class $\mathcal{P}^t_\hbar(Z(t,\hbar), S(t,\hbar))$, regularly depend on $\sqrt{\hbar}$ in the neighborhood of $\hbar = 0$ and *are to be determined*. Note that $\mathcal{P}^t_\hbar(Z(t,\hbar), S(t,\hbar)) \subset \mathbb{S}$. If this does not lead to misunderstanding, we use the contracted notation $\mathcal{P}^t_\hbar$ for $\mathcal{P}^t_\hbar(Z(t,\hbar), S(t,\hbar))$.

Here to construct symmetry operators we use the complex WKB–Maslov asymptotic solutions of the Cauchy problem for the GPE (2.1)

$$\Psi(\vec{x}, t)\big|_{t=s} = \psi(\vec{x}), \qquad \psi(\vec{x}) \in \mathcal{P}^0_\hbar, \tag{2.4}$$

where

$$\mathcal{P}^0_\hbar = \mathcal{P}^0_\hbar\big(Z_0(\hbar), S_0(\hbar)\big) = \left\{\phi : \phi(\vec{x},\hbar) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}}, t, \hbar\right) \exp\Big[\frac{i}{\hbar}(S_0(\hbar) + \langle\vec{P}_0(\hbar), \Delta\vec{x}_0\rangle)\Big]\right\},$$
$$Z_0(\hbar) = \big(\vec{P}_0(\hbar), \vec{X}_0(\hbar)\big), \qquad \Delta\vec{x}_0 = \vec{x} - \vec{X}_0(\hbar).$$

The definition of the class of trajectory-concentrated functions contains the phase trajectory $Z(t,\hbar)$ and the scalar function $S(t,\hbar)$ as "free parameters". The functions belonging to the class $\mathcal{P}^t_\hbar$, at any fixed time $t \in \mathbb{R}^1$ are *concentrated*, as $\hbar \to 0$ in the neighborhood of a point lying on the phase curve $z = Z(t,0)$, $t \in \mathbb{R}^1$ [3]. Therefore, it is natural to call the functions of the class $\mathcal{P}^t_\hbar$ *trajectory-concentrated functions.*

The WKB solutions of the form (2.3) are concentrated on a family of phase trajectories whose projections on the configuration space may intersect, giving rise to a caustic problem [25]. On the other hand, all semiclassical asymptotics of the class $\mathcal{P}^t_\hbar$ are concentrated on the same trajectory. So we do not face problems with caustics and collapse problem in constructing trajectory-concentrated solutions of the GPE.

Let $\widehat{O}(\hbar^\nu)$ be an operator $\hat{F}$ such that for any function Φ belonging to the space $\mathcal{P}^t_\hbar$ the following asymptotic estimate is valid:

$$\frac{\|\hat{F}\Phi\|}{\|\Phi\|} = O(\hbar^\nu), \qquad \hbar \to 0, \qquad \Delta\hat{z} = \hat{z} - Z(t,\hbar)$$

It may be shown (see [3, 6]) that for the functions belonging to $\mathcal{P}^t_\hbar$, the following asymptotic estimate is valid:

$$\Delta\hat{z} = \widehat{O}\big(\hbar^{1/2}\big), \qquad \hbar \to 0. \tag{2.5}$$

Let us expand the operators $\hat{H}(t) = H(\hat{z},t)$ and $\hat{V}(t) = V(\hat{z},\hat{w},t)$ in (2.1) as Taylor series in the operators $\Delta\hat{z} = \hat{z} - Z(t,\hbar)$ and $\Delta\hat{w} = \hat{w} - Z(t,\hbar)$, respectively, and restrict ourselves to quadratic terms. Then, in view of (2.5), the solution of the Cauchy problem (2.1) and (2.4) asymptotic in a formal small parameter $\hbar$ ($\hbar \to 0$) can be constructed[1] accurate to $O(\hbar^{3/2})$ (see [6]). The leading-order term of the asymptotics can be found by reducing the GPE (2.1) to a GPE with a quadratic nonlocal operator.

The higher-order corrections to the leading-order term can be found using perturbation theory [6]. Thus the study of GPEs with a quadratic nonlocal operator is crucial for the construction of semiclassical asymptotics for this type of GPE in the class of trajectory concentrated functions. Without loss of generality, we consider a GPE of the form

$$\left\{-i\hbar\partial_t + \hat{H}_{\rm qu}(\hat{z},t) + \varkappa\int_{\mathbb{R}^n} d\,\vec{y}\Psi^*(\vec{y},t)V_{\rm qu}(\hat{z},\hat{w},t)\Psi(\vec{y},t)\right\}\Psi(\vec{x},t) = 0, \tag{2.6}$$

where the linear operators $H_{\rm qu}(\hat{z},t)$ and $V_{\rm qu}(\hat{z},\hat{w},t)$ are Hermitian and quadratic in $\hat{z}$, $\hat{w}$, respectively:

$$H_{\rm qu}(\hat{z},t) = \frac{1}{2}\langle\hat{z}, \mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t), \hat{z}\rangle, \tag{2.7}$$

$$V_{\rm qu}(\hat{z},\hat{w},t) = \frac{1}{2}\langle\hat{z}, W_{zz}(t)\hat{z}\rangle + \langle\hat{z}, W_{zw}(t)\hat{w}\rangle + \frac{1}{2}\langle\hat{w}, W_{ww}(t)\hat{w}\rangle. \tag{2.8}$$

[1]Note that in the semiclassical trajectory-coherent approximation, if $\hbar$ is small enough ($\hbar \to 0$), all results are established for a finite time interval $[0,T]$. Evidently, this version of the semiclassical approach is not uniform in time as $T \to \infty$ (see [3, 6]). Therefore, the problem of long-time validity of the semiclassical trajectory coherent approximation (i.e., the two limits, as $\hbar \to 0$ and then as $T \to \infty$) should be the subject of special study.

Here $\mathcal{H}_{zz}(t)$, $W_{zz}(t)$, $W_{zw}(t)$, and $W_{ww}(t)$ are $2n \times 2n$ matrices; $\mathcal{H}_z(t)$ is a $2n$ vector; the angle brackets $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product of vectors:

$$\langle\vec{p},\vec{x}\rangle = \sum_{j=1}^{n} p_j x_j, \quad \vec{p},\vec{x} \in \mathbb{R}^n; \qquad \langle z, w\rangle = \sum_{j=1}^{2n} z_j w_j, \quad z, w \in \mathbb{R}^{2n}.$$

We call equation (2.6) with the linear operators $H_{\rm qu}$ and $V_{\rm qu}$ given by (2.7) and (2.8), respectively, *a reduced Gross–Pitaevskii equation* (RGPE).

An RGPE can be integrated explicitly [19, 33] and it possesses very rich symmetries. Analysis of these symmetries can provide a wealth of information about the equation and its solutions.

As an RGPE contains a nonlocal nonlinear term, its symmetry properties are of special interest in the symmetry analysis of partial differential equations. The matter is that the application of the standard methods of symmetry analysis [2, 14, 29, 30], developed basically for PDEs, leads to a number of difficulties when applied to equations different from PDEs: For instance, there are no regular rules for choosing an appropriate structure of symmetries for non-differential equations. This problem can be avoided by using an RGPE as its symmetry properties are closely related to the symmetry of the linear equation associated with the input nonlinear equation.

The key factor in symmetry analysis of the nonlinear equation $\hat{F}(\Psi)(\vec{x},t) = 0$ is the symmetry operator $\hat{A}$ that makes the set of solutions of the equation invariant (see, e.g., [21]):

$$\hat{F}(\Psi)(\vec{x},t) = 0 \quad \Rightarrow \quad \hat{F}(\hat{A}\Psi)(\vec{x},t) = 0. \tag{2.9}$$

Generally, it is impossible to find effectively a symmetry operator $\hat{A}$ for a given nonlinear operator $\hat{F}$ by solving the nonlinear operator equation (2.9). This situation is resolved in the group analysis of differential equations [2, 29, 30] where a symmetry $\hat{\sigma}$ (generator of a Lie group of symmetry operators) is the main object of analysis.

The symmetries are determined by the linear operator equation

$$\hat{F}(\Psi)(\vec{x},t) = 0 \quad \Rightarrow \quad \hat{F}'(\hat{\sigma}\Psi)(\vec{x},t) = 0.$$

Here $\hat{F}'(\Psi)$ is the Freshet derivative of $\hat{F}$ calculated for Ψ. For a linear operator $\hat{F}$, we have $\hat{F}' = \hat{F}$ and the symmetry operators being the same as the symmetries.

We assign the RGPE (2.6) to the class of nearly linear equations, following the definition given in [18]: A nearly linear equation determining a function Ψ has the form of a linear partial differential equation with coefficients depending on the moments of the function Ψ. This type of equation can be associated with a consistent system which includes a system of ordinary differential equations (ODEs) describing the evolution of the moments and RGPE.

Using the RGPE as an example, we can see that the class of symmetry operators for nearly linear equations can be found by solving the corresponding determining *linear* operator equations. In this sense, the symmetry properties of nearly linear equations are similar in many respects to those of linear equations.

Let us consider briefly a method for solving the Cauchy problem (2.4) for the RGPE (2.6), following the scheme described in [6]. We denote the Weyl-ordered symbol of an operator $\hat{A}(t) = A(\hat{z},t)$ by $A(z,t)$ and define the expectation value for $\hat{A}(t)$ over the state $\Psi(\vec{x},t)$ as

$$A_\Psi(t) = \frac{1}{\|\Psi\|^2}(\Psi, \hat{A}(t)\Psi) = \frac{1}{\|\Psi\|^2}\int_{\mathbb{R}^n} d\,\vec{x}\,\Psi^*(\vec{x},t)\hat{A}(t)\Psi(\vec{x},t).$$

As $\|\Psi\|^2$ does not depend on time, we have from (2.6), (2.7), and (2.8)

$$\dot{A}_\Psi(t) = \frac{1}{\|\Psi\|^2}\int_{\mathbb{R}^n} d\,\vec{x}\,\Psi^*(\vec{x},t)\bigg\{\frac{\partial\hat{A}(t)}{\partial t} + \frac{i}{\hbar}[H_{\rm qu}(\hat{z},t), \hat{A}(t)]_-$$

$$+\frac{i\tilde{\varkappa}}{\hbar}\int_{\mathbb{R}^n} \mathrm{d}\,\vec{y}\Psi^*(\vec{y},t)[V_{\mathrm{qu}}(\hat{z},\hat{w},t),\hat{A}(t)]_-\Psi(\vec{x},t)\Bigg\}, \tag{2.10}$$

where $\dot{A}_\Psi(t) = \mathrm{d}\,A_\Psi(t)/\,\mathrm{d}\,t$ and $\tilde{\varkappa} = \varkappa\|\Psi\|^2 = \varkappa\|\psi\|^2$.

We call (2.10) *the Ehrenfest equation* for the RGPE (2.6) as is common practice in quantum mechanics for the linear Schrödinger equation ($\varkappa = 0$ in (2.1)).

Let $z_\Psi(t) = (z_{\Psi l}(t))$ and $\Delta^{(2)}_\Psi(t) = \big(\Delta^{(2)}_{\Psi kl}(t)\big)$ denote the expectation values over $\Psi(\vec{x},t)$ for the operators

$$\hat{z}_l, \qquad \hat{\Delta}^{(2)}_{kl} = \frac{1}{2}\big(\Delta\hat{z}_k\Delta\hat{z}_l + \Delta\hat{z}_l\Delta\hat{z}_k\big), \qquad k,l = \overline{1,2n},$$

respectively. Here $\Delta\hat{z}_l = \hat{z}_l - (z_\Psi)_l(t)$. We call $z_\Psi(t)$ the first moments and $\Delta^{(2)}_\Psi(t)$ the second centered moments of $\Psi(\vec{x},t)$.

From (2.6), (2.7), (2.8), and (2.10) we immediately obtain a dynamical system in matrix notation:

$$\begin{aligned}
&\dot{z}_\Psi = J\big\{\mathcal{H}_z(t) + [\mathcal{H}_{zz}(t) + \tilde{\varkappa}(W_{zz}(t) + W_{zw}(t))]z_\Psi\big\},\\
&\dot{\Delta}^{(2)}_\Psi = J[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]\Delta^{(2)}_\Psi - \Delta^{(2)}_\Psi[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]J.
\end{aligned} \tag{2.11}$$

We call (2.11) *the Hamilton–Ehrenfest system* (HES) of the second order for the RGPE (2.6) as (2.11) contain the first and second moments.

For brevity, we use a shorthand notation for the total set of the first and second moments of $\Psi(\vec{x},t)$:

$$\mathfrak{g}_\Psi(t) = \big(z_\Psi(t), \Delta^{(2)}_\Psi(t)\big). \tag{2.12}$$

The functions $\mathfrak{g} = \mathfrak{g}_\Psi(t)$ describe phase orbits in the phase space of system (2.11).

Then the Cauchy problem (2.4) for the RGPE (2.6) can be written equivalently as

$$\hat{L}(t,\mathfrak{g}_\Psi(t))\Psi(\vec{x},t) = \big\{-i\hbar\partial_t + \hat{H}_q(t,\mathfrak{g}_\Psi(t))\big\}\Psi(\vec{x},t) = 0, \tag{2.13}$$

$$\begin{aligned}
\hat{H}_q(t,\mathfrak{g}_\Psi(t)) ={}& \frac{1}{2}\langle\hat{z},\mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t),\hat{z}\rangle + \frac{\tilde{\varkappa}}{2}\langle\hat{z},W_{zz}(t)\hat{z}\rangle\\
&+ \frac{\tilde{\varkappa}}{2}\langle z_\Psi(t),W_{ww}(t)z_\Psi(t)\rangle + \tilde{\varkappa}\langle\hat{z},W_{zw}(t)z_\Psi(t)\rangle + \frac{\tilde{\varkappa}}{2}\,\mathrm{Sp}\left[W_{ww}(t)\Delta^{(2)}_\Psi(t)\right],
\end{aligned} \tag{2.14}$$

$$\dot{\mathfrak{g}}_\Psi(t) = \Gamma(t,\mathfrak{g}_\Psi(t)), \tag{2.15}$$

$$\Psi(\vec{x},t)\Big|_{t=s} = \psi(\vec{x}), \qquad \mathfrak{g}_\Psi(t)\Big|_{t=s} = \mathfrak{g}_\psi. \tag{2.16}$$

Equation (2.15) is a concise form of the HES (2.11), and $\Gamma(t,\mathfrak{g}_\Psi(t))$ designates the r.h.s. of (2.11).

We call the reduced GPE (2.13) and the corresponding HES (2.15) *the consistent system* for the RGPE (2.6). The reduced GPE (2.13) can be assigned to the class of nearly linear equations [18], as the operator (2.14) of the RGPE (2.13) is a linear partial differential operator with coefficients depending only on the first and second moments $\mathfrak{g}_\Psi(t)$.

The consistent system (2.13), (2.15) allows us to reduce the Cauchy problem for the RGPE (2.13) to the Cauchy problem for a linear PDE, therefore the Cauchy problem (2.16) for HES (2.15) can be solved independently of equation (2.13).

Let

$$\mathfrak{g}(t,\mathbf{C}) = \big(z(t,\mathbf{C}), \Delta^{(2)}(t,\mathbf{C})\big)$$

be the general solution of the HES (2.15) and $\mathbf{C} = (C_1, C_2, \dots, C_N)$ denote the set of integration constants.

Consider a linear PDE with coefficients depending on the parameters $\mathbf{C}$:

$$\hat{L}(t, \mathbf{C})\Phi(\vec{x}, t, \mathbf{C}) = \big\{-i\hbar\partial_t + \hat{H}_q(t, \mathbf{C})\big\}\Phi(\vec{x}, t, \mathbf{C}) = 0, \tag{2.17}$$

where

$$\begin{aligned}\hat{H}_q(t, \mathbf{C}) &= \frac{1}{2}\langle\hat{z}, \mathcal{H}_{zz}(t)\hat{z}\rangle + \langle\mathcal{H}_z(t), \hat{z}\rangle + \frac{\tilde{\varkappa}}{2}\langle\hat{z}, W_{zz}(t)\hat{z}\rangle + \tilde{\varkappa}\langle\hat{z}, W_{zw}(t)Z(t, \mathbf{C})\rangle \\ &\quad + \frac{\tilde{\varkappa}}{2}\langle Z(t, \mathbf{C}), W_{ww}(t)Z(t, \mathbf{C})\rangle + \frac{\tilde{\varkappa}}{2}\,\mathrm{Sp}\left[W_{ww}(t)\Delta^{(2)}(t, \mathbf{C})\right].\end{aligned} \tag{2.18}$$

The operator $\hat{H}_q(t, \mathbf{C})$ of (2.17) is obtained from (2.14) where the general solution $\mathfrak{g}(t, \mathbf{C})$ of the HES (2.15) stands for the moments $\mathfrak{g}_\Psi(t)$. We call (2.17) *the associated linear equation* (ALE) for the RGPE (2.13).

Let $\Phi(\vec{x}, t, \mathbf{C}[\psi])$ denote the solution of the Cauchy problem for the ALE (2.17) with the initial condition

$$\Phi(\vec{x}, t, \mathbf{C}[\psi])\Big|_{t=s} = \psi(\vec{x}), \tag{2.19}$$

where the integration constants $\mathbf{C}$ have been replaced by the functionals $\mathbf{C} = \mathbf{C}[\psi]$ determined from the algebraic conditions

$$\mathfrak{g}(t, \mathbf{C})\Big|_{t=s} = \mathfrak{g}_\psi. \tag{2.20}$$

Then the solution of the Cauchy problem (2.13), (2.14) for the RGPE (see [6, 33] for details) is

$$\Psi(\vec{x}, t) = \Phi(\vec{x}, t, \mathbf{C}[\psi]). \tag{2.21}$$

Define $\mathbf{C}[\Psi](t)$ by the algebraic condition

$$\mathfrak{g}(t, \mathbf{C}[\Psi](t)) = \mathfrak{g}_\Psi(t). \tag{2.22}$$

From the uniqueness of the solution of the Cauchy problem for the HES (2.16) it follows that

$$\mathfrak{g}(t, \mathbf{C}[\Psi](t)) = \mathfrak{g}(t, \mathbf{C}[\psi])$$

and, hence,

$$\mathbf{C}[\Psi](t) = \mathbf{C}[\psi], \tag{2.23}$$

i.e., the functionals $\mathbf{C}[\Psi](t)$ are the integrals of (2.1).

Also, we have

$$\mathfrak{g}(t, \mathbf{C}[\psi]) = \mathfrak{g}_\psi(t), \tag{2.24}$$

where $\mathfrak{g}_\psi(t)$ is the solution of the HES (2.15) with the initial condition (2.20).

The 1D case of equation (2.23) is considered in more detail in [19]. Solving the associated linear equation (2.17) with the algebraic condition (2.20) we obtain a solution to the nonlinear equation (2.6).

Let us now turn to the construction of symmetry operators for the RGPE (2.6). By using an operator intertwining a pair of ALEs of the form (2.17). Analysis of the GPE of general form involves a great number of additional technical issues associated with the semiclassical approximation that requires a separate study. To illustrate the main ideas of the proposed approach, we restrict our discussion to the case of a quadratic operator for which equation (2.17) is integrable.

3 The intertwining operator and symmetry operators

In this section, we establish a relationship between the symmetry operators and the intertwining operator for the reduced Gross–Pitavevskii equation (2.6). A class of intertwining operators can be found as a set of products of the fundamental intertwining operator by the symmetry operators of the ALE (2.17).

According to definition (2.9), the nonlinear symmetry operator $\hat{A}(t)$ maps any solution $\Psi(\vec{x},t)$ of equation (2.13) into its another solution:

$$\Psi_A(\vec{x},t) = (\hat{A}(t)\Psi)(\vec{x},t).$$

For $\hat{a} = \hat{A}(t)\big|_{t=s}$ and $\psi(\vec{x})$ given by (2.4), we can set

$$\psi_a(\vec{x}) = \hat{a}\psi(\vec{x}) = \Psi_A(t)\big|_{t=s}$$

and use the notation $\mathfrak{g}_{\psi_a}$ for the first and second moments of $\psi_a(\vec{x})$ similar to (2.12).

From the solution of the Cauchy problem for the HES (2.16) with the initial condition $\mathfrak{g}_\Psi(t)\Big|_{t=s} = \mathfrak{g}_{\psi_a}$, analogously to (2.24), we have

$$\mathfrak{g}(t,\mathbf{C}[\psi_a]) = \mathfrak{g}_{\psi_a}(t). \tag{3.1}$$

According to (2.21), the solutions $\Psi(\vec{x},t)$ and $\Psi_A(\vec{x},t)$ of the RGPE (2.13) are found as

$$\Psi(\vec{x},t) = \Phi(\vec{x},t,\mathbf{C})\big|_{\mathbf{C}=\mathbf{C}[\psi]} \tag{3.2}$$

and

$$\Psi_A(\vec{x},t) = \Phi(\vec{x},t,\mathbf{C}')\big|_{\mathbf{C}'=\mathbf{C}'[\psi_a]},$$

where $\Phi(\vec{x},t,\mathbf{C})$ and $\Phi(\vec{x},t,\mathbf{C}')$ are the solutions of two ALEs of the form (2.17) with two different sets of integration constants $\mathbf{C}$ and $\mathbf{C}'$, respectively, and the corresponding linear operators $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$.

To construct the symmetry operator $\hat{A}(t)$ we relate the functions $\Phi(\vec{x},t,\mathbf{C}'[\psi_a])$ and $\Phi(\vec{x},t,\mathbf{C}[\psi])$ by a linear operator $\hat{M}(t,s,\mathbf{C}',\mathbf{C})$ intertwining the operators $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$:

$$\hat{L}(t,\mathbf{C}')\hat{M}(t,s,\mathbf{C}',\mathbf{C}) = \hat{R}(t,s,\mathbf{C}',\mathbf{C})\hat{L}(t,\mathbf{C}). \tag{3.3}$$

Here the linear operator $\hat{R}(t,s,\mathbf{C}',\mathbf{C})$ is a Lagrangian multiplier, and the initial condition is $\hat{M}(t,s,\mathbf{C}',\mathbf{C})|_{t=s} = \hat{a}$.

From (3.3) we have that $\Phi(\vec{x},t,\mathbf{C}') = \hat{M}(t,s,\mathbf{C}',\mathbf{C})\Phi(\vec{x},t,\mathbf{C})$ for two arbitrary sets of constants $\mathbf{C}'$ and $\mathbf{C}$, and this is especially true for $\Phi(\vec{x},t,\mathbf{C}'[\psi_a])$ and $\Phi(\vec{x},t,\mathbf{C}[\psi])$ with the constants $\mathbf{C}'[\psi_a]$ and $\mathbf{C}[\psi]$.

To find the operator $\hat{M}(t,s,\mathbf{C}',\mathbf{C})$, we consider a linear intertwining operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ for $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$ satisfying the conditions

$$\hat{L}(t,\mathbf{C}')\hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C}) = \hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\hat{L}(t,\mathbf{C}), \tag{3.4}$$

$$\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\Big|_{t=s} = \hat{\mathbb{I}}. \tag{3.5}$$

We call $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ *the fundamental intertwining operator* for $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$. Making use of $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$, the operator $\hat{M}(t,s,\mathbf{C}',\mathbf{C})$ involved into (3.3) can be presented as

$$\hat{M}(t,s,\mathbf{C}',\mathbf{C}) = \widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\widehat{B}(t,\mathbf{C}).$$

Here $\widehat{B}(t,\mathbf{C})\in\mathcal{B}$ is the linear symmetry operator of ALE (2.17) satisfying the conditions

$$[\hat{L}(t,\mathbf{C}),\widehat{B}(t,\mathbf{C})]_-=0,\qquad \widehat{B}(t,\mathbf{C})|_{t=s}=\hat{a}, \tag{3.6}$$

and $\mathcal{B}$ is the family of linear symmetry operators of the ALE (3.6).

Hence, given the operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ of (3.4) and the family $\mathcal{B}$ of linear symmetry operators of the ALE (3.6) we can construct the family of nonlinear symmetry operators for the GPE (2.1).

Thus, we arrive at

Theorem 1. *Let $\hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ be the fundamental intertwining operator (3.4) for $\hat{L}(t,\mathbf{C}')$ and $\hat{L}(t,\mathbf{C})$ and let $\widehat{B}(t,\mathbf{C})$ be the linear symmetry operator of the ALE (2.17) satisfying the conditions (3.6). Then*

$$(\hat{A}(t)\Psi)(\vec{x},t)=\widehat{\mathcal{D}}\big(t,s,\mathbf{C}'[\hat{a}\psi],\mathbf{C}[\Psi](t)\big)\widehat{B}(t,\mathbf{C}[\Psi](t))\Psi(\vec{x},t) \tag{3.7}$$

defines the family of nonlinear symmetry operators for the GPE (2.6). Here $\mathbf{C}'[\hat{a}\psi]$ and $\mathbf{C}[\Psi]$ $(=\mathbf{C}[\psi])$ can be found from (2.24) and (2.22), respectively, and $\widehat{B}\in\mathcal{B}$.

Note that the symmetry operator $\hat{A}(t)$ from (3.7) is nonlinear, as the operators $\hat{\mathcal{D}}$ and $\hat{B}$ depend on the parameters $\mathbf{C}$ being functionals of the function Ψ. To find the fundamental intertwining operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$, we introduce a function $\phi(\vec{x},t,\mathbf{C})$ by the conditions

$$\begin{aligned}&\Phi(\vec{x},t,\mathbf{C})=\hat{K}(\vec{x},t,s,\mathbf{C})\phi(\vec{x},t,\mathbf{C}),\\&\hat{K}(\vec{x},t,s,\mathbf{C})=\exp[-\langle\vec{X}(t,\mathbf{C}),\nabla\rangle]\exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C})+\langle\vec{P}(t,\mathbf{C}),\vec{x}\rangle]\Big\},\end{aligned} \tag{3.8}$$

where $\Phi(\vec{x},t,\mathbf{C})$ is a solution of equation (2.17), the vector $z=Z(t,\mathbf{C})=(\vec{P}(t,\mathbf{C}),\vec{X}(t,\mathbf{C}))$ satisfies equation (2.15), and $S(t,\mathbf{C})$ is a smooth function to be determined.

For $\phi(\vec{x},t,\mathbf{C})$ we have from (2.17)

$$\begin{aligned}&\hat{L}_0(\vec{x},t,\mathbf{C})\phi(\vec{x},t,\mathbf{C})=0,\\&\hat{L}_0(\vec{x},t,\mathbf{C})=\hat{K}^{-1}(\vec{x},t,s,\mathbf{C})\hat{L}(\vec{x},t,\mathbf{C})\hat{K}(\vec{x},t,s,\mathbf{C})\\&\qquad=-i\hbar\partial_t+\langle\dot{\vec{X}}(t,\mathbf{C}),i\hbar\nabla\rangle+\dot{S}(t,\mathbf{C})+\langle\dot{\vec{P}}(t,\mathbf{C}),\vec{x}\rangle-\langle\vec{P}(t,\mathbf{C}),\dot{\vec{X}}(t,\mathbf{C})\rangle\\&\qquad\quad+\frac{1}{2}\langle(\hat{z}+Z(t,\mathbf{C})),\mathcal{H}_{zz}(t)(\hat{z}+Z(t,\mathbf{C}))\rangle+\langle\mathcal{H}_z(t),(\hat{z}+Z(t,\mathbf{C}))\rangle\\&\qquad\quad+\tilde{\varkappa}\Big[\frac{1}{2}\langle(\hat{z}+Z(t,\mathbf{C})),W_{zz}(t)(\hat{z}+Z(t,\mathbf{C}))\rangle+\langle(\hat{z}+Z(t,\mathbf{C})),W_{zw}(t)Z(t,\mathbf{C})\rangle\\&\qquad\quad+\frac{1}{2}\langle Z(t,\mathbf{C}),W_{ww}(t)Z(t,\mathbf{C})\rangle+\frac{1}{2}\operatorname{Sp}\big[W_{ww}(t)\Delta^{(2)}(t,\mathbf{C})\big]\Big].\end{aligned}$$

Putting

$$S(t,\mathbf{C})=\int_s^t\Big\{\langle\vec{P}(t,\mathbf{C}),\dot{\vec{X}}(t,\mathbf{C})\rangle-H_\varkappa(t,\mathbf{C})\Big\}\,\mathrm{d}\,t, \tag{3.9}$$

where

$$\begin{aligned}H_\varkappa(t,\mathbf{C})=&\frac{1}{2}\langle Z(t,\mathbf{C}),[\mathcal{H}_{zz}(t)+\tilde{\varkappa}(W_{zz}(t)+2W_{zw}(t)+W_{ww}(t))]Z(t,\mathbf{C})\rangle\\&+\langle\mathcal{H}_z(t),Z(t,\mathbf{C})\rangle+\frac{1}{2}\tilde{\varkappa}\mathrm{Sp}[W_{ww}(t)\Delta^{(2)}(t,\mathbf{C})],\end{aligned}$$

and taking into account (2.15), we obtain an equation for the function $\phi(\vec{x},t,\mathbf{C})$:

$$\hat{L}_0(\vec{x},t)\phi(\vec{x},t,\mathbf{C})=0,\qquad \hat{L}_0(\vec{x},t)=-i\hbar\partial_t+\frac{1}{2}\langle\hat{z},\mathcal{H}_{zz}(t)\hat{z}\rangle+\tilde{\varkappa}\frac{1}{2}\langle\hat{z},W_{zz}(t)\hat{z}\rangle. \tag{3.10}$$

Theorem 2. *Let*

$$\hat{b}(t,s,\mathbf{C}',\mathbf{C}) = \langle b(t,s,\mathbf{C}',\mathbf{C}), J\hat{z}\rangle \tag{3.11}$$

and let the $2n$-component vector $b = b(t,s,\mathbf{C}',\mathbf{C})$ be a solution of the Cauchy problem for the system

$$\begin{aligned}&\dot{b} = J[\mathcal{H}_{zz}(t) + \tilde{\varkappa}W_{zz}(t)]b, \qquad b\big|_{t=s} = \delta Z_0(\mathbf{C},\mathbf{C}'),\\ &\delta Z_0(\mathbf{C},\mathbf{C}') = \big(\delta\vec{P}_0(\mathbf{C},\mathbf{C}'), \delta\vec{X}_0(\mathbf{C},\mathbf{C}')\big),\\ &\delta\vec{X}_0(\mathbf{C},\mathbf{C}') = \vec{X}_0(\mathbf{C}') - \vec{X}_0(\mathbf{C}), \qquad \delta\vec{P}_0(\mathbf{C},\mathbf{C}') = \vec{P}_0(\mathbf{C}') - \vec{P}_0(\mathbf{C}).\end{aligned}$$

Then the operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ involved into (3.4) can be presented as

$$\begin{aligned}\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C}) &= \exp\Big\{\frac{i}{2\hbar}\langle\delta\vec{X}_0(\mathbf{C},\mathbf{C}'), \delta\vec{P}_0(\mathbf{C},\mathbf{C}')\rangle\Big\}\\ &\quad\times \hat{K}(\vec{x},t,s,\mathbf{C}')\exp\Big\{\frac{i}{\hbar}\hat{b}(t,s,\mathbf{C}',\mathbf{C})\Big\}\hat{K}^{-1}(\vec{x},t,s,\mathbf{C}).\end{aligned} \tag{3.12}$$

The operator $\hat{K}(\vec{x},t,s,\mathbf{C})$ is defined in (3.8).

Proof. In view of (3.8) and (3.10), equation (3.4) for the fundamental intertwining operator can be written as

$$\begin{aligned}&\hat{K}(\vec{x},t,s,\mathbf{C}')\hat{L}_0(\vec{x},t)\hat{K}^{-1}(\vec{x},t,s,\mathbf{C}')\hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\\ &\quad = \hat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\hat{K}(\vec{x},t,s,\mathbf{C})\hat{L}_0(\vec{x},t)\hat{K}^{-1}(\vec{x},t,s,\mathbf{C}), \qquad \widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})\big|_{t=s} = \hat{\mathbb{I}}.\end{aligned}$$

Therefore, the operator $\hat{L}_0(\vec{x},t)$ given by (3.10) and the function $\phi(\vec{x},t,\mathbf{C})$ do not depend on the constants $\mathbf{C}$. Hence, we have

$$\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C}) = \hat{K}(\vec{x},t,s,\mathbf{C}')\hat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})\hat{K}^{-1}(\vec{x},t,s,\mathbf{C}),$$

where $\hat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})$ is the symmetry operator of equation (3.10), i.e.

$$\Big[\hat{L}_0(\vec{x},t), \hat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})\Big]_- = 0, \qquad \hat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})\Big|_{t=s} = \hat{\widehat{\mathcal{D}}}_0(\mathbf{C}',\mathbf{C}). \tag{3.13}$$

Here we used the notation

$$\begin{aligned}\hat{\widehat{\mathcal{D}}}_0(\mathbf{C}',\mathbf{C}) &= \hat{K}^{-1}(\vec{x},t,s,\mathbf{C}')\hat{K}(\vec{x},t,s,\mathbf{C})\Big|_{t=s}\\ &= \exp\Big[\langle\delta\vec{X}_0(\mathbf{C},\mathbf{C}'),\nabla\rangle - \frac{i}{\hbar}\langle\delta\vec{P}_0(\mathbf{C},\mathbf{C}'),\vec{x}\rangle\Big]\exp\Big\{\frac{i}{2\hbar}\langle\delta\vec{X}_0(\mathbf{C},\mathbf{C}'),\delta\vec{P}_0(\mathbf{C},\mathbf{C}')\rangle\Big\}.\end{aligned}$$

The solution of the Cauchy problem (3.13) for the operator $\hat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C})$ can be obtained with the standard methods (see, e.g., [3, 21]) as

$$\hat{\widehat{\mathcal{D}}}(t,s,\mathbf{C}',\mathbf{C}) = \exp\Big\{\frac{i}{2\hbar}\langle\delta\vec{X}_0(\mathbf{C},\mathbf{C}'),\delta\vec{P}_0(\mathbf{C}',\mathbf{C})\rangle\Big\}\exp\Big\{\frac{i}{\hbar}\hat{b}(t,s,\mathbf{C}',\mathbf{C})\Big\}, \qquad \blacksquare$$

Then the symmetry operator $\hat{A}(t)$ for equation (2.13) (or, equivalently, for equation (2.6)) can be presented as (3.7), where the intertwining operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ is defined by (3.12) and $\widehat{B}(t,\mathbf{C})$ is the symmetry operator for the ALE (2.17).

Using the explicit form (3.12) of the intertwining operator $\widehat{\mathcal{D}}(t,s,\mathbf{C}',\mathbf{C})$ and the operator $\hat{K}(\vec{x},t,s,\mathbf{C})$ from (3.8), we have

$$\Psi_A(\vec{x},t) = (\hat{A}(t)\Psi)(\vec{x},t)$$
$$= \exp\Big\{\frac{i}{\hbar}[S_A(t) + \langle\vec{P}_A(t),\vec{x}-\vec{X}_A(t)\rangle]\Big\}\widehat{B}\big(\vec{x}+\vec{X}(t)-\vec{X}_A(t),t\big)$$
$$\times \exp\Big\{-\frac{i}{\hbar}[S(t) + \langle\vec{P}(t),\vec{x}-\vec{X}(t)\rangle]\Big\}\Psi\big(\vec{x}+\vec{X}(t)-\vec{X}_A(t),t\big), \qquad (3.14)$$

where

$$\widehat{B}(\vec{x},t) = \widehat{B}\big(t,\mathbf{C}[\Psi](t)\big).$$

Note that expression (3.14) for the symmetry operators is not simple and requires further analysis, but other forms of symmetry operators for GPEs are unknown.

To obtain simplier examples of symmetry operators in explicit form, we consider the 1D case of equations (2.6), (2.7), and (2.8).

4 Symmetry operators in the 1D case

Based on the results of the previous section, here we construct in explicit form the symmetry operators for the RGPE (2.6) in the one-dimensional case and obtain two countable sets of exact solutions to the one-dimensional GPE using the symmetry operators.

Consider the reduced 1D GPE (2.6)

$$\hat{F}(\Psi)(\vec{x},t) = \big\{-i\hbar\partial_t + \widehat{H}_{\mathrm{qu}} + \varkappa\widehat{V}_{\mathrm{qu}}(\Psi)(t)\big\}\Psi(x,t) = 0, \qquad (4.1)$$
$$\Psi\big|_{t=0} = \psi(x), \qquad (4.2)$$

where we used the notations

$$\widehat{H}_{\mathrm{qu}} = \frac{1}{2}\left(\mu\hat{p}^2 + \rho(x\hat{p}+\hat{p}x) + \sigma x^2\right), \qquad \widehat{V}_{\mathrm{qu}}(\Psi) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\,y\left(ax^2 + 2bxy + cy^2\right)|\Psi(y)|^2,$$

$\hat{p} = -i\hbar\partial/\partial x$; a, b, and c are the real parameters of the nonlocal operator $\widehat{V}_{\mathrm{qu}}(\Psi)$; μ, σ, and ρ are the parameters of the linear operator $\widehat{H}_{\mathrm{qu}}$; $x,y\in\mathbb{R}^1$.

The Hamilton–Ehrenfest system (2.15) for the first-order moments becomes [5]

$$\dot{p} = -\rho p - \sigma_0 x,$$
$$\dot{x} = \mu p + \rho x, \qquad (4.3)$$

and for the second-order moments with $\Delta_{21}^{(2)} = \Delta_{12}^{(2)}$ we have

$$\dot{\Delta}_{11}^{(2)} = -2\rho\Delta_{11}^{(2)} - 2\tilde{\sigma}\Delta_{21}^{(2)},$$
$$\dot{\Delta}_{21}^{(2)} = \mu\Delta_{11}^{(2)} - \tilde{\sigma}\Delta_{22}^{(2)},$$
$$\dot{\Delta}_{22}^{(2)} = 2\mu\Delta_{21}^{(2)} + 2\rho\Delta_{22}^{(2)}, \qquad (4.4)$$

where

$$\sigma_0 = \sigma + \tilde{\varkappa}(a+b), \qquad \tilde{\sigma} = \sigma + \tilde{\varkappa}a.$$

We introduce the notation

$$\bar{\Omega} = \sqrt{\sigma_0\mu - \rho^2}, \qquad \Omega = \sqrt{\tilde{\sigma}\mu - \rho^2}$$

and assume that $\bar{\Omega}^2 = \sigma_0\mu - \rho^2 > 0$. Indeed, in this case, the general solution of system (4.3) is

$$\begin{aligned} X(t,\mathbf{C}) &= C_1 \sin\bar{\Omega}t + C_2\cos\bar{\Omega}t,\\ P(t,\mathbf{C}) &= \frac{1}{\mu}\big(\bar{\Omega}C_1 - \rho C_2\big)\cos\bar{\Omega}t - \frac{1}{\mu}\big(\bar{\Omega}C_2 + \rho C_1\big)\sin\bar{\Omega}t, \end{aligned}\tag{4.5}$$

and all solutions of system (4.3) are localized.

Assume that the wave packets that describe the evolution of particles by equation (4.1) do not spread. This takes place if $\Omega^2 = \tilde{\sigma}\mu - \rho^2 > 0$.

For system (4.4) we have

$$\begin{aligned} \Delta^{(2)}_{22}(t,\mathbf{C}) &= C_3\sin 2\Omega t + C_4\cos 2\Omega t + C_5,\\ \Delta^{(2)}_{21}(t,\mathbf{C}) &= \frac{1}{\mu}(\Omega C_3 - \rho C_4)\cos 2\Omega t - \frac{1}{\mu}(\Omega C_4 + \rho C_3)\sin 2\Omega t - \frac{\rho}{\mu}C_5,\\ \Delta^{(2)}_{11}(t,\mathbf{C}) &= \frac{1}{\mu^2}\big((\rho^2-\Omega^2)C_3 + 2\rho\Omega C_4\big)\sin 2\Omega t\\ &\qquad + \frac{1}{\mu^2}\big((\rho^2-\Omega^2)C_4 - 2\rho\Omega C_3\big)\cos 2\Omega t + \frac{\tilde{\sigma}}{\mu}C_5, \end{aligned}\tag{4.6}$$

and all solutions of system (4.4) are also localized. Here $\mathbf{C} = (C_1,\dots,C_5)$ and C_l, $l=\overline{1,5}$, are arbitrary integration constants.

The 1D associated linear equation (2.17) is

$$\hat{L}(t,\mathbf{C})\Phi(\vec{x},t,\mathbf{C}) = \big\{-i\hbar\partial_t + \hat{H}_q(t,\mathbf{C})\big\}\Phi(\vec{x},t,\mathbf{C}) = 0,\tag{4.7}$$

$$\hat{H}_q(t,\mathbf{C}) = \frac{\mu\hat{p}^2}{2} + \frac{\tilde{\sigma}x^2}{2} + \frac{\rho(x\hat{p}+\hat{p}x)}{2} + \tilde{\varkappa}bxX(t,\mathbf{C}) + \tilde{\varkappa}\frac{c}{2}\Big[X^2(t,\mathbf{C}) + \Delta^{(2)}_{22}(t,\mathbf{C})\Big].$$

We can immediately verify that for the associated linear equation (4.7) we can construct the following set of symmetry operators linear in x and $\hat{p}$:

$$\hat{a}(t,\mathbf{C}) = \frac{1}{\sqrt{2\hbar}}\big[C(t)\big(\hat{p} - P(t,\mathbf{C})\big) - B(t)\big(x - X(t,\mathbf{C})\big)\big],\tag{4.8}$$

$$\hat{a}^+(t,\mathbf{C}) = \frac{1}{\sqrt{2\hbar}}\big[C^*(t)\big(\hat{p} - P(t,\mathbf{C})\big) - B^*(t)\big(x - X(t,\mathbf{C})\big)\big].\tag{4.9}$$

Here the functions $B(t)$ and $C(t)$ are solutions of the linear Hamiltonian system

$$\begin{aligned} \dot{B} &= -\rho B - \tilde{\sigma}C,\\ \dot{C} &= \mu B + \rho C. \end{aligned}\tag{4.10}$$

The Cauchy matrix $\mathcal{X}(t)$ for system (4.10) can easily be found as

$$\mathcal{X}(t) = \begin{pmatrix} \cos\Omega t - \frac{\rho}{\Omega}\sin\Omega t & -\frac{1}{\mu\Omega}\big(\Omega^2+\rho^2\big)\sin\Omega t\\ \frac{\mu}{\Omega}\sin\Omega t & \cos\Omega t + \frac{\rho}{\Omega}\sin\Omega t \end{pmatrix}, \qquad \mathcal{X}(t)\Big|_{t=0} = \mathbb{I}_{2\times 2}.\tag{4.11}$$

The set of solutions normalized by the condition [24]

$$B(t)C^*(t) - C(t)B^*(t) = 2i\tag{4.12}$$

can be written as

$$B(t) = e^{i\Omega t}\frac{(-\rho + i\Omega)}{\sqrt{\Omega\mu}}, \qquad C(t) = e^{i\Omega t}\sqrt{\frac{\mu}{\Omega}}.\tag{4.13}$$

Equation (4.12) results in the following commutation relations for the symmetry operators (4.8) and (4.9):

$$\big[\hat{a}(t,\mathbf{C}),\hat{a}^+(t,\mathbf{C})\big]_- = 1.$$

For the function ϕ given by (3.8) in the 1D case, we obtain

$$\Phi(x,t,\mathbf{C}) = \hat{K}(\vec{x},t,s,\mathbf{C})\phi(\vec{x},t), \tag{4.14}$$

$$\hat{K}(x,t,\mathbf{C}) = \exp[-X(t,\mathbf{C})\partial_x]\exp\Big\{\frac{i}{\hbar}[S(t,\mathbf{C})+P(t,\mathbf{C})x]\Big\}, \tag{4.15}$$

where, according to (3.9),

$$S(t,\mathbf{C}) = \int_0^t \big\{P(t,\mathbf{C})\dot{X}(t) - H_\varkappa(t,\mathbf{C})\big\}\,\mathrm{d}\,t, \tag{4.16}$$

$$H_\varkappa(t,\mathbf{C}) = \frac{\mu}{2}P^2(t,\mathbf{C}) + \frac{1}{2}X^2(t,\mathbf{C})\big[\sigma_0+\tilde{\varkappa}(b+c)\big] + \rho P(t,\mathbf{C})X(t,\mathbf{C}) + \tilde{\varkappa}\frac{c}{2}\Delta_{22}^{(2)}(t,\mathbf{C}).$$

From (3.10) we find

$$\hat{L}_0(x,t)\phi = 0, \qquad \hat{L}_0(x,t) = -i\hbar\partial_t + \frac{\mu\hat{p}^2}{2} + \frac{(\sigma+\tilde{\varkappa}a)x^2}{2} + \frac{\rho(x\hat{p}+\hat{p}x)}{2}.$$

Then the symmetry operator $\hat{A}(t)$ (3.7) for equation (4.1) can be presented as

$$(\hat{A}(t)\Psi)(x,t) = \widehat{\mathcal{D}}(t,\mathbf{C}[\hat{a}\psi],\mathbf{C}[\Psi](t))\widehat{B}(t,\mathbf{C}[\Psi](t))\Psi(x,t), \tag{4.17}$$

where $\widehat{B}(t,\mathbf{C})$ is the symmetry operator of the associated linear equation (4.7).

The intertwining operator $\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C})$ presented, according to (3.12), as

$$\widehat{\mathcal{D}}(t,\mathbf{C}',\mathbf{C}) = \exp\Big\{i\frac{C_2'-C_2}{2\hbar\mu}\Big(\bar{\Omega}(C_1'-C_1)-\rho(C_2'-C_2)\Big)\Big\}$$
$$\times\hat{K}^{-1}(x,t,\mathbf{C}')\exp\Big\{\frac{i}{\hbar}\hat{b}(t,\mathbf{C}',\mathbf{C})\Big\}\hat{K}(x,t,\mathbf{C}), \tag{4.18}$$

where, according to (3.11),

$$\hat{b}(t,\mathbf{C}',\mathbf{C}) = b_x(t,\mathbf{C}',\mathbf{C})\hat{p} - b_p(t,\mathbf{C}',\mathbf{C})x = \langle b(t),J\hat{z}\rangle,$$

and the vector $b(t)$ is defined by

$$b(t,\mathbf{C}',\mathbf{C}) = \begin{pmatrix} b_p(t,\mathbf{C}',\mathbf{C})\\ b_x(t,\mathbf{C}',\mathbf{C})\end{pmatrix} = \frac{1}{\mu}\mathcal{X}(t)\begin{pmatrix}\bar{\Omega}(C_1'-C_1)-\rho(C_2'-C_2)\\ \mu(C_2'-C_2)\end{pmatrix}.$$

The matrix $\mathcal{X}(t)$ is given by (4.11).

The symmetry operator $\hat{A}(t)$ of the nonlinear equation (4.1) involved into (4.17) has the structure of a linear pseudodifferential operator whose parameters are functionals of the function on which the operator acts. Therefore, the explicit form of the operator $\hat{A}(t)$ is determined not only by the symmetry operator $\widehat{B}(t,\mathbf{C})$ of the associated linear equation, but also by the function $\Psi(x,t)$. Note that for some values of the parameters (more precisely, for the function $\Psi(x,t)$ that defines them) the pseudodifferential operator becomes a differential one.

We set

$$\widehat{B}(t,\mathbf{C}) = \widehat{B}_\nu(t,\mathbf{C}) = \frac{1}{\sqrt{\nu!}}\big[\hat{a}^+(t,\mathbf{C})\big]^\nu, \qquad \nu\in\mathbb{Z}_+, \tag{4.19}$$

where the operator $\hat{a}^+(t,\mathbf{C})$ is defined in (4.9).

Substituting (4.19) in (4.17) we obtain the symmetry operator, which we denote by $\hat{A}_\nu(t)$. Using a stationary solution of the Hamilton–Ehrenfest system (4.3), (4.4) we simplify the symmetry operators (4.8), (4.9) and generate a countable set of explicit solutions of the 1D GPE (4.1).

A stationary solution of equations (4.3), (4.4) is obtained from the general solution (4.5), (4.6) if we take integration constants as $\mathbf{C} = \mathbf{C}^0 = (C_1^0, \dots, C_5^0)$, where $C_1^0 = C_2^0 = C_3^0 = C_4^0 = 0$ and C_5^0 is an arbitrary real constant. The stationary solution is

$$\begin{aligned} &X(t,\mathbf{C}) = P(t,\mathbf{C}) = 0, \qquad \Delta_{22}^{(2)}(t,\mathbf{C}) = C_5^0, \\ &\Delta_{21}^{(2)}(t,\mathbf{C}) = -\frac{\rho}{\mu}C_5^0, \qquad \Delta_{11}^{(2)}(t,\mathbf{C}) = \frac{\tilde{\sigma}}{\mu}C_5^0. \end{aligned} \tag{4.20}$$

Substituting (4.20) in (4.7), we obtain the associated linear equation

$$\begin{aligned} &\hat{L}\big(x,t,\mathbf{C}^0\big)\Phi = 0, \\ &\hat{L}(x,t,\mathbf{C}^0) = \left[-i\hbar\partial_t + \frac{\mu\hat{p}^2}{2} + \frac{(\sigma+\tilde{\varkappa}a)x^2}{2} + \frac{\rho(x\hat{p}+\hat{p}x)}{2} + \tilde{\varkappa}\frac{c}{2}C_5^0\right]. \end{aligned} \tag{4.21}$$

The operator $\hat{K}(\vec{x},t,\mathbf{C}) = \hat{K}(\vec{x},t,\mathbf{C}^0)$ from (4.15) is the operator of multiplication by the function

$$\hat{K}(\vec{x},t,\mathbf{C}^0) = \exp\Big\{-\frac{i}{2\hbar}\tilde{\varkappa}cC_5^0t\Big\}.$$

The linear operators (4.8) and (4.9) then become

$$\hat{a}(t,\mathbf{C}^0) = \frac{1}{\sqrt{2\hbar}}\big[C(t)\hat{p} - B(t)x\big], \qquad \hat{a}^+(t,\mathbf{C}^0) = \frac{1}{\sqrt{2\hbar}}\big[C^*(t)\hat{p} - B^*(t)x\big]; \tag{4.22}$$

they are symmetry operators for equation (4.21); the functions $C(t)$ and $B(t)$ are defined in (4.13).

The function

$$\begin{aligned} \Phi_0(x,t,\mathbf{C}^0) = {}& \left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4}\exp\left\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\right\} \\ & \times\exp\left\{-\frac{i}{2}\Omega t - \frac{i}{2\hbar}\tilde{\varkappa}cC_5^0t\right\} \end{aligned} \tag{4.23}$$

is easily verified to be a solution of equation (4.21).

Upon direct substitution, we see that for the function (4.23), equations (2.19), (2.20), which determine the functionals $\mathbf{C}[\Psi](t)$, become

$$\begin{aligned} &X(0,\mathbf{C}) = x_\psi = 0, \qquad P(0,\mathbf{C}) = p_\psi = 0, \\ &\Delta_{22}^{(2)}(0,\mathbf{C}) = \big(\Delta_{22}^{(2)}\big)_\psi = \frac{\hbar}{2}\big|C(0)\big|^2 = \frac{\hbar\mu}{2\Omega}, \\ &\Delta_{11}^{(2)}(t,\mathbf{C}) = \big(\Delta_{11}^{(2)}\big)_\psi = \frac{\hbar}{2}\big|B(0)\big|^2 = \frac{\hbar(\varrho^2+\Omega^2)}{2\Omega\mu}, \\ &\Delta_{12}^{(2)}(0,\mathbf{C}) = \big(\Delta_{12}^{(2)}\big)_\psi = \frac{\hbar}{4}\big[B(0)C^*(0) + B^*(0)C(0)\big] = -\frac{\hbar\varrho}{2\Omega}, \\ &\psi(x) = \Phi_0\big(x,0,\mathbf{C}^0\big). \end{aligned} \tag{4.24}$$

From (4.24) and (4.20) it follows that $C_5^0 = (\hbar\mu/2\Omega)$. From (3.2) and (4.23) we find a particular solution $\Psi_0(x,t)$ of the GPE (4.1):

$$\Psi_0(x,t) = \Phi_0\big(x,t,\mathbf{C}^0\big)\Big|_{C_5^0=(\hbar\mu/2\Omega)}$$

$$= \left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4}\exp\left\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\right\}\exp\left\{-\frac{i}{2}\Omega t - \frac{i\mu}{4\Omega}\tilde{\varkappa}ct\right\}. \qquad (4.25)$$

The symmetry operators (4.19), (4.22) generate from (4.23) the solutions of the associated linear equation (4.21) that constitute a Fock basis in the space $L_2(\mathbb{R})$:

$$\Phi_\nu(x,t,\mathbf{C}^{0\prime}) = \widehat{B}_\nu(t,\mathbf{C^{0\prime}})\Phi_0(x,t,\mathbf{C}^{0\prime}) = \frac{1}{\sqrt{\nu!}}\left[\hat{a}^+(t,\mathbf{C}^{0\prime})\right]^\nu \Phi_0(x,t,\mathbf{C}^{0\prime})$$
$$= \frac{i^\nu}{\sqrt{\nu!}}\left(\frac{1}{\sqrt{2}}\right)^\nu H_\nu\left(\sqrt{\frac{\Omega}{\hbar\mu}}x\right)\Phi_0(x,t,\mathbf{C}^{0\prime})\exp\{-i\Omega\nu t\}, \qquad \nu\in\mathbb{Z}_+, \qquad (4.26)$$

where $H_\nu(\zeta)$ are the Hermite polynomials [12]

$$H_\nu(\zeta) = \left(2\zeta - \frac{\mathrm{d}}{\mathrm{d}\,\zeta}\right)^\nu \cdot 1.$$

The operator (4.18), intertwining the operators $\hat{L}(x,t,\mathbf{C}^0)$ and $\hat{L}(x,t,\mathbf{C}^{0\prime})$ of the form (4.21), reads

$$\widehat{\mathcal{D}}(t,\mathbf{C}^{0\prime},\mathbf{C}^0) = \exp\left\{\frac{i}{2\hbar}\tilde{\varkappa}c\left[C_5^0 - C_5^{0\prime}\right]t\right\}.$$

Equations (2.20) that determine the functionals $\mathbf{C}[\Psi_\nu](t)$ for the functions (4.26) can be written as

$$\begin{aligned}
&X(t,\mathbf{C}) = x_{\psi_a} = 0, \qquad P(0,\mathbf{C}) = p_{\psi_a} = 0,\\
&\Delta_{22}^{(2)}(0,\mathbf{C}) = \big(\Delta_{22}^{(2)}\big)_{\psi_a} = \frac{\hbar}{2}(2\nu+1)\big|C(0)\big|^2 = \frac{\hbar\mu}{2\Omega}(2\nu+1),\\
&\Delta_{11}^{(2)}(0,\mathbf{C}) = \big(\Delta_{11}^{(2)}\big)_{\psi_a} = \frac{\hbar}{2}\big|B(0)\big|^2(2\nu+1) = \frac{\hbar(\rho^2+\Omega^2)}{2\Omega\mu}(2\nu+1),\\
&\Delta_{12}^{(2)}(0,\mathbf{C}) = \big(\Delta_{12}^{(2)}\big)_{\psi_a} = \frac{\hbar}{4}\big[B(0)C^*(0) + B^*(0)C(0)\big](2\nu+1) = -\frac{\hbar\rho}{2\Omega}(2\nu+1),\\
&\psi_a(x) = \Phi_\nu(x,0,\mathbf{C}^{0\prime}) = \widehat{B}_\nu(0,\mathbf{C^{0\prime}})\psi(x).
\end{aligned} \qquad (4.27)$$

Here we have used the standard properties of Hermite polynomials [12]. Taking into account (4.20), we find from (4.27) that $C_5^{0\prime} = (\hbar\mu/\Omega)(\nu+1/2)$.

Then the symmetry operator $\hat{A}_\nu(t)$ determined by (4.17) transforms the solution $\Psi_0(x,t)$ of (4.25) into a solution $\Psi_\nu(x,t)$ of the nonlinear GPE (4.1) according to the following relation:

$$\begin{aligned}
\Psi_\nu(x,t) &= (\hat{A}_\nu(t)\Psi_0)(x,t)\\
&= \widehat{\mathcal{D}}(t,\mathbf{C}^0,\mathbf{C}^{0\prime})\frac{1}{\sqrt{\nu!}}\left[\hat{a}^+(t,\mathbf{C}^0)\right]^\nu\Big|_{C_5^0=(\hbar\mu/2\Omega),\,C_5^{0\prime}=(\hbar\mu/\Omega)(\nu+1/2)}\Psi_0(x,t)\\
&= \left(\frac{1}{\sqrt{2}}\right)^\nu\left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4}\exp\left\{-\frac{i}{2\hbar}\frac{\rho}{\mu}x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}x^2\right\}\\
&\quad\times H_\nu\left(\sqrt{\frac{\Omega}{\hbar\mu}}x\right)\exp\left\{-i\left(\nu+\frac{1}{2}\right)\left(\frac{\tilde{\varkappa}c\mu}{2\Omega}+\Omega\right)t\right\}.
\end{aligned} \qquad (4.28)$$

The functions (4.28) constitute a countable set of particular solutions to equation (4.1) which are generated from $\Psi_0(x,t)$ by the nonlinear symmetry operator $\hat{A}_\nu(t)$.

The symmetry operators $\hat{A}_\nu(t)$ in (4.28) generalize those of the linear equations used in the Maslov complex germ theory [4, 24], as in the limit $\varkappa \to 0$ ($\varkappa$ is the nonlinearity parameter in equation (4.2)), the operators $\hat{A}_\nu(t)$ become the creation operators of the Maslov complex germ theory. As in the linear case ($\varkappa = 0$), the operators $\hat{A}_\nu(t)$ generate a countable set of exact solutions $\Psi_\nu(x,t)$ to the nonlinear equation (4.2).

Assume that $\mathbf{C} = \mathbf{C}^1 = (C_1^1, C_2^1, 0, 0, C_5^1)$. This choice of the constants yields the following expression for the phase orbit (4.5), (4.6):

$$
\begin{aligned}
&X\big(t, C_1^1, C_2^1\big) = C_1^1 \sin\bar{\Omega}t + C_2^1\cos\bar{\Omega}t,\\
&P\big(t, C_1^1, C_2^1\big) = \frac{1}{\mu}\big(\bar{\Omega}C_1^1 - \rho C_2^1\big)\cos\bar{\Omega}t - \frac{1}{\mu}\big(\bar{\Omega}C_2^1 + \rho C_1^1\big)\sin\bar{\Omega}t,\\
&\Delta_{22}^{(2)}\big(t,\mathbf{C}^1\big) = C_5^1, \qquad \Delta_{21}^{(2)}\big(t,\mathbf{C}^1\big) = -\frac{\rho}{\mu}C_5^1, \qquad \Delta_{11}^{(2)}\big(t,\mathbf{C}^1\big) = \frac{\tilde{\sigma}}{\mu}C_5^1.
\end{aligned}
\tag{4.29}
$$

In view of (4.29) and (4.15), we have

$$
\hat{K}(x,t,\mathbf{C}^1) = \exp\big[-X\big(t,\mathbf{C}^1\big)\partial_x\big]\exp\Big\{\frac{i}{\hbar}\big[S\big(t,\mathbf{C}^1\big) + P\big(t,\mathbf{C}^1\big)x\big]\Big\}.
$$

Consider the action of the operator $\hat{A}_0(t)$ involved into (4.17), (4.19) on the functions (4.28). Let us write the operator (4.18) intertwining the operators $L(x,t,\mathbf{C}^0)$ and $L(x,t,\mathbf{C}^1)$ determined by (4.21) as

$$
\begin{aligned}
\widehat{\mathcal{D}}\big(t,\mathbf{C},\mathbf{C}^0\big) = \exp[-X(t,\mathbf{C})\partial_x]\exp\left\{\frac{i}{\hbar}\left[S(t,\mathbf{C}) - \frac{1}{2}\tilde{\varkappa}cC_5^0 t + P\big(t,\mathbf{C}^1\big)x\right]\right\}\\
\times\exp\Big\{\frac{i}{2\hbar\mu}C_2\big(\tilde{\Omega}C_1 - \rho C_2\big)\Big\}\exp\Big\{\frac{i}{\hbar}\hat{b}\big(t,\mathbf{C}^1,\mathbf{C}^0\big)\Big\},
\end{aligned}
$$

where

$$
\hat{b}\big(t,\mathbf{C},\mathbf{C}^0\big) = b_x\big(t,\mathbf{C},\mathbf{C}^0\big)\hat{p} - b_p(t,\mathbf{C},\mathbf{C}^0)x = \langle b(t), J\hat{z}\rangle,
$$

and the vector $b(t)$ is defined by the expression

$$
b\big(t,\mathbf{C},\mathbf{C}^0\big) = \begin{pmatrix} b_p\big(t,\mathbf{C},\mathbf{C}^0\big)\\ b_x\big(t,\mathbf{C},\mathbf{C}^0\big)\end{pmatrix} = \frac{1}{\mu}\mathcal{X}(t)\begin{pmatrix}\tilde{\Omega}C_1 - \rho C_2\\ \mu C_2\end{pmatrix}.
$$

The matrix $\mathcal{X}(t)$ is given by (4.11).

Let us construct a nonlinear symmetry operator $\hat{A}(t,\alpha)$ corresponding to the nonstationary phase orbit (4.5), (4.6). The operator $\hat{A}(t,\alpha)$ maps the nonstationary solution of equation (4.1), $\Psi_\nu(x,t)$ given by (4.28), into another nonstationary solution of this equation, $\widetilde{\Psi}_\nu(x,t)$. Consider the shift operator

$$
\widehat{B}\big(t,\mathbf{C}^1\big) = \widehat{B}\big(t,\alpha,\mathbf{C}^1\big) = \exp\{\alpha\hat{a}^+(t) - \alpha^*\hat{a}(t)\}, \qquad \alpha\in\mathbb{C},
\tag{4.30}
$$

where the operators $\hat{a}(t,\mathbf{C})$ and $\hat{a}^+(t,\mathbf{C})$ are defined by expressions (4.8), (4.9). The operator (4.30) should be substituted in (4.17) for the symmetry operator $\widehat{B}(t,\mathbf{C})$.

Let us write the operator $\widehat{B}(t,\alpha,\mathbf{C}^1)$ involved into (4.30) as

$$
\widehat{B}(t,\alpha,\mathbf{C}^1) = \exp\{\beta(t)\hat{p} + \gamma(t)x\} = \exp\Big\{-\frac{i\hbar}{2}\beta(t)\gamma(t)\Big\}\exp\{\gamma(t)x\}\exp\{\beta(t)\hat{p}\},
$$

where

$$
\beta(t) = \frac{1}{\sqrt{2\hbar}}\big[C^*(t)\alpha - C(t)\alpha^*\big], \qquad \gamma(t) = \frac{1}{\sqrt{2\hbar}}\big[B(t)\alpha^* - B^*(t)\alpha\big].
$$

Thus, we have

$$\begin{aligned}
&\psi_a(x) = \widehat{B}\big(0, \alpha, \mathbf{C}^1\big)\Phi_\nu\big(x, 0, \mathbf{C}^{0\prime}\big)\\
&\qquad = \exp\Big\{-\frac{i\hbar}{2}\beta(0)\gamma(0)\Big\}\exp\{\gamma(0)x\}\Phi_\nu\big(x - i\hbar\beta(0), 0, \mathbf{C}^{0\prime}\big), \qquad (4.31)\\
&\Phi_\nu\big(x - i\hbar\beta(0), 0, \mathbf{C}^{0\prime}\big) = \frac{i^\nu}{\sqrt{\nu!}}\left(\frac{1}{\sqrt{2}}\right)^\nu H_\nu\bigg(\sqrt{\frac{\Omega}{\hbar\mu}}\Big[x - i\hbar\beta(0)\Big]\bigg)\Phi_0\big(x - i\hbar\beta(0), 0, \mathbf{C}^{0\prime}\big),\\
&\Phi_0\big(x - i\hbar\beta(0), 0, \mathbf{C}^{0\prime}\big)\\
&\qquad = \Phi_0\big(x, 0, \mathbf{C}^{0\prime}\big)\exp\left\{\left[\left(\frac{-\rho + i\Omega}{\mu}\right)x\beta(0) - i\frac{\hbar}{2}\left(\frac{-\rho + i\Omega}{\mu}\right)\beta^2(0)\right]\right\}.
\end{aligned}$$

Note that

$$\begin{aligned}
\gamma(0) &= \frac{1}{\sqrt{2\hbar}}\big[-B^*(0)\alpha + B(0)\alpha^*\big] = \frac{1}{\sqrt{2\hbar}}\left[\frac{(\rho + i\Omega)}{\sqrt{\Omega\mu}}\alpha + \frac{(-\rho + i\Omega)}{\sqrt{\Omega\mu}}\alpha^*\right]\\
&= i\frac{\sqrt{2}}{\sqrt{\hbar\Omega\mu}}\big[\rho\alpha_2 + \Omega\alpha_1\big],\\
\beta(0) &= \frac{1}{\sqrt{2\hbar}}\big[C^*(0)\alpha - C(0)\alpha^*\big] = \frac{1}{\sqrt{2\hbar}}\sqrt{\frac{\mu}{\Omega}}(\alpha - \alpha^*) = i\sqrt{\frac{2\mu}{\hbar\Omega}}\alpha_2,\\
\gamma(0) + \frac{-\rho + i\Omega}{\mu}\beta(0) &= \frac{1}{\sqrt{2\hbar}}\left[-B^*(0)\alpha + B(0)\alpha^* + \frac{-\rho + i\Omega}{\mu}\big(C^*(0)\alpha - C(0)\alpha^*\big)\right]\\
&= \frac{1}{\sqrt{2\hbar\Omega\mu}}\big[(\rho + i\Omega)\alpha + (-\rho + i\Omega)\alpha^*) + (-\rho + i\Omega)(\alpha - \alpha^*)\big] = \frac{2i\sqrt{\Omega}\alpha}{\sqrt{2\hbar\mu}}.
\end{aligned}$$

Here $\alpha_1 = \operatorname{Re}\alpha$ and $\alpha_2 = \operatorname{Im}\alpha$. Similarly, we have

$$\begin{aligned}
-\frac{i\hbar}{2}\left[\beta(0)\gamma(0) + \frac{-\rho + i\Omega}{\mu}\beta^2(0)\right] &= -\frac{i\hbar}{2}\frac{2i\sqrt{\Omega}\alpha}{\sqrt{2\hbar\mu}}\beta(0) = \frac{1}{2}\alpha\big[\alpha - \alpha^*\big]\\
&= \frac{1}{2}\big[\alpha^2 - |\alpha|^2\big] = i\alpha\alpha_2. \qquad (4.32)
\end{aligned}$$

Substituting (4.32) in (4.31), we obtain

$$\begin{aligned}
\psi_a(x) &= \widehat{B}\big(0, \alpha, \mathbf{C}^1\big)\Phi_\nu(x, 0, \mathbf{C}^1)\\
&= \exp\left\{\frac{i\alpha\alpha_2}{2}\right\}\exp\left\{\frac{2i\sqrt{\Omega}}{\sqrt{2\hbar\mu}}\alpha x\right\}\frac{i^\nu}{\sqrt{\nu!}}\left(\frac{1}{\sqrt{2}}\right)^\nu H_\nu\bigg(\sqrt{\frac{\Omega}{\hbar\mu}}\Big[x + \frac{\sqrt{2\hbar\mu}}{\Omega}\alpha_2\Big]\bigg)\\
&\quad\times\left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4}\exp\left\{\frac{i}{2\hbar}\left(\frac{-\rho + i\Omega}{\mu}\right)x^2\right\}. \qquad (4.33)
\end{aligned}$$

From (4.33), in particular, it follows that

$$|\psi_a(x)|^2 = \sqrt{\frac{\Omega}{\pi\hbar\mu}}\exp\left\{-\frac{\Omega}{\hbar\mu}\left(x + \frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2\right)^2\right\}\frac{1}{\nu!}\Big(\frac{1}{2}\Big)^\nu H^2_\nu\bigg(\sqrt{\frac{\Omega}{\hbar\mu}}\left[x + \frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2\right]\bigg).$$

Similar to (4.27), we write equations (2.22) determining the functionals $\mathbf{C}[\psi_a](t)$ for the functions (4.33) as

$$X\big(0, C^1_1, C^1_2\big) = C^1_2 = x_\psi = -\frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2,$$

$$P\big(0, C_1^1, C_2^1\big) = \frac{1}{\mu}\big(\bar{\Omega}C_1^1 - \rho C_2^1\big) = p_\psi = \frac{\sqrt{2\hbar\Omega}}{\sqrt{\mu}}\alpha_1,$$
$$\Delta_{22}^{(2)}(0, \mathbf{C}) = \big(\Delta_{22}^{(2)}\big)_\psi = \frac{\hbar}{2}(2\nu+1)\big|C(0)\big|^2 = \frac{\hbar\mu}{2\Omega}(2\nu+1),$$
$$\Delta_{11}^{(2)}(0, \mathbf{C}) = \big(\Delta_{11}^{(2)}\big)_\psi = \frac{\hbar}{2}\big|B(0)\big|^2(2\nu+1) = \frac{\hbar(\varrho^2+\Omega^2)}{2\Omega\mu}(2\nu+1),$$
$$\Delta_{12}^{(2)}(0, \mathbf{C}) = \big(\Delta_{12}^{(2)}\big)_\psi = \frac{\hbar}{4}\big[B(0)C^*(0) + B^*(0)C(0)\big](2\nu+1) = -\frac{\hbar\varrho}{2\Omega}(2\nu+1).$$

From equations (4.5) and (4.12), in view of (3.1), we obtain

$$C_1^1 = C_1^1(\alpha) = \frac{\sqrt{2\hbar\mu\Omega}}{\bar{\Omega}}\Big(\alpha_1 - \frac{\rho}{\Omega}\alpha_2\Big), \qquad C_2^1 = C_2^1(\alpha) = -\frac{\sqrt{2\hbar\mu}}{\sqrt{\Omega}}\alpha_2,$$
$$C_3^1 = C_4^1 = 0, \qquad C_5^1 = \frac{\hbar\mu}{2\Omega}(2\nu+1). \tag{4.34}$$

Then the nonlinear symmetry operator for the nonlinear GPE (4.1), $\hat{A}(t,\alpha)$ determined by (4.17) and (4.30) transforms the solution $\Psi_\nu(x,t)$ (4.28) into a nonstationary solution $\widetilde{\Psi}_\nu(x,t)$:

$$\begin{aligned}
\widetilde{\Psi}_\nu(x,t,\alpha) &= (\hat{A}(t,\alpha)\Psi_\nu)(x,t) = \widehat{\widehat{\mathcal{D}}}\big(t, \mathbf{C}^1, \mathbf{C}^0\big)\Big|_{C_5^0 = C_5^1 = (\hbar\mu/\Omega)(\nu+1/2),} \Psi_\nu(x,t) \\
&= \frac{i^\nu}{\sqrt{\nu!}}\left(\frac{1}{\sqrt{2}}\right)^\nu \left(\frac{1}{\pi\hbar}\right)^{1/4}\left(\frac{\Omega}{\mu}\right)^{1/4} \exp\left\{-\frac{i}{2\hbar}\frac{\rho}{\mu}\Delta x^2 - \frac{1}{2\hbar}\frac{\Omega}{\mu}\Delta x^2\right\} \\
&\quad \times H_\nu\left(\sqrt{\frac{\Omega}{\hbar\mu}}\Delta x\right)\exp\left\{-i\Big(\nu+\frac{1}{2}\Big)\Big(\frac{\tilde{\varkappa} c\mu}{2\Omega} + \Omega\Big)t\right\} \\
&\quad \times \exp\left\{\frac{i}{\hbar}\big[S(t, C_1^1(\alpha), C_2^1(\alpha)) + P(t, C_1^1(\alpha), C_2^1(\alpha))\Delta x\big]\right\},
\end{aligned} \tag{4.35}$$

which is localized around the phase orbit $\big(P\big(t, C_1^1(\alpha), C_2^1(\alpha)\big), X\big(t, C_1^1(\alpha), C_2^1(\alpha)\big)\big)$. Here $\Delta x = x - X(t, C_1^1(\alpha), C_2^1(\alpha))$ with the constants $C_1^1(\alpha)$, $C_2^1(\alpha)$ determined by equation (4.34), and the function $S(t, C_1^1(\alpha), C_2^1(\alpha))$ is determined by (4.16) where $C_1 = C_1^1(\alpha)$, $C_2 = C_2^1(\alpha)$, $C_3 = 0$, $C_4 = 0$, and $C_5^0 = (\hbar\mu/\Omega)(\nu+1/2)$.

In the linear case ($\varkappa = 0$), the operators (4.35) with $\alpha \in \mathbb{C}$ form a representation of the Heisenberg–Weyl group [21, 31]. The functions $\widetilde{\Psi}_\nu(x,t,\alpha)$ determined by (4.35) minimize the Schrödinger uncertainty relation for $\nu = 0$ [32], and, hence, they describe squeezed coherent states [11].

5 Discussion

Direct calculation of symmetry operators for a nonlinear equation is, as a rule, a severe problem because of the nonlinearity and complexity of the determining equations [26]. However, for nearly linear equations [18] a wide class of symmetry operators can be constructed by solving linear determining equations for operators of this type much as symmetry operators are found for linear PDEs. We have illustrated this situation with the example of the generalized multi-dimensional Gross–Pitaevskii equation (2.1). The formalism of semiclassical asymptotics leads to the semiclassically reduced GPE (2.6) (or (2.13)), which belongs to the class of nearly linear equations. Note that the solutions of GPE can be found in a special class of functions decreasing at infinity [6]. The reduced GPE is the quadratic one in the space coordinates and derivatives and contains a nonlocal term of special form. In constructing the symmetry operators for the reduced Gross–Pitaevskii equation (2.13), we use the fact that this equation can be associated with the linear equation (2.17). The symmetry operator $\hat{A}(t)$ of the reduced GPE (2.13), which is a particular case of (3.7), has the structure of a linear pseudodifferential operator with coefficients $\mathbf{C}$ depending on the function Ψ on which the operator acts. The operator $\hat{A}$ is determined in terms of the linear intertwining operator $\widehat{\mathcal{D}}$ and of the symmetry operators of the associated linear equation (2.17). The dependence of the coefficients $\mathbf{C}$ on Ψ arises from the algebraic condition (3.1), and therefore the operator $\hat{A}(t)$ is nonlinear. This is the key point of the presented approach. The 1D examples considered show that for a special choice of the parameters $\mathbf{C}$ we can construct symmetry operators and generate the families of solutions to the nonlinear equation (4.1) written in explicit form.

The further development of the study of symmetry operators is seen as a generalization to the approach for integro-differential GPEs of more general form and to systems of equations of this type.

Acknowledgements

We would like to thank the anonymous referees who gave a relevant contribution to improve the paper. The work was supported in part by the Russian Federation programs "Kadry" (contract No. 16.740.11.0469) and "Nauka" (contract No. 1.604.2011) and by Tomsk State University project No. 2.3684.2011.

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