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Solvable Many-Body Models of Goldfish Type with One-, Two- and Three-Body Forces

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Abstract. The class of *solvable* many-body problems "of goldfish type" is extended by including (the additional presence of) *three-body* forces. The solvable N-body problems thereby identified are characterized by Newtonian equations of motion featuring 19 arbitrary "coupling constants". Restrictions on these constants are identified which cause these systems – or appropriate variants of them – to be *isochronous* or *asymptotically isochronous*, i.e. *all* their solutions to be *periodic* with a *fixed* period (independent of the initial data) or to have this property up to contributions vanishing exponentially as $t \to \infty$.

Key words: many-body problems; N-body problems; partial differential equations; isochronous systems

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1 Introduction

Over three decades ago [3] a class of solvable N-body problems featuring several free parameters ("coupling constants") was introduced by identifying the coordinates $z_n(t)$ of the moving particles with the N zeros of the time-dependent (monic) polynomial $\psi(z,t)$ (of degree N in the independent variable z),

$$\psi(z,t) = \prod_{n=1}^{N} [z - z_n(t)] = z^N + \sum_{m=1}^{N} c_m(t) z^{N-m}, \tag{1.1}$$

itself evolving according to a *linear* partial differential equation (PDE) – suitably restricted to guarantee that it feature a polynomial solution of degree N in z. The simplest dynamical system belonging to this class displays remarkably neat properties and was therefore considered a "goldfish" (for a justification of this terminology, see [5]; subsequently this terminology has been often employed to identify N-body models belonging to this class, and this justifies its use also in the present paper, including its title). The behavior of the solutions of these N-body problems has been variously investigated and also used to arrive at related mathematical results, such as finite-dimensional representations of differential operators and Diophantine properties of the zeros of certain polynomials: see the two monographs [4] and [6] and the references quoted there (including the more recent ones added to the 2012 paperback version of [6]), and the more recent papers [1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The class of *linear* PDEs satisfied by the polynomial $\psi(z,t)$ which subtended these investigations was so far restricted to PDEs featuring derivatives up to *second* order. The restriction to *time*-differentiations of *second order* was motivated by the interest in N-body problems featuring equations of motion of *Newtonian* type (i.e., "acceleration equals force"); the more general restriction to differentiations of at most *second-order* implied that the N-body problems under consideration only involved *one-body* and *two-body* (generally velocity-dependent) forces. In the present paper we extend our consideration to *third-order* z-derivatives – while maintaining the restriction to *second-order time*-differentiations so as to only treat N-body problems of *Newtonian* type. This entails that the corresponding *solvable* N-body problems thereby identified involve, additionally, *three-body* forces.

In the following Section 2 we report our main results, namely we display the Newtonian equations of motion of the N-body problems identified in this paper and we indicate how they are *solved by algebraic operations*. In a terse Section 3 these findings are proven: these developments rely on identities reported and proven in Appendix A, constituting a substantial part of this paper. In Section 4 we highlight some special cases in which these N-body problems – or appropriate variants of them – are *isochronous* or *asymptotically isochronous*, i.e. *all* their solutions are *periodic* with a *fixed* period (independent of the initial data) or they feature this property up to contributions vanishing exponentially as $t \to \infty$. A terse Section 5 entitled "Outlook" outlines possible future developments.

2 Main results

In this section we report the main results of this paper, which are then proven in the following section. But firstly let us specify our notation.

Notation 2.1. The N coordinates ("dependent variables") of the point, unit-mass, moving particles which are the protagonists of the N-body problem treated in this paper are generally denoted as $z_n \equiv z_n(t)$, with t ("time") the "independent variable". As usual differentiation with respect to time is denoted by a superimposed dot, hence $\dot{z}_n \equiv dz_n/dt$, $\ddot{z}_n \equiv d^2 z_n/dt^2$. As we just did here, often the indication of the t-dependence is not explicitly displayed. We generally assume that these coordinates z_n are *complex* numbers, so that the points with coordinates $z_n(t)$ move in the *complex* z-plane; but special subcases in which the coordinates z_n are *real* are of course possible; and it is of course also possible to reinterpret motions taking place in the *complex* z-plane as instead taking place in the *real* horizontal plane by identifying the real and imaginary parts of the *complex* numbers $z_n = x_n + iy_n$ as the Cartesian components of the *real* 2-vectors $\vec{r}_n = (x_n, y_n)$ (as explained in Chapter 4 of [4], entitled "Solvable and/or integrable many-body problems in the plane, obtained by complexification"). Here and hereafter i is the imaginary unit, $i^2 = -1$, N is an *arbitrary* positive integer (generally $N \geq 2$), and it is understood that subscripts such as n (and also m, k, ℓ, but not j; see below) run over the positive integers from 1 to N (unless otherwise specified); the reader is often (but not always) explicitly reminded of this fact. We also use occasionally the notation $\underline{z}$ to denote the (generally *complex*) N-vector of components z_n, $\underline{z} \equiv (z_1, \ldots, z_N)$; and likewise for other underlined letters (see below). A key role in our treatment is played by the time-dependent (monic) polynomial $\psi(z,t)$ of degree N in the (*scalar*, generally *complex*) variable z which features the N coordinates $z_n(t)$ as its N zeros, see (1.1).

The *solvable* N-body problem treated in this paper is characterized by the following Newtonian equations of motion:

$$\ddot{z}_n + E\dot{z}_n = B_0 + B_1 z_n - (N-1)[2A_3 + 3(N-2)F_4]z_n^2 + (N-1)(N-2)G_3\dot{z}_n z_n$$
$$+ \sum_{\ell=1;\,\ell\neq n}^{N} \big\{(z_n - z_\ell)^{-1}\big[2\dot{z}_n\dot{z}_\ell + 2\big(A_0 + A_1 z_n + A_2 z_n^2 + A_3 z_n^3\big)$$
$$- (\dot{z}_n + \dot{z}_\ell)(D_0 + D_1 z_n) - D_2 z_n(\dot{z}_n z_\ell + \dot{z}_\ell z_n)\big]\big\}$$

$$+ \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left[\frac{3(F_0 + F_1 z_n + F_2 z_n^2 + F_3 z_n^3 + F_4 z_n^4)}{(z_n - z_\ell)(z_n - z_k)} \right. \tag{2.1}$$
$$\left. - \left(\frac{G_0 + G_1 z_n + G_2 z_n^2 + G_3 z_n^3}{z_n - z_\ell} \right) \left(\frac{\dot z_n + \dot z_k}{z_n - z_k} + \frac{\dot z_\ell + \dot z_k}{z_\ell - z_k} \right) \right], \qquad n = 1, \ldots, N.$$

Here and hereafter the 19 upper-case letters A_j, B_j, D_j, E, F_j, G_j denote time-independent parameters ("coupling constants"). They are *a priori arbitrary* (possibly *complex*) numbers; special assignments are occasionally considered below. This notation is chosen consistently with equation (2.3.3-2) of [4] (with $C = 1$), to which this system of N equations of motion reduces when the 9 "new" parameters vanish, i.e.

$$F_0 = F_1 = F_2 = F_3 = F_4 = G_0 = G_1 = G_2 = G_3 = 0. \tag{2.2}$$

Note that the terms associated with the 4 new parameters F_0, F_1, F_2, F_3, represent *velocity-independent three-body* forces, the terms associated with the 3 new parameters G_0, G_1, G_2 represent *velocity-dependent three-body* forces, and the terms associated with the new parameters F_4 respectively G_3 represent *velocity-independent* respectively *velocity-dependent one-body* and *three-body* forces. (Of course terms representing *one-body* forces can always be absorbed in those representing *many-body* forces: for instance the *one-body* term $-2(N-1)A_3 z_n^2$ in the right-hand side of (2.1) could be eliminated by replacing z_n^3 with $z_n^2 z_\ell$ in the *two-body* term multiplying A_3 in the first sum in the right-hand side of (2.1)).

This Newtonian N-body system is *solvable* by *algebraic* operations because – as shown in the following Section 3 – the coordinates $z_n(t)$ evolving according to this system of N Ordinary Differential Equations (ODEs) coincide with the N zeros of the time-dependent polynomial (1.1) of degree N in z, itself evolving according to the following *linear* PDE:

$$\begin{aligned} &\psi_{tt} + \{E - (N-1)[D_2 + (N-2)G_3]z\}\psi_t \\ &\quad + \big(G_0 + G_1 z + G_2 z^2 + G_3 z^3\big)\psi_{zzt} + \big(D_0 + D_1 z + D_2 z^2\big)\psi_{zt} \\ &\quad + \big(F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4\big)\psi_{zzz} + \big(A_0 + A_1 z + A_2 z^2 + A_3 z^3\big)\psi_{zz} \\ &\quad + \big\{B_0 + B_1 z - (N-1)[2A_3 + 3(N-2)F_4]z^2\big\}\psi_z \\ &\quad + N\big\{ - B_1 - (N-1)[A_2 + (N-2)F_3] + (N-1)[A_3 + 2(N-2)F_4]z\big\}\psi = 0. \end{aligned} \tag{2.3}$$

Here the 19 upper-case letters are of course the same time-independent parameters featured by the Newtonian equations of motion (2.1). Again, this notation is consistent with that used in [4]: indeed this PDE is a natural generalization of equation (2.3.3-1) of [4] (with $C = 1$) to which it clearly reduces when the *new* parameters vanish, see (2.2).

This PDE implies that the N coefficients $c_m \equiv c_m(t)$, see (1.1), evolve according to the following system of N *linear* ODEs:

$$\begin{aligned} &\ddot c_m + (N+2-m)(N+1-m)G_0 \dot c_{m-2} + (N+1-m)[D_0 + (N-m)G_1]\dot c_{m-1} \\ &\quad + \{E + (N-m)[D_1 + (N-1-m)G_2]\}\dot c_m - m[D_2 + (2N-3-m)G_3]\dot c_{m+1} \\ &\quad + (N+3-m)(N+2-m)(N+1-m)F_0 c_{m-3} \\ &\quad + (N+2-m)(N+1-m)[A_0 + (N-m)F_1]c_{m-2} \\ &\quad + (N+1-m)\big\{(N-m)[A_1 + (N-1-m)F_2] + B_0\big\}c_{m-1} \\ &\quad - m\big\{(2N-1-m)A_2 + B_1 + \big[3N^2 - 6N + 2 - 3(N-1)m + m^2\big]F_3\big\}c_m \\ &\quad + m(m+1)[A_3 + (3N-5-m)F_4]c_{m+1} = 0, \qquad m = 1, \ldots, N, \end{aligned} \tag{2.4}$$

of course with $c_0 = 1$ and $c_m = 0$ for $m < 0$ and for $m > N$.

Remark 2.1. This system of N ODEs satisfied by the N coefficients $c_m(t)$ justifies the assertion made above that the linear PDE (2.3) admits a polynomial solution: specifically, the polynomial (1.1) of degree N in z. Of course this system of N ODEs reduces to equation (2.3.3-8) of [4] (with $C = 1$) when the *new* parameters vanish, see (2.2). Likewise, it reduces to equation (4.54) of [6] (up to the correction of a trivial misprint in that equation, and to an obvious notational change). Note however that the analogous equation has been wrongly reported (as equation (3)) in [13]: due to a trivial misprint (a multiplicative factor c_{m-2} missing in the first term in the second line) and the mistake of inserting two terms in the right-hand side (which should instead just be zero); fortunately this mistake has no consequence on the remaining part of that paper – except for the mistaken Remark 1.1 which should of course be ignored.

This system, (2.4), of N *autonomous linear* ODEs is of course solvable by *algebraic* operations. Indeed its *general* solution reads

$$\underline{c}(t) = \sum_{m=1}^{N} \left[b_m^{(+)} \underline{v}^{(+)(m)} \exp\left(\lambda_m^{(+)} t\right) + b_m^{(-)} \underline{v}^{(-)(m)} \exp\left(\lambda_m^{(-)} t\right) \right], \tag{2.5}$$

where the N-vector $\underline{c}(t)$ has the N components $c_m(t)$, the $2N$ (time-independent) coefficients $b_m^{(\pm)}$ are *a priori* arbitrary – to be fixed *a posteriori* in order to satisfy the $2N$ initial conditions $c_m(0)$ and $\dot{c}_m(0)$ – while the $2N$ (time-independent) N-vectors $\underline{v}^{(\pm)(m)}$ respectively the $2N$ (time-independent) numbers $\lambda_m^{(\pm)}$ are the $2N$ eigenvectors, respectively the $2N$ eigenvalues, of the following (time-independent) generalized matrix-vector eigenvalue problem:

$$\left(\lambda^2 \underline{I} + \lambda \underline{U} + \underline{V}\right) \underline{v} = 0. \tag{2.6a}$$

Here of course $\underline{I}$ is the $N \times N$ unit matrix ($I_{mn} = \delta_{m,n}$ where, here and below, $\delta_{m,n}$ is the Kronecker symbol) and the two $N \times N$ matrices $\underline{U}$ and $\underline{V}$ are defined componentwise as follows:

$$\begin{aligned} U_{mn} &= (N+1-m)(N+2-m)G_0 \delta_{n,m-2} + (N+1-m)[D_0 + (N-m)G_1]\delta_{n,m-1} \\ &\quad + \{E + (N-m)[D_1 + (N-1-m)G_2]\}\delta_{n,m} \\ &\quad - m[D_2 + (2N-3-m)G_3]\delta_{n,m+1}, \qquad n,m = 1,\dots,N; \end{aligned} \tag{2.6b}$$

$$\begin{aligned} V_{mn} &= (N+3-m)(N+2-m)(N+1-m)F_0 \delta_{n,m-3} \\ &\quad + (N+2-m)(N+1-m)[A_0 + (N-m)F_1]\delta_{n,m-2} \\ &\quad + (N+1-m)\big\{(N-m)[A_1 + (N-1-m)F_2] + B_0\big\}\delta_{n,m-1} \\ &\quad - m\big\{(2N-1-m)A_2 + B_1 + \big[3N^2 - 6N + 2 - 3(N-1)m + m^2\big]F_3\big\}\delta_{n,m} \\ &\quad + m(m+1)[A_3 + (3N-5-m)F_4]\delta_{n,m+1} \qquad n,m = 1,\dots,N. \end{aligned} \tag{2.6c}$$

This implies of course that the $2N$ eigenvalues $\lambda_m^{(\pm)}$ are the $2N$ roots of the following polynomial equation (of degree $2N$ in λ):

$$\det\left(\lambda^2 \underline{I} + \lambda \underline{U} + \underline{V}\right) = 0. \tag{2.6d}$$

These findings show that the solution of the system (2.4) is achieved by the *algebraic* operation of determining the eigenvalues and eigenvectors of the matrix-vector generalized eigenvalue problem (2.6a).

The algebraic equation (2.6d) can be *explicitly* solved (for arbitrary N) in the two special cases in which the two $N \times N$ matrices $\underline{U}$ and $\underline{V}$ are either both *upper triangular* or both *lower triangular*.

The *first* of these two special cases obtains if, of the 19 parameters in (2.4), the following 4 vanish:

$$A_3 = D_2 = F_4 = G_3 = 0. \tag{2.7}$$

The *second* of these two special cases obtains if instead, of the 19 parameters in (2.4), the following 9 vanish:

$$A_0 = A_1 = B_0 = D_0 = F_0 = F_1 = F_2 = G_0 = G_1 = 0. \tag{2.8}$$

It is then easily seen that – in both these two cases – the $2N$ eigenvalues $\lambda_m^{(\pm)}$ read

$$\lambda_m^{(\pm)} = -\frac{1}{2}\big\{E + (N-m)[D_1 + (N-1-m)G_2] \pm \Delta_m\big\}, \tag{2.9a}$$

$$\Delta_m^2 = \{E + (N-m)[D_1 + (N-1-m)G_2]\}^2 + 4m\big\{(2N-1-m)A_2 + B_1 + \big[3N^2 - 6N + 2 - 3(N-1)m + m^2\big]F_3\big\}. \tag{2.9b}$$

These findings imply that – at least in these cases – specific predictions on the actual behavior of the solutions of the Newtonian N-body problem (2.1) can be easily made. In Section 4 we identify in particular the cases in which this Newtonian N-body problem is *isochronous* or *asymptotically isochronous*, or an appropriate variant of it is *isochronous*; while the findings described in this Section 2 are proven in the following Section 3.

3 Derivation of the equations of motion of the Newtonian N-body problem

Our task in this section is to obtain the equations of motion (2.1) characterizing the new N-body problem of *goldfish* type.

The starting point is the *linear* third-order PDE (2.3) satisfied by the polynomial (1.1). We already saw in Section 2 the implication of this evolution PDE for the coefficients $c_m(t)$ of its polynomial solution (1.1), leading to the identification – via (1.1) and (2.5) – of its solution by algebraic operations. In this section we show that the fact that the polynomial (1.1) satisfies the PDE (2.3) implies that its zeros $z_n(t)$ indeed evolve according to the Newtonian equations of motion (2.1).

This is in fact an immediate consequence – via trivial, if somewhat cumbersome, algebra – of (some of) the identities reported in Appendix A of [6] and of the additional identities (A.2) and (A.3) reported (and proven) in Appendix A of this paper, see below. Indeed these identities – valid for an *arbitrary* time-dependent polynomial $\psi(z,t)$ of degree N in z, see (1.1)) – allow to transform the *linear* PDE (2.3) satisfied by the polynomial $\psi(z,t)$ into the system of *nonlinear* ODEs (2.1) satisfied by its zeros $z_n(t)$. Note that the fact that this outcome obtains is both a *consequence* and a *confirmation* of the fact that the PDE (2.3) admits as its solution the polynomial $\psi(z,t)$, see (1.1), of degree N in z, hence featuring N zeros $z_n(t)$.

4 Isochronous and asymptotically isochronous cases

In this section we identify cases in which the Newtonian N-body problem (2.1) is *isochronous* or *asymptotically isochronous*, or variants of it are *isochronous*.

First of all we note, see (2.5), that if the $2N$ eigenvalues $\lambda_m^{(\pm)}$ are *all imaginary* and read

$$\lambda_m^{(\pm)} = i r_m^{(\pm)} \omega, \qquad r_m^{(\pm)} \equiv \frac{p_m^{(\pm)}}{q_m^{(\pm)}}, \qquad m = 1, \dots, N, \tag{4.1}$$

– with ω a *positive real* number, the N numbers $q_m^{(\pm)}$ *all positive integers*, the N numbers $p_m^{(\pm)}$ *all integers* (positive, negative or vanishing, with $p_m^{(\pm)}$ and $q_m^{(\pm)}$ coprimes and the $2N$ *real rational* numbers $r_m^{(\pm)} \equiv p_m^{(\pm)}/q_m^{(\pm)}$ *all different among themselves*) – then the N coefficients $c_m(t)$ evolve *isochronously*, i.e.

$$c_m(t+T) = c_m(t), \qquad m = 1, \ldots, N, \tag{4.2a}$$

with the period

$$T = \frac{2\pi q}{\omega} \tag{4.2b}$$

independent of the initial data. Here of course q is the *minimum common multiple* of the $2N$ denominators of the $2N$ *rational* numbers $r_m^{(\pm)} \equiv p_m^{(\pm)}/q_m^{(\pm)}$. And it is plain that the same property of *isochrony* is then shared by the N coordinates $z_n(t)$, namely the Newtonian N-body problem (2.1) is then *isochronous* as well; with the possibility that in some open regions of its phase space the periodicity only holds for a period which is a (generally *small*) *integer multiple* of T due to the fact that some of the zeros of the polynomial $\psi(z,t)$ – itself evolving *isochronously* with period T, see (1.1) – might "exchange their roles" over the time evolution (for an analysis of this phenomenology, also explaining the meaning of the assertion made above that the *integer multiple* in question is generally *small*, see [18]).

Let us then focus on the two cases – as identified at the end of Section 2, see (2.7) and (2.8) – in which the $2N$ eigenvalues $\lambda_m^{(\pm)}$ can be *explicitly* obtained, see (2.9). It is then easy to identify the additional restrictions on the parameters which are *necessary* – and also *sufficient*, up to some minor additional restrictions to exclude the coincidence of eigenvalues – to guarantee that the Newtonian N-body problem (2.1) be *isochronous* with period T. They read:

$$E = -2ir_1\omega, \qquad D_1 = -2ir_2\omega, \qquad G_2 = -2ir_3\omega, \tag{4.3a}$$

$$A_2 = a_2\omega^2, \qquad B_1 = b_1\omega^2, \qquad F_3 = f_3\omega^2, \tag{4.3b}$$

with

$$f_3 = 2r_3(s_2r_4 - R_2), \tag{4.3c}$$

$$a_2 = 2(s_1s_2 - 1)R_1r_3 - R_2^2 + r_4^2 - 3(N-1)f_3, \tag{4.3d}$$

$$b_1 = 2R_1(s_1r_4 - R_2) - (2N-1)a_2 - \big(3N^2 - 6N + 2\big)f_3, \tag{4.3e}$$

$$R_1 = r_1 + N[r_2 + (N-1)r_3], \qquad R_2 = r_2 + (2N-1)r_3. \tag{4.3f}$$

Here r_1, r_2, r_3, r_4 are 4 arbitrary *rational* numbers and s_1 and s_2 are two *arbitrary signs* ($+$ or $-$). Indeed with these assignments $\lambda_m^{(\pm)}$ clearly satisfies the condition (4.1) with

$$r_m^{(\pm)} = (1 \pm s_1)R_1 - (R_2 \pm r_4)m + (1 \pm 1)r_3m^2. \tag{4.4}$$

It is moreover plain that, if the $2N$ eigenvalues $\lambda_m^{(\pm)}$ satisfy, instead of the condition (4.1), the less restrictive condition

$$\lambda_m^{(\pm)} = \big(ir_m^{(\pm)} - \rho_m^{(\pm)}\big)\omega \tag{4.5}$$

– with ω again a *positive real* number, the $2N$ numbers $r_m^{(\pm)}$ again *all real* and *rational* and the $2N$ numbers $\rho_m^{(\pm)}$ *all real* and *nonnegative*, $\rho_m^{(\pm)} \geq 0$, with *at least one of them vanishing* – then the N coefficients $c_m(t)$ are *asymptotically isochronous*,

$$\lim_{t\to\infty}\big[c_m(t) - c_m^{(\text{asy})}(t)\big] = 0, \tag{4.6a}$$

with $c_m^{(\mathrm{asy})}(t)$ periodic,

$$c_m^{(\mathrm{asy})}(t+T) = c_m^{(\mathrm{asy})}(t). \tag{4.6b}$$

Here T is given again by (4.2b), but now with q being the *minimum common multiple* of the denominators $q_m^{(\pm)}$'s of the rationals $r_m^{(\pm)}$'s associated with *vanishing* $\rho_m^{(\pm)}$'s, $\rho_m^{(\pm)} = 0$, see (4.5) and (4.1). And it is again plain that the same property is then shared by the N coordinates $z_n(t)$, namely that the Newtonian N-body problem (2.1) is then *asymptotically isochronous* as well (again, with a period which might be a, generally *small*, *integer multiple* of T [18]).

Next, less us investigate the cases in which – via a well-known trick, see for instance Section 2.1 (entitled "The trick") of [6] – *isochronous* variants can be manufactured of the Newtonian N-body problem (2.1). To this end, it is convenient to re-write the equations of motion (2.1) via the *formal* replacement of dependent and independent variables $z_n(t) \Rightarrow \zeta_n(\tau)$, so that the equations of motion read as follows:

$$\begin{aligned}
\zeta_n'' + E\zeta_n' = {} & B_0 + B_1\zeta_n - (N-1)[2A_3 + 3(N-2)F_4]\zeta_n^2 + (N-1)(N-2)G_3\zeta_n'\zeta_n \\
& + \sum_{\ell=1;\,\ell\neq n}^{N} \big\{(\zeta_n-\zeta_\ell)^{-1}\big[2\zeta_n'\zeta_\ell' + 2\big(A_0 + A_1\zeta_n + A_2\zeta_n^2 + A_3\zeta_n^3\big) \\
& - \big(\zeta_n' + \zeta_\ell'\big)(D_0 + D_1\zeta_n) - D_2\zeta_n(\zeta_n'\zeta_\ell + \zeta_\ell'\zeta_n)\big]\big\} \\
& + \sum_{k,\ell=1;\,k\neq n,\,\ell\neq n,\,k\neq\ell}^{N} \Bigg[\frac{3(F_0 + F_1\zeta_n + F_2\zeta_n^2 + F_3\zeta_n^3 + F_4\zeta_n^4)}{(\zeta_n-\zeta_\ell)(\zeta_n-\zeta_k)} \\
& - \left(\frac{G_0 + G_1\zeta_n + G_2\zeta_n^2 + G_3\zeta_n^3}{\zeta_n-\zeta_\ell}\right)\left(\frac{\zeta_n'+\zeta_k'}{\zeta_n-\zeta_k} + \frac{\zeta_\ell'+\zeta_k'}{\zeta_\ell-\zeta_k}\right)\Bigg], \qquad n = 1,\dots,N.
\end{aligned} \tag{4.7}$$

Here and hereafter (in this section) appended primes denote differentiation with respect to the variable τ.

We now perform the following change of dependent and independent variables ("the trick"):

$$z_n(t) = \exp(i\alpha\omega t)\zeta_n(\tau), \qquad \tau \equiv \tau(t) = \frac{\exp(i\omega t) - 1}{i\omega}. \tag{4.8a}$$

Here and below ω is a positive constant, and α a nonvanishing number that we reserve to assign, see below.

It is then plain that there hold the following formulas:

$$\dot z_n - i\alpha\omega z_n = \exp[i(\alpha+1)\omega t]\zeta_n', \tag{4.8b}$$

$$\ddot z_n - (2\alpha+1)i\omega\dot z_n - \alpha(\alpha+1)\omega^2 z_n = \exp[i(\alpha+2)\omega t]\zeta_n''. \tag{4.8c}$$

And via these formulas one can easily obtain the equations of motion implied for the dependent variables $z_n(t)$ by the equations of motion (4.7) satisfied by the variables $\zeta_n(\tau)$, and thereby ascertain for which assignments of the parameter α, and for which corresponding restrictions on the 19 coupling constants featured by these equations of motion, the resulting equations of motion satisfied by the N coordinates $z_n(t)$ are *autonomous*, i.e. they feature no *explicit* time-dependence. We list below all these cases, on the understanding that *all* the coupling constants which do *not* appear in these new equations of motion have been set to *zero* (while those that do appear are *arbitrary*).

For $\alpha = -2$, these Newtonian equations of motion read

$$\ddot z_n + 3i\omega\dot z_n - 2\omega^2 z_n = B_0 + 2\sum_{\ell=1;\,\ell\neq n}^{N} \frac{(\dot z_n + 2i\omega z_n)(\dot z_\ell + 2i\omega z_\ell)}{z_n - z_\ell}$$

$$+ \sum_{\ell,k=1;\, \ell\neq n,\, k\neq n,\, \ell\neq k}^{N} \frac{3F_2 z_n^2}{(z_n - z_\ell)(z_n - z_k)}. \tag{4.9}$$

For $\alpha = -1$, these Newtonian equations of motion read

$$\begin{aligned} \ddot{z}_n + i\omega\dot{z}_n = & \sum_{\ell=1;\, \ell\neq n}^{N} \frac{2(\dot{z}_n + i\omega z_n)(\dot{z}_\ell + i\omega z_\ell) + 2A_0 - D_0[\dot{z}_n + \dot{z}_\ell + i\omega(z_n + z_\ell)]}{z_n - z_\ell} \\ & + \sum_{\ell,k=1;\, \ell\neq n,\, k\neq n,\, \ell\neq k}^{N} \left[\frac{3F_1 z_n}{(z_n - z_\ell)(z_n - z_k)} \right. \\ & \left. - \frac{G_1 z_n}{z_n - z_\ell} \left(\frac{\dot{z}_n + \dot{z}_k + i\omega(z_n + z_k)}{z_n - z_k} + \frac{\dot{z}_\ell + \dot{z}_k + i\omega(z_\ell + z_k)}{z_\ell - z_k} \right) \right]. \end{aligned} \tag{4.10}$$

For $\alpha = -2/3$, these Newtonian equations of motion read

$$\begin{aligned} \ddot{z}_n + \frac{i\omega}{3}\dot{z}_n + \frac{2}{9}\omega^2 z_n = 2 & \sum_{\ell=1;\, \ell\neq n}^{N} \frac{(\dot{z}_n + 2i\omega z_n/3)(\dot{z}_\ell + 2i\omega z_\ell/3)}{z_n - z_\ell} \\ & + \sum_{\ell,k=1;\, \ell\neq n,\, k\neq n,\, \ell\neq k}^{N} \frac{3F_0}{(z_n - z_\ell)(z_n - z_k)}. \end{aligned} \tag{4.11}$$

For $\alpha = -1/2$, these Newtonian equations of motion read

$$\begin{aligned} \ddot{z}_n + \frac{\omega^2}{4} z_n = 2 & \sum_{\ell=1;\, \ell\neq n}^{N} \frac{(\dot{z}_n + i\omega z_n/2)(\dot{z}_\ell + i\omega z_\ell/2)}{z_n - z_\ell} \\ & - \sum_{\ell,k=1;\, \ell\neq n,\, k\neq n,\, \ell\neq k}^{N} \left[\frac{G_0}{z_n - z_\ell} \left(\frac{\dot{z}_n + \dot{z}_k + i\omega(z_n + z_k)/2}{z_n - z_k} \right. \right. \\ & \left. \left. + \frac{\dot{z}_\ell + \dot{z}_k + i\omega(z_\ell + z_k)/2}{z_\ell - z_k} \right) \right]. \end{aligned} \tag{4.12}$$

For $\alpha = 1$, these Newtonian equations of motion read

$$\begin{aligned} & \ddot{z}_n - 3i\omega\dot{z}_n - 2\omega^2 z_n = (N-1)(N-2)G_3(\dot{z}_n - i\omega z_n) z_n \\ & \quad + \sum_{\ell=1;\, \ell\neq n}^{N} \left[(z_n - z_\ell)^{-1} \left\{ 2(\dot{z}_n - i\omega z_n)(\dot{z}_\ell - i\omega z_\ell) \right. \right. \\ & \quad \left. \left. - D_2 z_n [(\dot{z}_n - i\omega z_n) z_\ell + (\dot{z}_\ell - i\omega z_\ell) z_n] \right\} \right] \\ & \quad - \sum_{\ell,k=1;\, \ell\neq n,\, k\neq n,\, \ell\neq k}^{N} \left[\frac{G_3 z_n^3}{z_n - z_\ell} \left(\frac{\dot{z}_n + \dot{z}_k - i\omega(z_n + z_k)}{z_n - z_k} + \frac{\dot{z}_\ell + \dot{z}_k - i\omega(z_\ell + z_k)}{z_\ell - z_k} \right) \right]. \end{aligned} \tag{4.13}$$

For $\alpha = 2$, these Newtonian equations of motion read

$$\begin{aligned} & \ddot{z}_n - 5i\omega\dot{z}_n - 6\omega^2 z_n = -(N-1)[2A_3 + 3(N-2)F_4] z_n^2 \\ & \quad + \sum_{\ell=1;\, \ell\neq n}^{N} \frac{2(\dot{z}_n - 2i\omega z_n)(\dot{z}_\ell - 2i\omega z_\ell) + 2A_3 z_n^3}{z_n - z_\ell} + \sum_{\ell,k=1;\, \ell\neq n,\, k\neq n,\, \ell\neq k}^{N} \frac{3F_4 z_n^4}{(z_n - z_\ell)(z_n - z_k)}. \end{aligned} \tag{4.14}$$

And it is plain that *all* these Newtonian models are *isochronous*: see (4.8a), and note that the solutions $\zeta_n(\tau)$ of the *solvable* N-body model (4.7) have at most *algebraic* singularities as functions of τ.

5 Outlook

The search for more general *solvable* models of goldfish type should continue: they might still yield interesting results.

Another development likely to yield interesting findings is the investigation of the behavior, in the infinitesimal neighborhood of its equilibria, of the N-body model introduced above, especially in the *isochronous* cases. This investigation might yield new *Diophantine* findings for the zeros of interesting polynomials. We plan to pursue these results, which shall eventually be submitted to a journal devoted to special functions if they turn out to be sufficiently interesting to justify their publication.

A Appendix: identities involving the zeros of a polynomial

In this Appendix we report (and then prove) several *identities* for the time-dependent polynomial $\psi \equiv \psi(z,t)$, see (1.1), of degree N in z. We of course use hereafter the notation introduced in Section 2, see Notation 2.1, and in addition the following convenient shorthand notation: as in [6] (see there equations (A.4) and (A.5))

$$D\psi \Longleftrightarrow F_n(\underline{z},\underline{\dot{z}}) \tag{A.1a}$$

– with D a differential operator acting on the independent variables z and t of the polynomial $\psi(z,t)$, see (1.1) – stands for the *identity*

$$D\psi(z,t) = \psi(z,t)\sum_{n=1}^{N}[z-z_n(t)]^{-1}F_n(\underline{z},\underline{\dot{z}}). \tag{A.1b}$$

Below we often, for notational simplicity, omit to indicate explicitly the dependence on their arguments of $\psi \equiv \psi(z,t)$, $z_n \equiv z_n(t)$ and $c_m(t)$ (see (1.1)).

We now list the following identities, which complement those reported in Appendix A of [6] (see in particular the 2012 paperback version, where the formulas denoted in [6] as (A.8k) and (A.8l) are corrected):

$$z^p\psi_{zzz} \Longleftrightarrow 3z_n^p \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq \ell}^{N} \left[(z_n-z_k)^{-1}(z_n-z_\ell)^{-1}\right], \qquad p=0,1,2, \tag{A.2a}$$

$$z^3\psi_{zzz} - N(N-1)(N-2)\psi \Longleftrightarrow 3z_n^3 \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq \ell}^{N} \left[(z_n-z_k)^{-1}(z_n-z_\ell)^{-1}\right], \tag{A.2b}$$

$$\begin{aligned} &z^4\psi_{zzz} + (N-1)(N-2)(3c_1 - Nz)\psi \\ &\qquad \Longleftrightarrow 3z_n^4 \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq \ell}^{N} \left[(z_n-z_k)^{-1}(z_n-z_\ell)^{-1}\right]; \end{aligned} \tag{A.2c}$$

$$z^p\psi_{zzt} \Longleftrightarrow -\sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq \ell}^{N} \left[\left(\frac{z_n^p}{z_n-z_k}\right)\left(\frac{\dot{z}_n+\dot{z}_\ell}{z_n-z_\ell}+\frac{\dot{z}_k+\dot{z}_\ell}{z_k-z_\ell}\right)\right], \qquad p=0,1,2, \tag{A.3a}$$

$$\begin{aligned} &z^3\psi_{zzt} - (N-1)(N-2)z\psi_t \Longleftrightarrow (N-1)(N-2)\dot{z}_n z_n \\ &\qquad - \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq \ell}^{N} \left[\left(\frac{z_n^3}{z_n-z_k}\right)\left(\frac{\dot{z}_n+\dot{z}_\ell}{z_n-z_\ell}+\frac{\dot{z}_k+\dot{z}_\ell}{z_k-z_\ell}\right)\right]. \end{aligned} \tag{A.3b}$$

Of course in the last two, (A.3a) and (A.3b), superimposed dots indicate t-differentiations. Also note that in the third, (A.2c), of these identities, consistently with (1.1)

$$c_1 \equiv c_1(t) = -\sum_{n=1}^{N}[z_n(t)]. \tag{A.4}$$

But remarkably – due to a neat cancellation, see (A.2c) and equation (A.6b) of [6] – this quantity does not appear in the equations of motion (2.1).

Our task in this Appendix is to prove these *identities*. To perform these proofs it is convenient to introduce the following shorthand notation denoting sums over (dummy) indices restricted to take *different* values (among themselves):

$$\sum\nolimits'_{nk} \equiv \sum_{n,k=1;\, n\neq k}^{N}; \qquad \sum\nolimits'_{nk\ell} \equiv \sum_{n,k,\ell=1;\, n\neq k,\, k\neq \ell,\, \ell\neq n}^{N}. \tag{A.5}$$

We moreover introduce the following shorthand notation for the "denominator" $\operatorname{den}(z_n,z_k,z_\ell)$,

$$\operatorname{den}(z_n, z_k, z_\ell) \equiv (z_n - z_k)(z_n - z_\ell)(z_k - z_\ell), \tag{A.6a}$$

which clearly has the property to be *invariant* under the cyclic exchange $n \to k \to \ell$ of the three indices n, k, ℓ,

$$\operatorname{den}(z_n, z_k, z_\ell) = \operatorname{den}(z_k, z_\ell, z_n), \tag{A.6b}$$

and to be instead *antisymmetric* under the exchange of any two of its three arguments z_n, z_k, z_ℓ,

$$\operatorname{den}(z_n, z_k, z_\ell) = -\operatorname{den}(z_k, z_n, z_\ell) = -\operatorname{den}(z_\ell, z_k, z_n) = -\operatorname{den}(z_n, z_\ell, z_k). \tag{A.6c}$$

Likewise, we denote corresponding "numerators" as $\operatorname{num}(n, k, \ell)$ (i.e., $\operatorname{num}(z_n, z_k, z_\ell) \equiv \operatorname{num}(n, k, \ell)$) and, whenever one of them is the sum of an *arbitrary* number of terms $\operatorname{num}_j(n, k, \ell)$, i.e.

$$\operatorname{num}(n, k, \ell) \equiv \sum_{j=0} \operatorname{num}_j(n, k, \ell), \tag{A.7}$$

we take advantage (if need be) of the following

Lemma A.1. *If each of the addends* $\operatorname{num}_j(n, k, \ell)$ *in the right-hand side of* (A.7) *is invariant under the exchange of any two of the three indices* n, k, ℓ, *namely if each of the addends* $\operatorname{num}_j(n, k, \ell)$ *satisfies at least one of the following three relations:*

$$\operatorname{num}_j(n, k, \ell) = \operatorname{num}_j(k, n, \ell), \tag{A.8a}$$

or

$$\operatorname{num}_j(n, k, \ell) = \operatorname{num}_j(\ell, k, n), \tag{A.8b}$$

or

$$\operatorname{num}_j(n, k, \ell) = \operatorname{num}_j(n, \ell, k), \tag{A.8c}$$

then the triple sum $\sum'_{nk\ell}[\operatorname{num}(n, k, \ell)/\operatorname{den}(z_n, z_k, z_\ell)]$ *vanishes:*

$$\begin{aligned}\sum\nolimits'_{nk\ell}\left[\frac{\operatorname{num}(n,k,\ell)}{\operatorname{den}(z_n,z_k,z_\ell)}\right] &\equiv \sum\nolimits'_{nk\ell}\left\{\sum_{j=0}\left[\frac{\operatorname{num}_j(n,k,\ell)}{\operatorname{den}(z_n,z_k,z_\ell)}\right]\right\}\\ &= \sum_{j=0}\left\{\sum\nolimits'_{nk\ell}\left[\frac{\operatorname{num}_j(n,k,\ell)}{\operatorname{den}(z_n,z_k,z_\ell)}\right]\right\} = 0.\end{aligned} \tag{A.9}$$

The validity of this assertion is an obvious consequence of the *antisymmetry* – see (A.6c) and (A.8) – under the exchange of two appropriately chosen dummy indices in the triple sums $\sum'_{nk\ell}[\mathrm{num}_j(n,k,\ell)/\,\mathrm{den}(z_n,z_k,z_\ell)]$, which therefore vanish (for all values of j).

Next, let us report and prove the following identities, valid for any set of N arbitrary numbers z_n (for convenience, we always assume them to be *all different among themselves*).

$$\sum\nolimits'_{nk}\left(\frac{z_n}{z_n-z_k}\right)=\frac{N(N-1)}{2}, \tag{A.10a}$$

$$\sum\nolimits'_{nk\ell}\left(\frac{z_k}{z_k-z_\ell}\right)=\frac{N(N-1)(N-2)}{2}, \tag{A.10b}$$

$$\sum\nolimits'_{nk\ell}\left[\frac{z_n^p}{(z_n-z_k)(z_n-z_\ell)}\right]=0, \qquad p=0,1, \tag{A.10c}$$

$$\sum\nolimits'_{nk\ell}\left[\frac{z_n^2}{(z_n-z_k)(z_n-z_\ell)}\right]=\frac{N(N-1)(N-2)}{3}, \tag{A.10d}$$

$$\sum\nolimits'_{nk\ell}\left(\frac{z_nz_k}{z_k-z_\ell}\right)=-\frac{1}{2}(N-1)(N-2)c_1, \tag{A.10e}$$

$$\sum\nolimits'_{nk\ell}\left(\frac{z_n^2}{z_n-z_\ell}\right)=-(N-1)(N-2)c_1, \tag{A.10f}$$

$$\sum\nolimits'_{nk\ell}\left[\frac{z_n^3}{(z_n-z_k)(z_n-z_\ell)}\right]=-(N-1)(N-2)c_1. \tag{A.10g}$$

In the last three formulas of course c_1 is defined by (A.4).

The proof of the (well-known) identity (A.10a) is trivial:

$$\sum\nolimits'_{nk}\left(\frac{z_n}{z_n-z_k}\right)=\frac{1}{2}\sum\nolimits'_{nk}\left(\frac{z_n-z_k}{z_n-z_k}\right)=\frac{N(N-1)}{2}. \tag{A.11}$$

The first step is justified by adding to the left-hand side of (A.10a) what is obtained by the exchange of the dummy indices k and ℓ (which does not change the result) and dividing by 2 (this operation is often repeated below without describing it in as much detail as done here); the second step is immediately implied by the first definition (A.5).

Then the proof of (A.10b) is no less trivial: it follows from (A.10a) and the second definition (A.5).

The proof of (A.10c) goes as follows. For $p=0$ by adding to the right-hand side of this formula the sums obtained by performing *sequentially* two cyclic transformations of the three indices n, k, ℓ and by taking advantage of the invariance property (A.6a)) – and by dividing the sum of the three equal sums thereby obtained by 3 – we clearly get

$$\sum\nolimits'_{nk\ell}\left[\frac{1}{(z_n-z_k)(z_n-z_\ell)}\right]=\frac{1}{3}\sum\nolimits'_{nk\ell}\left[\frac{(z_k-z_\ell)+(z_\ell-z_n)+(z_n-z_k)}{\mathrm{den}(z_n,z_k,z_\ell)}\right], \tag{A.12}$$

and it is then plain that this quantity vanishes. For $p=1$ via (A.6a)

$$\sum\nolimits'_{nk\ell}\left[\frac{z_n}{(z_n-z_k)(z_n-z_\ell)}\right]=\sum\nolimits'_{nk\ell}\left[\frac{z_nz_k-z_nz_\ell}{\mathrm{den}(z_n,z_k,z_\ell)}\right] \tag{A.13}$$

and it is plain that this quantity vanishes thanks to Lemma A.1, since in the right-hand side the first term in the numerator is invariant under the exchange of dummy indices $n\leftrightarrow k$ and in the second under the exchange $n\leftrightarrow\ell$.

The proof of (A.10d) goes through the following steps:

$$\sum\nolimits'_{nk\ell}\left[\frac{z_n^2}{(z_n-z_k)(z_n-z_\ell)}\right]=\sum\nolimits'_{nk\ell}\left[\frac{z_n^2}{(z_n-z_k)(z_n-z_\ell)}-\frac{z_k}{z_k-z_\ell}\right]+\sum\nolimits'_{nk\ell}\frac{z_k}{z_k-z_\ell}$$

$$= -\sum\nolimits'_{nk\ell}\left[\frac{z_k^2 z_\ell - (z_n^2 z_k + z_k^2 z_n - z_n z_k z_\ell)}{\mathrm{den}(z_n,z_k,z_\ell)}\right] + \frac{N(N-1)(N-2)}{2}$$
$$= -\sum\nolimits'_{nk\ell}\left[\frac{z_k^2 z_\ell}{\mathrm{den}(z_n,z_k,z_\ell)}\right] + \frac{N(N-1)(N-2)}{2}$$
$$= \sum\nolimits'_{nk\ell}\left[\frac{z_n^2 z_\ell}{\mathrm{den}(z_n,z_k,z_\ell)}\right] + \frac{N(N-1)(N-2)}{2}$$
$$= -\frac{1}{2}\sum\nolimits'_{nk\ell}\left[\frac{z_n^2}{(z_n-z_k)(z_n-z_\ell)}\right] + \frac{N(N-1)(N-2)}{2}, \qquad \text{(A.14)}$$

the first of which is quite trivial, the second is given by standard algebra together with the definition (A.6a) and the formula (A.10b) just proven, the third of which is implied by the formula (A.9) (since $z_n^2 z_k + z_k^2 z_n - z_n z_k z_\ell$ is symmetrical under the exchange of dummy indices $n \leftrightarrow k$), the fourth of which obtains via the exchange $n \leftrightarrow k$ (see (A.6c)), and the last of which – which clearly implies (A.10d) – is clearly implied by the property (A.6c) and by the definition (A.6a) of $\mathrm{den}(z_n,z_k,z_\ell)$.

The proof of (A.10e) is again quite trivial:

$$\sum\nolimits'_{nk\ell}\left(\frac{z_n z_k}{z_k - z_\ell}\right) = \frac{1}{2}\sum\nolimits'_{nk\ell}(z_n) = -\frac{(N-1)(N-2)}{2}c_1, \qquad \text{(A.15)}$$

the first step being implied by the antisymmetry of the summand under the exchange of the dummy indices $k \leftrightarrow \ell$ and the second by the definitions of the symbol $\sum'_{nk\ell}$ (see (A.5)) and of c_1 (see (A.4)).

The proof of (A.10f) is analogous:

$$\sum\nolimits'_{nk\ell}\left(\frac{z_n^2}{z_n - z_\ell}\right) = \frac{1}{2}\sum\nolimits'_{nk\ell}(z_n + z_\ell) = \sum\nolimits'_{nk\ell}(z_n) = -(N-1)(N-2)c_1, \qquad \text{(A.16)}$$

with the first step justified by the replacement $z_n^2 \to (z_n^2 - z_\ell^2)/2$ due to the antisymmetry of the summand under the exchange of dummy indices $n \leftrightarrow \ell$, the second step justified by the symmetry of the summand, and the third step justified as just above.

Finally, the proof of (A.10g) is analogous:

$$\sum\nolimits'_{nk\ell}\left[\frac{z_n^3}{(z_n-z_k)(z_n-z_\ell)}\right] = \sum\nolimits'_{nk\ell}\left[\frac{z_n^3}{(z_n-z_k)(z_n-z_\ell)} - \frac{z_n^2}{z_k - z_\ell}\right]$$
$$= -\sum\nolimits'_{nk\ell}\left[\frac{z_n^2(z_n-z_k)}{(z_n-z_\ell)(z_k-z_\ell)}\right] = -\frac{1}{2}\sum\nolimits'_{nk\ell}\left[\frac{(z_n^2-z_k^2)(z_n-z_k)}{(z_n-z_\ell)(z_k-z_\ell)}\right]$$
$$= \frac{1}{2}\sum\nolimits'_{nk\ell}\left(\frac{z_n^2-z_k^2}{z_n-z_\ell} - \frac{z_n^2-z_k^2}{z_k-z_\ell}\right) = \frac{1}{2}\sum\nolimits'_{nk\ell}\left(\frac{z_n^2}{z_n-z_\ell} + \frac{z_k^2}{z_k-z_\ell}\right)$$
$$= \sum\nolimits'_{nk\ell}\left(\frac{z_n^2}{z_n-z_\ell}\right) = -(N-1)(N-2)c_1. \qquad \text{(A.17)}$$

Here the first step is justified by the vanishing of (the sum over) the added term (due to the antisymmetry of the summand under the exchange of dummy indices $k \leftrightarrow \ell$), the second step follows by trivial algebra, the third step is justified by the symmetry of the summand under the exchange of dummy indices $n \leftrightarrow k$, the fourth step is justified by the identity

$$-\frac{z_n - z_k}{(z_n-z_\ell)(z_k-z_\ell)} = \frac{1}{z_n-z_\ell} - \frac{1}{z_k-z_\ell}, \qquad \text{(A.18)}$$

the fifth step by the elimination of two addends antisymmetric under the exchanges of dummy indices $n \leftrightarrow \ell$ respectively $k \leftrightarrow \ell$, the sixth step by the symmetry of the summand under the exchange of dummy indices $n \leftrightarrow \ell$, and the last step by (A.10f).

Now we can finally proceed and prove the formulas (A.2) and (A.3).

We start from reporting equations (A.1) (which coincides with (1.1)), (A.2), (A.3), (A.8a) and (A.9a) of [6]:

$$\psi = \prod_{n=1}^{N} [z - z_n] = z^N + \sum_{m=1}^{N} c_m z^{N-m} = \sum_{m=0}^{N} c_m z^{N-m}, \qquad c_0 = 1, \tag{A.19}$$

$$\psi_z = \psi \sum_{n=1}^{N} (z - z_n)^{-1}, \tag{A.20}$$

$$\psi_t = -\psi \sum_{n=1}^{N} (z - z_n)^{-1} \dot{z}_n, \tag{A.21}$$

$$\psi_{zz} = 2\psi \sum_{n=1}^{N} \left[(z - z_n)^{-1} \sum_{\ell=1;\, \ell\neq n}^{N} (z_n - z_\ell)^{-1} \right], \tag{A.22}$$

$$\psi_{zt} = -\psi \sum_{n=1}^{N} \left\{ (z - z_n)^{-1} \sum_{\ell=1;\, \ell\neq n}^{N} \left[(\dot{z}_n + \dot{z}_\ell)(z_n - z_\ell)^{-1} \right] \right\}. \tag{A.23}$$

The last two equations correspond of course to (A8.a) and (A.9a) via the convention defining the symbol $\Longleftrightarrow$, see (A.1).

Partial differentiation of (A.22) with respect to z gives

$$\psi_{zzz} = 2\psi_z \sum_{n=1}^{N} (z - z_n)^{-1} \sum_{\ell=1;\, \ell\neq n}^{N} (z_n - z_\ell)^{-1} - 2\psi \sum_{n=1}^{N} (z - z_n)^{-2} \sum_{\ell=1;\, \ell\neq n}^{N} (z_n - z_\ell)^{-1}. \tag{A.24a}$$

Via (A.20) this becomes (after a convenient cancellation and change of dummy index from n to k)

$$\psi_{zzz} = 2\psi \sum_{n=1}^{N} \left[(z - z_n)^{-1} \sum_{k=1;\, k\neq n}^{N} (z - z_k)^{-1} \sum_{\ell=1;\, \ell\neq n}^{N} (z_n - z_\ell)(z_n - z_\ell) \right]. \tag{A.24b}$$

We then use the identity

$$(z - z_n)^{-1}(z - z_k)^{-1} = (z_n - z_k)^{-1}\left[(z - z_n)^{-1} - (z - z_k)^{-1} \right], \tag{A.25}$$

getting thereby

$$\psi_{zzz} = 2\psi \sum_{n,k=1;\, k\neq n}^{N} \left\{ \left[(z - z_n)^{-1} - (z - z_k)^{-1} \right] (z_n - z_k)^{-1} \sum_{\ell=1;\, \ell\neq n}^{N} (z_n - z_\ell)^{-1} \right\}. \tag{A.26a}$$

We then exchange the dummy indices n and k in the second of the two sums over these indices, getting thereby

$$\psi_{zzz} = 2\psi \sum_{n,k=1;\, k\neq n}^{N} \left\{ \frac{1}{(z - z_n)(z_n - z_k)} \left[\sum_{\ell=1;\, \ell\neq n}^{N} \left(\frac{1}{z_n - z_\ell} \right) + \sum_{\ell=1;\, \ell\neq k}^{N} \left(\frac{1}{z_k - z_\ell} \right) \right] \right\}, \tag{A.26b}$$

which can also be written as follows:

$$\psi_{zzz} = 2\psi \sum_{n=1}^{N} \left\{ \left(\frac{1}{z - z_n} \right) \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq \ell}^{N} \left[\left(\frac{1}{z_n - z_k} \right) \left(\frac{1}{z_n - z_\ell} + \frac{1}{z_k - z_\ell} \right) \right] \right\}. \tag{A.26c}$$

This implies (using the definition of the symbols $\Leftrightarrow$ and $\sum'_{nk\ell}$, see above)

$$\psi_{zzz} \iff 2 \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left\{ (z_n - z_k)^{-1} \left[(z_n - z_\ell)^{-1} + (z_k - z_\ell)^{-1} \right] \right\}, \tag{A.27a}$$

$$\psi_{zzz} \iff 2 {\sum}'_{nk\ell} \left[(z_n + z_k - 2z_\ell)(z_k - z_\ell)^{-1}(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right]; \tag{A.27b}$$

and finally, taking advantage of the fact that the part of the summand antisymmetric under the exchange of the two dummy indices k and ℓ can be eliminated – so that $(z_n + z_k - 2z_\ell)/(z_k - z_\ell) \equiv 3/2 + (2z_n - z_k - z_\ell)/[2(z_k - z_\ell)]$ can be replaced by $3/2$ – one gets (A.2a) with $p = 0$, which is thereby proven. This is the first of the new formulas analogous to those reported in Appendix A of [6].

To proceed it is convenient to write in longhand the formula we just proved:

$$\psi_{zzz} = \psi \sum_{n=1}^{N} \left\{ 3(z - z_n)^{-1} \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left[(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \right\}. \tag{A.28}$$

Multiplication by z^p then yields

$$z^p \psi_{zzz} = \psi \sum_{n=1}^{N} \left\{ 3z^p (z - z_n)^{-1} \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left[(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \right\}. \tag{A.29}$$

By replacing the z^p in the numerator with $z_n^p + (z^p - z_n^p)$ we get

$$\begin{aligned} z^p \psi_{zzz} = {} & \psi \sum_{n=1}^{N} \left\{ \frac{3z_n^p}{z - z_n} \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left[(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \right\} \\ & + 3\psi {\sum}'_{nk\ell} \left[\frac{z^p - z_n^p}{z - z_n} (z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \end{aligned} \tag{A.30}$$

(see (A.5)). It is then plain that, for $p = 1$ and $p = 2$, the formula (A.10c) implies that the last sum in this equation vanishes. Hence (A.2a) is now proven also for $p = 1$ and $p = 2$.

To prove (A.2b) we set $p = 3$ in (A.30), which then reads

$$\begin{aligned} z^3 \psi_{zzz} = {} & \psi \sum_{n=1}^{N} \left\{ \frac{3z_n^3}{z - z_n} \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left[(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \right\} \\ & + 3\psi {\sum}'_{nk\ell} \left[(z^2 + zz_n + z_n^2)(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \end{aligned} \tag{A.31}$$

(via the identity $z^3 - z_n^3 = (z - z_n)(z^2 + zz_n + z_n^2)$). And it is then plain that this formula, via (A.10c) and (A.10d), yields (A.2b), which is thereby proven.

Likewise, to prove (A.2c) we set $p = 4$ in (A.30), which then reads

$$\begin{aligned} z^4 \psi_{zzz} = {} & \psi \sum_{n=1}^{N} \left\{ 3(z - z_n)^{-1} z_n^4 \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, k\neq\ell}^{N} \left[(z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right] \right\} \\ & + 3\psi {\sum}'_{nk\ell} \left[\left(z^3 + z^2 z_n + z z_n^2 + z_n^3 \right) (z_n - z_k)^{-1}(z_n - z_\ell)^{-1} \right]. \end{aligned} \tag{A.32}$$

Then it is easily seen that, via (A.10c), (A.10d) and (A.10g), this equation yields (A.2c), which is thereby proven.

Next, to prove (A.3a) (to begin with, with $p=0$), we z-differentiate (A.23) getting thereby

$$\psi_{zzt} = -\psi_z \sum_{n=1}^{N} \left[(z-z_n)^{-1} \sum_{\ell=1;\,\ell\neq n}^{N} \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} \right) \right] + \psi \sum_{n=1}^{N} \left[(z-z_n)^{-2} \sum_{k=1;\,k\neq n}^{N} \left(\frac{\dot z_n + \dot z_k}{z_n - z_k} \right) \right]$$
$$= -\psi \sum_{k=1}^{N} \left\{ (z-z_k)^{-1} \sum_{n=1;\,n\neq k}^{N} \left[(z-z_n)^{-1} \sum_{\ell=1;\,\ell\neq n}^{N} \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} \right) \right] \right\}$$
$$= -\psi {\sum}'_{nk\ell} \left\{ \left[(z-z_k)^{-1} - (z-z_n)^{-1} \right] (z_k - z_n)^{-1} \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} \right) \right\}$$
$$= -\psi \sum_{n=1}^{N} \frac{1}{z-z_n} \left\{ \sum_{k,\ell=1;\,k\neq n,\,\ell\neq n,\,\ell\neq k}^{k} \left[\frac{1}{z_n - z_k} \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} + \frac{\dot z_k + \dot z_\ell}{z_k - z_\ell} \right) \right] \right\}. \tag{A.33}$$

The first equality corresponds clearly to the differentiation of (A.23); the second obtains via (A.20) (also taking account of the cancellation occurring for $n=k$); the third equality obtains via the identity

$$(z-z_k)^{-1}(z-z_n)^{-1} = \left[(z-z_k)^{-1} - (z-z_n)^{-1} \right] (z_k - z_n)^{-1}; \tag{A.34}$$

and the fourth equality obtains via the exchange of dummy indices $k \leftrightarrow n$ in the sum containing the term $(z-z_k)^{-1}$ (and note the cancellations of the term with $\ell = n$ in the resulting sum, with the term with $\ell = k$ in the second sum, justifying the exclusion of the 3 addends with $k=n$, $\ell = n$ and $\ell = k$ in the sum in the last formula). The final result corresponds – of course, via the notation (A.1) – to (A.3a) with $p=0$, which is thereby proven.

To prove (A.3a) with $p=1$ and $p=2$ we start from the longhand version of the equation we just proved, multiplied by $z^p \equiv z_n^p + (z^p - z_n^p)$:

$$z^p \psi_{zzt} = -\psi {\sum}'_{nk\ell} \left\{ \frac{z_n^p}{z-z_n} \left[(z_n - z_k)^{-1} \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} + \frac{\dot z_k + \dot z_\ell}{z_k - z_\ell} \right) \right] \right\}$$
$$- \psi {\sum}'_{nk\ell} \left\{ \left(\frac{z^p - z_n^p}{z - z_n} \right) \left[(z_n - z_k)^{-1} \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} + \frac{\dot z_k + \dot z_\ell}{z_k - z_\ell} \right) \right] \right\}. \tag{A.35}$$

From this formula it is plain that (A.3a) is proven also for $p=1$ and $p=2$ if, for these two values of p, there holds the *identity*

$$\sigma(p;\underline{z},\underline{\dot z}) \equiv {\sum}'_{nk\ell} \left[\left(\frac{z^p - z_n^p}{z - z_n} \right) \left(\frac{1}{z_n - z_k} \right) \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} + \frac{\dot z_k + \dot z_\ell}{z_k - z_\ell} \right) \right] = 0. \tag{A.36}$$

Note that we have now employed the convenient notation (A.5).

Indeed, for $p=1$,

$$\sigma(1;\underline{z},\underline{\dot z}) = {\sum}'_{nk\ell} \left[\left(\frac{1}{z_n - z_k} \right) \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} + \frac{\dot z_k + \dot z_\ell}{z_k - z_\ell} \right) \right]$$
$$= {\sum}'_{nk\ell} \left[\frac{\dot z_n z_k + \dot z_k z_n + \dot z_\ell (z_n + z_k) - (\dot z_n + \dot z_k + 2\dot z_\ell) z_\ell}{\mathrm{den}(z_n, z_k, z_\ell)} \right]. \tag{A.37}$$

The second equality obtains via trivial algebra and the definition (A.6a) of $\mathrm{den}(z_n, z_k, z_\ell)$; and it clearly implies that $\sigma(1;\underline{z},\underline{\dot z})$ vanishes via Lemma A.1, since the numerator of the summand in the sum in the right-hand side of the second equality is clearly invariant under the exchange of dummy indices $n \leftrightarrow k$.

Likewise, for $p=2$,

$$\sigma(2;\underline{z},\underline{\dot z}) = {\sum}'_{nk\ell} \left[\left(\frac{z + z_n}{z_n - z_k} \right) \left(\frac{\dot z_n + \dot z_\ell}{z_n - z_\ell} + \frac{\dot z_k + \dot z_\ell}{z_k - z_\ell} \right) \right] = z\sigma(1;\underline{z},\underline{\dot z})$$

$$
\begin{aligned}
&+ \sum\nolimits'_{nk\ell} \left\{ \left[\frac{z_n[\dot{z}_n z_k + \dot{z}_k z_n + \dot{z}_\ell(z_n + z_k) - (\dot{z}_n + \dot{z}_k + 2\dot{z}_\ell) z_\ell]}{\operatorname{den}(z_n, z_k, z_\ell)} \right] \right\} \\
&= \sum\nolimits'_{nk\ell} \left\{ \left[\frac{z_n[\dot{z}_n z_k + \dot{z}_k z_n + \dot{z}_\ell(z_n + z_k) - (\dot{z}_n + \dot{z}_k + 2\dot{z}_\ell) z_\ell]}{\operatorname{den}(z_n, z_k, z_\ell)} \right] \right\} \\
&= \sum\nolimits'_{nk\ell} \big\{ \big[\operatorname{den}(z_n, z_k, z_\ell) \big]^{-1} \times \\
&\quad \times \big[(\dot{z}_k + \dot{z}_\ell) z_n^2 - (\dot{z}_n + \dot{z}_\ell) z_n z_\ell + \dot{z}_n z_n z_k + \dot{z}_\ell z_k z_n - (\dot{z}_k + \dot{z}_\ell) z_n z_\ell \big] \big\} \\
&= \sum\nolimits'_{nk\ell} \left\{ \left[\frac{\dot{z}_n z_n z_k + \dot{z}_\ell z_k z_n - \dot{z}_k z_n z_\ell - \dot{z}_\ell z_n z_\ell}{\operatorname{den}(z_n, z_k, z_\ell)} \right] \right\} \\
&= \sum\nolimits'_{nk\ell} \left\{ \left[\frac{\dot{z}_n z_n (z_k + z_\ell) + 2\dot{z}_\ell z_k z_n}{\operatorname{den}(z_n, z_k, z_\ell)} \right] \right\} = 0. \qquad \text{(A.38)}
\end{aligned}
$$

The first equality is implied by the definition (A.36) with $p = 2$ and the identity $z^2 - z_n^2 = (z + z_n)(z - z_n)$; the second equality is a consequence of the vanishing of $\sigma(1; \underline{z}, \underline{\dot{z}})$ proven above; the third equality is obtained via trivial algebra; the fourth equality obtains – thanks to Lemma A.1 – because the first two addends in the numerator in the right-hand side are invariant under the exchanges of dummy indices $k \leftrightarrow \ell$ respectively $n \leftrightarrow \ell$; the fifth equality obtains by performing on the fourth term in the numerator the exchange of dummy indices $n \leftrightarrow \ell$ (entailing a change of sign, since the denominator changes sign under this exchange of indices, see (A.6c)) and on the third term the exchange of dummy indices $k \leftrightarrow \ell$ (entailing likewise a change of sign, see (A.6c)). Finally the last equality is implied by Lemma A.1, since the first term in the numerator is now invariant under the exchange of dummy indices $k \leftrightarrow \ell$ and the second under the exchange $n \leftrightarrow k$.

Finally, let us prove (A.3b). Now we start from the identity

$$
\begin{aligned}
z^2 \psi_{zzt} - (N-1)(N-2)\psi_t &= (N-1)(N-2)\psi \sum_{n=1}^{N} \left(\frac{\dot{z}_n}{z - z_n} \right) \\
&\quad - \psi \sum\nolimits'_{nk\ell} \left\{ \frac{z_n^2}{z - z_n} \left[(z_n - z_k)^{-1} \left(\frac{\dot{z}_n + \dot{z}_\ell}{z_n - z_\ell} + \frac{\dot{z}_k + \dot{z}_\ell}{z_k - z_\ell} \right) \right] \right\} \\
&= -\psi \sum\nolimits'_{nk\ell} \left\{ \frac{1}{z - z_n} \left[\frac{z_n^2}{z_n - z_k} \left(\frac{\dot{z}_n + \dot{z}_\ell}{z_n - z_\ell} + \frac{\dot{z}_k + \dot{z}_\ell}{z_k - z_\ell} \right) - \dot{z}_n \right] \right\}. \qquad \text{(A.39)}
\end{aligned}
$$

This identity is justified by (A.21) and by (the longhand version of) the identity we just proved, (A.3a). To write the second equality we also used the trivial observation that the very definition of the symbol $\sum'_{nk\ell}$, see (A.5), implies the identity $\sum'_{nk\ell} f(n) = (N-1)(N-2) \sum_{n=1}^{N} f(n)$ for any function $f(n)$ (and we will feel free to use this identity again below).

We now multiply this identity by $z \equiv (z - z_n) + z_n$, and we thereby obtain

$$
\begin{aligned}
z^2 \psi_{zzt} - (N-1)(N-2)\psi_t &= -\tilde{\sigma}(3; \underline{z}, \underline{\dot{z}})\psi + \psi \sum_{n=1}^{N} \Bigg[\left(\frac{1}{z - z_n} \right) \Bigg\{ (N-1)(N-2) \dot{z}_n z_n \\
&\quad - \sum_{k,\ell=1;\, k\neq n,\, \ell\neq n,\, \ell\neq k} \left[\frac{z_n^3}{z - z_n} \left(\frac{\dot{z}_n + \dot{z}_\ell}{z_n - z_\ell} + \frac{\dot{z}_k + \dot{z}_\ell}{z_k - z_\ell} \right) \right] \Bigg\} \Bigg], \qquad \text{(A.40a)}
\end{aligned}
$$

with

$$
\tilde{\sigma}(3; \underline{z}, \underline{\dot{z}}) \equiv \sum\nolimits'_{nk\ell} \left[\frac{z_n^2}{z_n - z_k} \left(\frac{\dot{z}_n + \dot{z}_\ell}{z_n - z_\ell} + \frac{\dot{z}_k + \dot{z}_\ell}{z_k - z_\ell} \right) - \dot{z}_n \right]. \qquad \text{(A.40b)}
$$

It is now plain that (A.3b) is proven if we show that $\tilde{\sigma}(3;\underline{z},\underline{\dot{z}})$ vanishes, $\tilde{\sigma}(3;\underline{z},\underline{\dot{z}})=0$. To prove this we make the following steps:

$$\begin{aligned}\tilde{\sigma}(3;\underline{z},\underline{\dot{z}}) &= \sum\nolimits'_{nk\ell}\big([\mathrm{den}(z_n,z_k,z_\ell)]^{-1}\big\{z_n^2[(\dot{z}_n+\dot{z}_\ell)(z_k-z_\ell)+(\dot{z}_k+\dot{z}_\ell)(z_n-z_\ell)]\\ &\quad-\dot{z}_n(z_n-z_k)(z_n-z_\ell)(z_k-z_\ell)\big\}\big)\\ &= \sum\nolimits'_{nk\ell}\big([\mathrm{den}(z_n,z_k,z_\ell)]^{-1}\big\{\dot{z}_n(z_k-z_\ell)\big[z_n^2-(z_n-z_k)(z_n-z_\ell)\big]\\ &\quad+\dot{z}_k z_n^2(z_n-z_\ell)+\dot{z}_\ell z_n^2(z_n+z_k-2z_\ell)\big\}\big)\\ &= \sum\nolimits'_{nk\ell}\left(\frac{\dot{z}_n}{\mathrm{den}(z_n,z_k,z_\ell)}\right.\\ &\quad\left.\times\big\{(z_k-z_\ell)\big[z_n^2-(z_n-z_k)(z_n-z_\ell)-z_k^2\big]-z_\ell^2(z_\ell+z_k-2z_n)\big\}\right)\\ &= \sum\nolimits'_{nk\ell}\left\{\dot{z}_n\frac{-(z_k^3+z_\ell^3)+z_n(z_k^2+z_\ell^2)}{\mathrm{den}(z_n,z_k,z_\ell)}\right\}=0.\end{aligned} \qquad (A.41)$$

Here the first equality is justified by the definition (A.6a) and a bit of trivial algebra, the second equality obtains by trivial algebra, the third equality obtains by the exchange of the dummy indices $n\leftrightarrow k$ in the term multiplying $\dot{z}_k$ and likewise the exchange $n\leftrightarrow\ell$ in the term multiplying $\dot{z}_\ell$ (under these exchanges the denominator changes sign, see (A.6c)), the fourth equality obtains by trivial algebra, and the final equality to zero is yielded by Lemma A.1 since the numerator is invariant under the exchange $k\leftrightarrow\ell$.

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