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On Addition Formulae for Sigma Functions of Telescopic Curves

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Abstract. A telescopic curve is a certain algebraic curve defined by $m-1$ equations in the affine space of dimension m, which can be a hyperelliptic curve and an (n,s) curve as a special case. We extend the addition formulae for sigma functions of (n,s) curves to those of telescopic curves. The expression of the prime form in terms of the derivative of the sigma function is also given.

Key words: sigma function; tau function; Schur function; Riemann surface; telescopic curve; gap sequence

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1 Introduction

In this paper we study the multivariate sigma functions of telescopic curves and derive addition formulae together with their degenerate limits.

The multivariate sigma function originally introduced by F. Klein [12, 13] for hyperelliptic curves is generalized and extensively studied for the last decade (see [6] and references therein). Compared with Riemann's theta function, the sigma function is more algebraic and is directly related with the defining equation of an algebraic curve. A typical example where this nature of the sigma function is exhibited is the inversion problem of algebraic integrals. It is well known that the solution of Jacobi's inversion problem for a hyperelliptic curve has a simple description by hyperelliptic $\wp$-functions, the second logarithmic derivatives of the sigma function [2, 6]. The inversion of hyperelliptic or more general algebraic integrals of genus g on the Abel–Jacobi image W_k of the k-th symmetric products of the curve with $k<g$ is extensively studied in connection with the problem of mathematical physics (see [3, 6] and references therein). This problem is intimately related with the problem on the vanishing of the derivatives of the sigma function on W_k. Recently it is recognized that the approach from the view point of tau functions of integrable hierarchies provides a general and effective method to study such a problem [21].

In Sato's theory of Kadomtsev–Petviashvili (KP) hierarchy [24] the tau function is constructed from a point of the universal Grassmann manifold (UGM). For a solution corresponding to an algebraic curve the point of UGM is specified by expanding functions or sections of bundles on the curve using a local coordinate at a given point. The point of UGM obtained in this way belongs to the cell of UGM labeled by the partition λ determined from the gap sequence at the point. The series expansion of the corresponding tau function begins from Schur function associated with λ. The tau function corresponding to the point of UGM specified by the affine ring of an (n,s) curve [7] had been used to study the sigma function in [20, 21].

Therefore it is important to have such a pair (X, p_∞) of an algebraic curve X and a point $p_\infty \in X$ that satisfies the following two conditions. The first is that a basis, as a vector space, of the space of regular functions on $X\backslash\{p_\infty\}$ can be explicitly described. The second is that the gap sequence at p_∞ can be computable. A traditional example is a non-singular plane algebraic curve which can be completed by one point ∞ $(= p_\infty)$ such as a hyperelliptic curve of odd degree or more generally an (n, s) curve. Telescopic curves give new examples. They can be hyperelliptic and (n, s) curves as special cases and are not realized as non-singular plane algebraic curves in general. Before explaining telescopic curves let us briefly explain the origin of the term "telescopic".

Let $a_1, \ldots, a_m$ be relatively prime positive integers. For a nonnegative integer n the problem of determining nonnegative integers $x_1, \ldots, x_m$ satisfying the Diophantine equation

$$n = a_1 x_1 + \cdots + a_m x_m \tag{1}$$

has been studied in number theory since early times. It is well known that the equation (1) has a solution if n is sufficiently large. The greatest number n for which the equation (1) has no solution is called Frobenius number and we denote it by $F(a_1, \ldots, a_m)$. Brauer [4] gave an upper bound of the Frobenius number as

$$F(a_1, \ldots, a_m) \leq -a_1 + \sum_{i=2}^{m} a_i \left(\frac{d_{i-1}}{d_i} - 1 \right), \tag{2}$$

where $d_i = \gcd(a_1, \ldots, a_i)$. Also Brauer [4] and Brauer and Seelbinder [5] showed that the equality in (2) holds if and only if

$$\frac{a_i}{d_i} \in \frac{a_1}{d_{i-1}} \mathbb{Z}_{\geq 0} + \cdots + \frac{a_{i-1}}{d_{i-1}} \mathbb{Z}_{\geq 0}, \qquad 2 \leq i \leq m. \tag{3}$$

The condition (3) is introduced in Brauer [4] for the first time and now it is called telescopic condition. For $a_1, \ldots, a_m$ satisfying (3) the semigroup $S := a_1 \mathbb{Z}_{\geq 0} + \cdots + a_m \mathbb{Z}_{\geq 0}$ is called telescopic semigroup. The telescopic semigroup has many nice structures and has many applications in algebraic geometric code, algebraic curve cryptography, and commutative algebra (see for instance [11, 17, 18]).

In [18] Miura introduced a certain canonical form, Miura canonical form, for defining equations of any non-singular algebraic curve. A telescopic curve [18] is a special curve for which Miura canonical form is easy to determine. Let $m \geq 2$ and $(a_1, \ldots, a_m)$ a sequence of relatively prime positive integers satisfying the telescopic condition (3). Then the telescopic curve associated with $(a_1, \ldots, a_m)$ or the $(a_1, \ldots, a_m)$ curve is the algebraic curve defined by certain $m-1$ equations in $\mathbb{C}^m$. The form of defining equations is explicitly computable from $(a_1, \ldots, a_m)$ (see (5)). If a telescopic curve is non-singular, then it can be completed by adding one point, say ∞, and the gap sequence at ∞ becomes the complement of the telescopic semigroup [1, 18]. In such a case the genus of the curve is also explicitly computable (see (6)).

In [21] the vanishing and the expansion of the sigma function of an (n, s) curve on the Abel–Jacobi image W_k for $k < g$ are studied using the properties of the tau function of the KP-hierarchy. Those results are then applied to study the restriction of the addition formulae on W_k to the lower strata $W_{k'}$ with $k' < k$. On the other hand the sigma function of the telescopic curve is explicitly constructed in [1]. In this paper we show that almost results in [21] are extended to the case of telescopic curves. The results imply two things. The first is that telescopic curves are natural objects to study sigma functions. We expect, more generally, the Miura canonical form is suitable to describe properties of the sigma functions. The second is that the tau function approach is effective in a more general case than that of (n, s) curves.

We expect that the method by integrable hierarchies can equally be efficient to study sigma functions with characteristics of arbitrary Riemann surfaces [15].

Finally we comment that the sigma functions of certain space curves, which are not telescopic, are studied in [14, 16].

The present paper is organized as follows. In Section 2 the definition and examples of telescopic curves are given. The construction of the sigma function, up to the normalization constant, associated with telescopic curves is reviewed in Section 3. The local coordinate z at ∞ is specified and the expression in terms of z of the variables appearing in the defining equations of the curve is given. This is necessary to determine constants appearing in every formula in later sections. In Section 4 the expression of the tau function is given using the sigma function. The normalization constant necessary in the definition of the sigma function is specified with the help of it. In Section 5 main results including the addition formulae are given. Their proofs are indicated in Section 6. The example of addition formulae is given for a $(4,6,5)$ curve in Section 7. In Appendix A the detailed properties on the series expansion of the sigma function are given for the sake of completeness of the construction of the sigma function.

2 Telescopic curves

In this section we briefly review the definition and properties of telescopic curves following [1, 18] and give some examples.

Let $m \geq 2$, $(a_1, \ldots, a_m)$ a sequence of positive integers such that $\gcd(a_1, \ldots, a_m) = 1$ and $d_i = \gcd(a_1, \ldots, a_i)$ for $1 \leq i \leq m$. We call $(a_1, \ldots, a_m)$ telescopic if

$$\frac{a_i}{d_i} \in \frac{a_1}{d_{i-1}}\mathbb{Z}_{\geq 0} + \cdots + \frac{a_{i-1}}{d_{i-1}}\mathbb{Z}_{\geq 0}, \qquad 2 \leq i \leq m.$$

The following examples of telescopic sequences are given in [22].

Example 1.

(i) (a_1, a_2), s.t. $\gcd(a_1, a_2) = 1$.

(ii) $(k, k+2, k+1)$, s.t. k even.

(iii) $(ab, bc, a+c)$, s.t. $\gcd(a, c) = 1$, $\gcd(b, a+c) = 1$.

(iv) $(a_1, \ldots, a_m)$, s.t. $a_i = a^{m-i}b^{i-1}$, $a > b$, $\gcd(a, b) = 1$.

Notice that whether a sequence is telescopic depends on the order of the numbers. For example, $(4,6,5)$ is telescopic while $(4,5,6)$ is not.

In the following we assume that $A_m = (a_1, \ldots, a_m)$ is telescopic unless otherwise stated.

Let

$$B(A_m) = \left\{ (l_1, \ldots, l_m) \in \mathbb{Z}_{\geq 0}^m \,|\, 0 \leq l_i \leq \frac{d_{i-1}}{d_i} - 1 \text{ for } 2 \leq i \leq m \right\}.$$

Lemma 1 ([1, 18])**.** *For any $a \in a_1\mathbb{Z}_{\geq 0} + \cdots + a_m\mathbb{Z}_{\geq 0}$, there exists a unique element $(k_1, \ldots, k_m)$ of $B(A_m)$ such that*

$$\sum_{i=1}^m a_i k_i = a.$$

By this lemma, for any $2 \leq i \leq m$, there exists a unique sequence $(l_{i1}, \ldots, l_{im}) \in B(A_m)$ satisfying

$$\sum_{j=1}^m a_j l_{ij} = a_i \frac{d_{i-1}}{d_i}. \tag{4}$$

Consider $m-1$ polynomials in m variables $x_1,\ldots,x_m$ given by

$$F_i(x)=x_i^{d_{i-1}/d_i}-\prod_{j=1}^{m}x_j^{l_{ij}}-\sum\kappa^{(i)}_{j_1\cdots j_m}x_1^{j_1}\cdots x_m^{j_m},\quad 2\le i\le m,\tag{5}$$

where $\kappa^{(i)}_{j_1\cdots j_m}\in\mathbb{C}$ and the sum of the right hand side is over all $(j_1,\ldots,j_m)\in B(A_m)$ such that

$$\sum_{k=1}^{m}a_kj_k<a_i\frac{d_{i-1}}{d_i}.$$

Let X^{aff} be the common zeros of $F_2,\ldots,F_m$:

$$X^{\mathrm{aff}}=\big\{(x_1,\ldots,x_m)\in\mathbb{C}^m\,|\,F_i(x_1,\ldots,x_m)=0,\ 2\le i\le m\big\}.$$

In [1, 18] X^{aff} is proved to be an affine algebraic curve. We assume that X^{aff} is nonsingular. Let X be the compact Riemann surface corresponding to X^{aff}. Then X is obtained from X^{aff} by adding one point, say ∞ [1, 18]. It is proved in [1, 18] that x_i has a pole of order a_i at ∞. The genus of X is given by [1, 18]

$$g=\frac{1}{2}\left(1+\sum_{i=2}^{m}a_i\frac{d_{i-1}}{d_i}-\sum_{i=1}^{m}a_i\right).\tag{6}$$

We call X the $(a_1,\ldots,a_m)$ curve or the telescopic curve associated with $(a_1,\ldots,a_m)$. The numbers $a_1,\ldots,a_m$ are a generator of the semigroup of non-gaps at ∞.

Example 2.

(i) The telescopic curve associated with a pair of relatively prime integers (n,s) is the (n,s) curve introduced in [7].

(ii) For $A_3=(2k,2k+2,2k+1)$, $k\ge2$, in (ii) of Example 1, polynomials F_i are given by

$$F_2(x)=x_2^k-x_1^{k+1}-\sum\nolimits^{(2)}\kappa^{(2)}_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3},$$
$$F_3(x)=x_3^2-x_1x_2-\sum\nolimits^{(3)}\kappa^{(3)}_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3},$$

where $\sum^{(i)}$, $i=2,3$ signify the sum over all $(i_1,i_2,i_3)\in B(A_3)$ such that

$$2ki_1+2(k+1)i_2+(2k+1)i_3<\begin{cases}2k(k+1) & \text{for } \sum^{(2)},\\ 2(2k+1) & \text{for } \sum^{(3)}.\end{cases}$$

The genus of X is $g=k^2$.

(iii) For $A_3=(ab,bc,a+c)$, $a\ne1$, in (iii) of Example 1, we have

$$F_2(x)=x_2^a-x_1^c-\sum\nolimits^{(2)}\kappa^{(2)}_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3},$$
$$F_3(x)=x_3^b-x_1x_2-\sum\nolimits^{(3)}\kappa^{(3)}_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3},$$

where $\sum^{(i)}$, $i=2,3$ denote the sum over all $(i_1,i_2,i_3)\in B(A_3)$ such that

$$abi_1+bci_2+(a+c)i_3<\begin{cases}abc & \text{for } \sum^{(2)},\\ b(a+c) & \text{for } \sum^{(3)}.\end{cases}$$

The genus of X is

$$g=\frac{1+abc-a-c}{2}.$$

(*iv*) For $A_m=(a_1,\dots,a_m)$ in (*iv*) of Example 1, we have

$$F_i(x)=x_i^a-x_{i-1}^b-\sum_{a_1j_1+\cdots+a_mj_m<aa_i}\kappa^{(i)}_{j_1\cdots j_m}x_1^{j_1}\cdots x_m^{j_m}.$$

The genus of X is

$$g=\frac{a-b+(b-1)a^m-(a-1)b^m}{2(a-b)}.$$

3 Sigma function of telescopic curves

An algebraic bilinear differential of a telescopic curve associated with $(a_1,\dots,a_m)$ is explicitly constructed in [1]. Consequently an expression of the sigma function in terms of Riemann's theta function and some algebraic data had been given. We recall the results of [1] and add some necessary results for our purpose.

Let X be a telescopic curve of genus $g\geq 1$ associated with $(a_1,\dots,a_m)$ and $(l_{i1},\dots,l_{im})$ the element of $B(A_m)$ specified by (4).

Lemma 2. *For any i we have $l_{ij}=0$ for $j\geq i$.*

Proof. Since A_m is telescopic, there exist $k_1,\dots,k_{i-1}\in\mathbb{Z}_{\geq 0}$ such that

$$a_i\frac{d_{i-1}}{d_i}=a_1k_1+\cdots+a_{i-1}k_{i-1}. \tag{7}$$

We prove that we can take $0\leq k_j<d_{j-1}/d_j$ for any $j\geq 2$ by changing k_j appropriately if necessary.

Suppose that $k_{j'}\geq d_{j'-1}/d_{j'}$ for some j'. Take the largest number j satisfying this condition. Let us write

$$k_j=\frac{d_{j-1}}{d_j}q+r,$$

with $q\geq 1$, $0\leq r<d_{j-1}/d_j$. Since A_m is telescopic, there exist $u_1,\dots,u_{j-1}\in\mathbb{Z}_{\geq 0}$ such that

$$a_j\frac{d_{j-1}}{d_j}=a_1u_1+\cdots+a_{j-1}u_{j-1}.$$

Then we have

$$a_jk_j=a_j\frac{d_{j-1}}{d_j}q+a_jr=a_1qu_1+\cdots+a_{j-1}qu_{j-1}+a_jr.$$

Substituting this into (7) we get the expression of the form (7) with $0\leq k_l<d_{l-1}/d_l$ for any $l\geq j$. Repeating similar change of $k_{j'}$ for j' smaller than j successively we finally get the expression of a_id_{i-1}/d_i of the form (7) with $0\leq k_j<d_{j-1}/d_j$ for any $j\geq 2$.

By the definition of $B(A_m)$, $(k_1,\dots,k_{i-1},0,\dots,0)\in B(A_m)$. Since the element of $B(A_m)$ satisfying (4) is unique by Lemma 1, $(l_{i1},\dots,l_{im})=(k_1,\dots,k_{i-1},0,\dots,0)$. ■

For the defining equations (5), we assign degrees as

$$\deg\kappa^{(i)}_{j_1\cdots j_m}=a_id_{i-1}/d_i-\sum_{k=1}^m a_kj_k.$$

Lemma 3. *It is possible to take a local parameter z around ∞ such that*

$$x_1 = \frac{1}{z^{a_1}}, \qquad x_k = \frac{1}{z^{a_k}}\left(1 + \sum_{l=1}^{\infty} e_{kl} z^l\right), \qquad 2 \le k \le m, \tag{8}$$

where e_{kl} belongs to $\mathbb{Q}\big[\{\kappa^{(i)}_{j_1\cdots j_m}\}\big]$ and is homogeneous of degree l if $e_{kl} \neq 0$.

Proof. It is possible to take a local parameter z_0 around ∞ such that

$$x_1 = \frac{1}{z_0^{a_1}}.$$

Let $\zeta = \exp\big(2\pi\sqrt{-1}/a_1\big)$ and $i \ge 0$. Then $z_i := \zeta^i z_0$ is also a local parameter around ∞. Let $e^{(i)}_k$ be the coefficient of the first term of the series expansion of x_k around ∞ with respect to z_i:

$$x_k = \frac{e^{(i)}_k}{z_i^{a_k}}(1 + O(z_i)), \qquad 2 \le k \le m. \tag{9}$$

We prove that there exists i such that $e^{(i)}_2 = \cdots = e^{(i)}_m = 1$.

Let $e^{(i)} = \big(e^{(i)}_2, \ldots, e^{(i)}_m\big)$ for $0 \le i < a_1$. First we show $e^{(i)} \neq e^{(j)}$ for $i \neq j$. Suppose $e^{(i)} = e^{(j)}$. Since $e^{(i)}_k = \zeta^{a_k i} e^{(0)}_k$, we have $\zeta^{a_k(i-j)} = 1$ for $k = 2, \ldots, m$. From $\gcd(a_1, \ldots, a_m) = 1$ and $0 \le i, j < a_1$, we have $i = j$.

By Lemma 2 the defining equations of X are as follows:

$$x_k^{d_{k-1}/d_k} = x_1^{l_{k1}} \cdots x_{k-1}^{l_{kk-1}} + \sum \kappa^{(k)}_{j_1\ldots j_m} x_1^{j_1} \cdots x_m^{j_m}, \qquad 2 \le k \le m. \tag{10}$$

By substituting (9) to (10) and comparing the coefficients of $z_i^{-a_k d_{k-1}/d_k}$, we have

$$\big(e^{(i)}_2\big)^{d_1/d_2} = 1, \qquad \big(e^{(i)}_k\big)^{d_{k-1}/d_k} = \big(e^{(i)}_2\big)^{l_{k2}} \cdots \big(e^{(i)}_{k-1}\big)^{l_{kk-1}}, \qquad 3 \le k \le m.$$

Let

$$S = \big\{(s_2, \ldots, s_m) \in \mathbb{C}^{m-1} \mid s_2^{d_1/d_2} = 1,\ s_k^{d_{k-1}/d_k} = s_2^{l_{k2}} \cdots s_{k-1}^{l_{kk-1}},\ 3 \le k \le m\big\}.$$

Since $\sharp S = (d_1/d_2) \cdots (d_{m-1}/d_m) = (d_1/d_m) = a_1$ and $e^{(i)} \in S$ for $i = 0, \ldots, a_1 - 1$, we have

$$S = \big\{e^{(0)}, \ldots, e^{(a_1-1)}\big\}.$$

Since $(1, \ldots, 1) \in S$, there exists i such that $e^{(i)} = (1, \ldots, 1)$. For $z := z_i$, x_k is expanded as

$$x_1 = \frac{1}{z^{a_1}}, \qquad x_k = \frac{1}{z^{a_k}}\left(1 + \sum_{l=1}^{\infty} e_{kl} z^l\right), \qquad e_{kl} \in \mathbb{C}.$$

Let us prove that e_{kl} belongs to $\mathbb{Q}\big[\{\kappa^{(i)}_{j_1\ldots j_m}\}\big]$ and is homogeneous of degree l if $e_{kl} \neq 0$. We define the order $<$ in the set $\{e_{kl}\}$ so that $e_{k'l'} < e_{kl}$ if

1) $l' < l$ or
2) $l' = l$ and $k' < k$.

We prove the statement by induction on this order. By (5) and Lemma 2 we have

$$\left(1+\sum_{j=1}^{\infty}e_{kj}z^j\right)^{\frac{d_{k-1}}{d_k}} = \prod_{s=2}^{k-1}\left(1+\sum_{j=1}^{\infty}e_{sj}z^j\right)^{l_{ks}} + \sum \kappa^{(k)}_{j_1\cdots j_m} z^{\frac{a_k d_{k-1}}{d_k}-\sum_{s=1}^{m}a_s j_s} \prod_{s=2}^{m}\left(1+\sum_{j=1}^{\infty}e_{sj}z^j\right)^{j_s}, \qquad (11)$$

where we define the empty product from $s=2$ to 1 to be one in the first term of the right hand side.

In (11) for $k=2$, the coefficient of z of the left hand side is $(d_1/d_2)e_{21}$ and that of the right hand side is the sum of $\kappa^{(2)}_{j_1\cdots j_m}$ with $(j_1,\ldots,j_m)$ satisfying the equation $(a_2d_1/d_2)-\sum_{s=1}^{m}a_sj_s=1$. Therefore the statement is valid for the minimal element e_{21}.

Assume that the statement holds for any $e_{k'l'}$ satisfying $e_{k'l'}<e_{kl}$. The coefficient of z^l of the left hand side of (11) is $(d_{k-1}/d_k)e_{kl}+T$, where T is a sum of $\prod_i e_{kq_i}$ satisfying $\sum_i q_i = l$ and $q_i<l$. In the right hand side of (11), the coefficient of z^l of the first term is the sum of $\prod_i e_{p_iq_i}$ satisfying $2\le p_i<k$ and $\sum_i q_i=l$, and that of the second term is the sum of $\kappa^{(k)}_{j_1\cdots j_m}\prod_i e_{p_iq_i}$ with $(j_1,\ldots,j_m)$ satisfying $\sum_i q_i = l-(a_kd_{k-1}/d_k)+\sum_{s=1}^{m}a_sj_s$. Therefore, by the assumption of induction, we find that e_{kl} belongs to $\mathbb{Q}\big[\{\kappa^{(i)}_{j_1\cdots j_m}\}\big]$ and is homogeneous of degree l if $e_{kl}\neq 0$. ■

For a meromorphic function f on X we denote by $\mathrm{ord}_\infty(f)$ the order of a pole at ∞. Then we have $\mathrm{ord}_\infty(x_i)=a_i$. We enumerate the monomials $x_1^{\alpha_1}\cdots x_m^{\alpha_m}$, $(\alpha_1,\ldots,\alpha_m)\in B(A_m)$ according as the order of a pole at ∞ and denote them by φ_i, $i\ge 1$. In particular we have $\varphi_1=1$.

Let $(w_1,\ldots,w_g)$ be the gap sequence at ∞:

$$\{w_i\,|\,1\le i\le g\}=\mathbb{Z}_{\ge 0}\Big\backslash\left\{\sum_{i=1}^{m}a_i\mathbb{Z}_{\ge 0}\right\}, \qquad w_1<\cdots<w_g.$$

In particular $w_1=1$, since $g\ge 1$.

A basis of holomorphic one forms is given by

$$du_{w_i}=-\frac{\varphi_{g+1-i}}{\det G(x)}dx_1, \qquad (12)$$

where $G(x)$ is the Jacobian matrix

$$G(x)=\left(\frac{\partial F_i}{\partial x_j}\right)_{2\le i,j\le m}.$$

The following lemma is proved in [1].

Lemma 4. *We have $w_g=2g-1$. In particular du_{2g-1} has a zero of order $2g-2$ at ∞.*

More precisely we have the following properties.

Proposition 1.

(*i*) *The following expansion is valid around ∞:*

$$du_{2g-1}=z^{2g-2}\left(1+\sum_{l=1}^{\infty}e'_lz^l\right)dz,$$

where e'_l belongs to $\mathbb{Q}\big[\{\kappa^{(i)}_{j_1\cdots j_m}\}\big]$ and is homogeneous of degree l if $e'_l\neq 0$.

(ii) For $1 \le i \le g$ the expansion of du_{w_i} at ∞ is of the form

$$du_{w_i} = z^{w_i-1}(1+O(z))dz.$$

Proof. (i) From Lemmas 2, 3, we have, around ∞,

$$\det G(x) = a_1 z^{-\sum_{i=2}^{m}((d_{i-1}/d_i)-1)a_i}\left(1+\sum_{l=1}^{\infty} e_l'' z^l\right)dz,$$

where e_l'' belongs to $\mathbb{Q}[\{\kappa^{(i)}_{j_1\cdots j_m}\}]$ and is homogeneous of degree l if $e_l'' \neq 0$. Therefore, from (6), we obtain the assertion.

(ii) Let $w_i^* = \mathrm{ord}_\infty(\varphi_i)$, $W = \{w_1,\dots,w_g\}$, and $W' = \{w_1^*,\dots,w_g^*\}$. Note that $W \cup W' = \{0,1,\dots,2g-1\}$. If $w \in W'$, then $w_g - w \in W$. In fact, if $w_g - w \in W'$, then $w_g \in W'$, which is contradiction. Since $w_g = 2g-1$, we have $2g-1-w^*_{g+1-i} = w_i$ for any i. Therefore, from (12), Lemma 3, and Proposition 1(i), we obtain the assertion. ■

The algebraic bilinear differential takes the form

$$\widehat{\omega}(x,y) = d_y\Omega(x,y) + \sum_{i=1}^{g} du_{w_i}(x)dr_i(y),$$

where $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)$ are points on X,

$$\Omega(x,y) = \frac{\det H(x,y)}{(x_1-y_1)\det G(x)}dx_1,$$

$H=(h_{ij})_{2\le i,j\le m}$ with

$$h_{ij} = \frac{F_i(y_1,\dots,y_{j-1},x_j,x_{j+1},\dots,x_m) - F_i(y_1,\dots,y_{j-1},y_j,x_{j+1},\dots,x_m)}{x_j-y_j},$$

and dr_i is a second kind differential with a pole only at ∞. By construction $\{du_{w_i}, dr_i\}$ becomes a symplectic basis of the cohomology group $H^1(X,\mathbb{C})$ (see [1, 19]).

Take a symplectic basis $\{\alpha_i,\beta_i\}$ of the homology group and define the period matrices by

$$2\omega_1 = \left(\int_{\alpha_j} du_{w_i}\right), \qquad 2\omega_2 = \left(\int_{\beta_j} du_{w_i}\right),$$
$$-2\eta_1 = \left(\int_{\alpha_j} dr_i\right), \qquad -2\eta_2 = \left(\int_{\beta_j} dr_i\right).$$

The normalized period matrix is given by $\tau = \omega_1^{-1}\omega_2$.

Let $\delta = \tau\delta' + \delta''$, $\delta',\delta'' \in \mathbb{R}^g$ be the Riemann's constant with respect to the choice $(\{\alpha_i,\beta_i\},\infty)$. We set $\delta = {}^t({}^t\delta', {}^t\delta'')$. Since du_{2g-1} has a zero of order $2g-2$ at ∞ by Lemma 4, we have $\delta \in (\mathbb{Z}/2)^{2g}$.

We define the function $\widehat{\sigma}(u)$, $u=(u_{w_1},\dots,u_{w_g})$ by

$$\widehat{\sigma}(u) = \exp\left(\frac{1}{2}{}^t u\eta_1\omega_1^{-1}u\right)\theta[\delta]\big((2\omega_1)^{-1}u,\tau\big),$$

where $\theta[\delta](u)$ is the Riemann's theta function with the characteristic δ.

4 Relation with tau function

In the case of (n, s) curves the expression of the tau function of the KP-hierarchy in terms of the sigma function is given in [8, 10, 20]. For the tau function corresponding to the point of Sato's universal Grassmann manifold (UGM) specified by the affine ring of a telescopic curve a similar expression holds.

Let A be the affine ring of X,

$$A = \mathbb{C}[x_1, \dots, x_m]/I,$$

where I is the ideal generated by $F_i(x)$, $2 \le i \le m$. As a vector space $A = \oplus_{i=1}^{\infty} \mathbb{C}\varphi_i$.

We embed A in UGM as in Section 5 of [20] using the local parameter z in (8) and denote U^A the image of it. Let ξ^A be the normalized frame of U^A and $\tau(t; \xi^A)$ the tau function corresponding to ξ^A. Then we have the expansion of the form

$$\tau(t; \xi^A) = s_\lambda(t) + \sum_{\lambda < \mu} \xi_\mu s_\mu(t), \tag{13}$$

where $\lambda = (\lambda_1, \dots, \lambda_g)$ is the partition defined by

$$\lambda_i = w_{g+1-i} - g + i, \qquad 1 \le i \le g, \tag{14}$$

$s_\mu(t)$ is the Schur function corresponding to the partition μ and, in general, for two partitions $\mu = (\mu_1, \dots, \mu_l)$ and $\nu = (\nu_1, \dots, \nu_l)$, $\mu \le \nu$ if and only if $\mu_i \le \nu_i$ for any i.

Define b_{ij}, $\widehat{q}_{ij}$ and c_i by the expansions

$$du_{w_i} = \sum_{j=1}^{\infty} b_{ij} z^{j-1} dz, \qquad \widehat{\omega}(p_1, p_2) = \left(\frac{1}{(z_1 - z_2)^2} + \sum_{i,j=1}^{\infty} \widehat{q}_{ij} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2,$$

$$\log \left(z^{-g+1} \sqrt{\frac{du_{2g-1}}{dz}} \right) = \sum_{i=1}^{\infty} c_i \frac{z^i}{i},$$

where $z_i = z(p_i)$.[1] Let us set

$$B = (b_{ij})_{1 \le i \le g, 1 \le j}, \qquad \widehat{q}(t) = \sum_{i,j=1}^{\infty} \widehat{q}_{ij} t_i t_j, \qquad t = {}^t(t_1, t_2, \dots).$$

Then we have

Theorem 1.

(i) *There exists a constant C such that*

$$\tau(t; \xi^A) = C \exp \left(- \sum_{i=1}^{\infty} c_i t_i + \frac{1}{2} \widehat{q}(t) \right) \widehat{\sigma}(Bt).$$

(ii) *The tau function $\tau(t; \xi^A)$ is a solution to the a_1-reduced KP-hierarchy.*

Proof. The proof of this theorem is completely pararell to the case of (n, s) curves in [20]. In fact the special property of (n, s) curves we use in [20] is the existence of a holomorphic one form which vanishes at ∞ to the order $2g - 2$. In the present case du_{2g-1} has such a property by Lemma 4. ∎

[1] In the right hand side of the defining equation of c_i in [20], z_i should be corrected as z^i/i.

Combining the expansion (13) with Theorem 1 we have

Corollary 1. *We have the following expansion*

$$C\widehat{\sigma}(u) = s_\lambda(t)|_{t_{w_i}=u_{w_i}} + \cdots ,$$

where λ *is defined by* (14) *and* $\cdots$ *part is a series in* $\prod_{i=1}^{g} {u_{w_i}}^{\gamma_i}$, $\sum_{i=1}^{g} \gamma_i w_i > \sum_{i=1}^{g} \lambda_i$.

Definition 1. We define the sigma function by

$$\sigma(u) = C\widehat{\sigma}(u).$$

It follows from this definition and Corollary 1 that the series expansion of $\sigma(u)$ at the origin begins from Schur function corresponding to the gap sequence at ∞. It is possible to give more precise properties of the expansion of $\sigma(u)$ which is similar to the case of (n, s) curves (see Appendix A).

5 Addition formulae

Our result is that all properties of the sigma functions of (n, s) curves given in Sections 4 and 5 of the paper [21] are valid, formally without any change, for sigma functions of telescopic curves. The strategy of the proofs of theorems in this section is explained in the next section.

In order to state the results precisely we need the prime function of a telescopic curve which was introduced in [19] for (n, s) curves.

Let $E(p_1, p_2)$ be the prime form of a telescopic curve X. Since du_{2g-1} has a zero of order $2g - 2$ at ∞ by Lemma 4, it is possible to define, as in the case of (n, s) curves, the prime function $\tilde{E}(p_1, p_2)$ by

$$\tilde{E}(p_1, p_2) = -E(p_1, p_2) \prod_{i=1}^{2} \sqrt{du_{2g-1}(p_i)} \exp\left(\frac{1}{2} \int_{p_1}^{p_2} {}^t d\mathbf{u} \eta_1 \omega_1^{-1} \int_{p_1}^{p_2} d\mathbf{u} \right).$$

It is a multi-valued analytic function on $X \times X$ which has similar properties to that for (n, s) curves.

For a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ and $0 \le k \le l$ we set

$$\begin{aligned}
&(w_l, \dots, w_1) = (\lambda_1 + l - 1, \lambda_2 + l - 2, \dots, \lambda_l), \\
&N_{\lambda,k} = \lambda_{k+1} + \cdots + \lambda_l, \qquad N'_{\lambda,1} = \lambda_2 + \cdots + \lambda_l - l + 1, \\
&c'_{\lambda,k} = \frac{N_{\lambda,k}!}{\prod_{i=1}^{l-k} w_i!} \prod_{i<j}^{l-k} (w_j - w_i), \qquad \tilde{c}_\lambda = \frac{N'_{\lambda,1}!}{\prod_{i=1}^{l-1} (w_i - 1)!} \prod_{i<j}^{l-1} (w_j - w_i),
\end{aligned}$$

where $c'_{\lambda,l}$ is considered to be 1.

The following theorems give the properties of the sigma function restricted to the Abel–Jacobi image W_k of the k-th symmetric products of X for $k < g$.

Theorem 2. *Let* $1 \le k \le g$ *and* $p_1, \dots, p_k \in X$. *Then*

(*i*) *We have*

$$\partial_{u_1}^{N_{\lambda,k}} \sigma\left(\sum_{i=1}^{k} p_i \right) = c'_{\lambda,k} S_{(\lambda_1,\dots,\lambda_k)}(z_1, \dots, z_k) + \cdots ,$$

where $\cdots$ *part is a series in* $z_1, \ldots, z_k$ *containing only terms proportional to* $\prod_{i=1}^{k} z_i^{\gamma_i}$ *with* $\sum_{i=1}^{k} \gamma_i > \sum_{i=1}^{k} \lambda_i$.

(*ii*) *The following expansion in* z_k *holds:*

$$\partial_{u_1}^{N_{\lambda,k}} \sigma\left(\sum_{i=1}^{k} p_i\right) = \frac{c'_{\lambda,k}}{c'_{\lambda,k-1}} \partial_{u_1}^{N_{\lambda,k-1}} \sigma\left(\sum_{i=1}^{k-1} p_i\right) z_k^{\lambda_k} + O\big(z_k^{\lambda_k+1}\big).$$

Theorem 3.

(*i*) *If* $n < N'_{\lambda,1}$ *we have, for* $p_1, p_2 \in X$,

$$\partial_{u_1}^{n} \sigma(p_1 - p_2) = 0.$$

(*ii*) *The following expansion with respect to* $z_i = z(p_i)$, $i = 1, 2$ *is valid:*

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2) = (-1)^{g-1} \tilde{c}_\lambda (z_1 z_2)^{g-1} (z_1 - z_2)(1 + \cdots),$$

where $\cdots$ *part is a series in* z_1, z_2 *which contains only terms proportional to* $z_1^i z_2^j$ *with* $i + j > 0$.

(*iii*) *We have*

$$\partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2) = (-1)^{g-1} \frac{\tilde{c}_\lambda}{c'_{\lambda,1}} \partial_{u_1}^{N_{\lambda,1}} \sigma(p_1) z_2^{g-1} + O\big(z_2^{g}\big).$$

The next theorem gives the expression of the prime function as a derivative of the sigma function.

Theorem 4. *Let* $\lambda = (\lambda_1, \ldots, \lambda_g)$ *be the partition corresponding to the gap sequence at* ∞ *of an* $(a_1, \ldots, a_m)$ *curve* X. *Then*

$$\tilde{E}(p_1, p_2) = (-1)^{g-1} \tilde{c}_\lambda^{-1} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_1 - p_2).$$

Corollary 2. *For* $p \in X$ *we have*

$$\tilde{E}(\infty, p) = {c'_{\lambda,1}}^{-1} \partial_{u_1}^{N_{\lambda,1}} \sigma(p).$$

Finally the addition formulae of sigma functions for telescopic curves are given by

Theorem 5.

(*i*) *For* $n \geq g$ *and* $p_i \in X$, $1 \leq i \leq n$,

$$\frac{\sigma\left(\sum_{i=1}^{n} p_i\right) \prod_{i<j} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)}{\prod_{i=1}^{n} \left(\partial_{u_1}^{N_{\lambda,1}} \sigma(p_i)\right)^n} = \tilde{b}_{\lambda,n} \det\left(\varphi_i(p_j)\right)_{1 \leq i,j \leq n},$$

with

$$\tilde{b}_{\lambda,n} = (-1)^{\frac{1}{2}gn(n-1)} \tilde{c}_\lambda^{\frac{1}{2}n(n-1)} (c'_{\lambda,1})^{-n^2}.$$

(*ii*) *For* $n < g$

$$\frac{\partial_{u_1}^{N_{\lambda,n}} \sigma\left(\sum_{i=1}^{n} p_i\right) \prod_{i<j} \partial_{u_1}^{N'_{\lambda,1}} \sigma(p_j - p_i)}{\prod_{i=1}^{n} \left(\partial_{u_1}^{N_{\lambda,1}} \sigma(p_i)\right)^n} = b'_{\lambda,n} \det\left(\varphi_i(p_j)\right)_{1\le i,j\le n},$$

with

$$b'_{\lambda,n} = (-1)^{\frac{1}{2}gn(n-1)} \tilde{c}_\lambda^{\frac{1}{2}n(n-1)} (c'_{\lambda,1})^{-n^2} c'_{\lambda,n}.$$

The addition formulae of this kind were firstly derived by Ônishi [23] for the hyperelliptic sigma functions. We remark that the formulae in this theorem are written using algebraic data only, the sigma function, its derivatives and the algebraic functions φ_i. This fact makes it possible to study the restriction of the formulae on W_k to lower strata $W_{k'}$ for $k' < k$ as in (*ii*). This is a main difference of our formulae and Fay's general addition formulae [9].

6 Proofs

In [21] properties of the sigma function of an (n, s) curve have been proved by establishing the corresponding properties of Schur and tau functions. In this paper we exclusively consider t_1-derivatives and u_1-derivatives. We omit the results on "a-derivatives" in [21], since they are not used in addition formulae for general telescopic curves.

As far as t_1-derivatives are concerned, all the statements for Schur and tau functions in [21] hold, as stated there, for Schur function $s_\lambda(t)$ associated with any partition λ and the tau functions $\tau(t)$ which have the expansion of the form

$$\tau(t) = s_\lambda(t) + \sum_{\lambda<\mu} \xi_\mu s_\mu(t).$$

Then Theorems 2, 3, 4, Corollary 2 can be proved in a similar manner to the case of (n, s) curves using Theorem 1.

By Theorem 4 the prime function of a telescopic curve can be written as a derivative of the sigma function. Conversely the sigma function can be expressed by using the prime function and algebraic functions φ_i as in the case of an (n, s) curve.

Theorem 6. *For* $n \ge g$ *and* $p_i \in X$, $1 \le i \le n$,

$$\sigma\left(\sum_{i=1}^{n} p_i\right) = \frac{\prod_{i=1}^{n} \tilde{E}(\infty, p_i)^n}{\prod_{i<j}^{n} \tilde{E}(p_i, p_j)} \det\left(\varphi_i(p_j)\right)_{1\le i,j\le n}. \tag{15}$$

Expanding (15) in $z(p_n)$ successively with the help of Theorem 2 we get

Corollary 3. *For* $n < g$ *we have*

$$\partial_{u_1}^{N_{\lambda,n}} \sigma\left(\sum_{i=1}^{n} p_i\right) = c'_{\lambda,n} \frac{\prod_{i=1}^{n} \tilde{E}(\infty, p_i)^n}{\prod_{i<j}^{n} \tilde{E}(p_i, p_j)} \det\left(\varphi_i(p_j)\right)_{1\le i,j\le n}.$$

Theorem 5 can be obtained from Theorem 6 and Corollary 3 by substituting the sigma function expression of the prime function given by Theorem 4 and Corollary 2.

7 Example: $(4, 6, 5)$-curve

In this section we give an explicit example of the addition formulae in the case of a $(4,6,5)$-curve X. By Example 2(ii) in Section 2, the genus of X is 4. The gap sequence at ∞ is $(w_1, w_2, w_3, w_4) = (1, 2, 3, 7)$. The partition corresponding to the gap sequence at ∞ is $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 1, 1, 1)$. Therefore we have $N_{\lambda,0} = 7$, $N_{\lambda,1} = 3$, $N_{\lambda,2} = 2$, $N_{\lambda,3} = 1$, $N'_{\lambda,1} = 0$, $c'_{\lambda,1} = c'_{\lambda,2} = c'_{\lambda,3} = \tilde{c}_\lambda = 1$. On the other hand we have $\varphi_1 = 1$, $\varphi_2 = x_1$, $\varphi_3 = x_3$, $\varphi_4 = x_2$. Therefore the addition formulae given in Theorem 5 for $n = 2, 3, 4$ are as follows:

(i) For $n = 2$,

$$\frac{\left(\partial_{u_1}^2 \sigma(p_1 + p_2)\right) \sigma(p_2 - p_1)}{(\partial_{u_1}^3 \sigma(p_1))^2 (\partial_{u_1}^3 \sigma(p_2))^2} = x_1(p_2) - x_1(p_1).$$

(ii) For $n = 3$,

$$\frac{\partial_{u_1} \sigma(p_1 + p_2 + p_3) \prod\limits_{1 \le i < j \le 3} \sigma(p_j - p_i)}{\prod\limits_{j=1}^{3} (\partial_{u_1}^3 \sigma(p_j))^3} = \begin{vmatrix} 1 & 1 & 1 \\ x_1(p_1) & x_1(p_2) & x_1(p_3) \\ x_3(p_1) & x_3(p_2) & x_3(p_3) \end{vmatrix}.$$

(iii) For $n = 4$,

$$\frac{\sigma(p_1 + p_2 + p_3 + p_4) \prod\limits_{1 \le i < j \le 4} \sigma(p_j - p_i)}{\prod\limits_{j=1}^{4} (\partial_{u_1}^3 \sigma(p_j))^4} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1(p_1) & x_1(p_2) & x_1(p_3) & x_1(p_4) \\ x_3(p_1) & x_3(p_2) & x_3(p_3) & x_3(p_4) \\ x_2(p_1) & x_2(p_2) & x_2(p_3) & x_2(p_4) \end{vmatrix}.$$

A Series expansion of sigma function

Using Theorem 6 in a similar manner to the case of (n, s) curves (cf. [19]), we can show the following theorem.

Theorem 7. *The expansion of $\sigma(u)$ at the origin takes the form*

$$\sigma(u) = s_\lambda(t)|_{t_{w_i} = u_{w_i}} + \sum_{\sum\limits_{i=1}^{g} \gamma_i w_i > \sum\limits_{i=1}^{g} \lambda_i} \tilde{e}_{\gamma_1 \ldots \gamma_g} u_{w_1}^{\gamma_1} \cdots u_{w_g}^{\gamma_g},$$

where $\tilde{e}_{\gamma_1 \ldots \gamma_g}$ belongs to $\mathbb{Q}\big[\big\{\kappa^{(i)}_{j_1 \ldots j_m}\big\}\big]$ and is homogeneous of degree $\sum\limits_{i=1}^{g} \gamma_i w_i - \sum\limits_{i=1}^{g} \lambda_i$ if $\tilde{e}_{\gamma_1 \ldots \gamma_g} \neq 0$.

Remark 1. It is possible to prove the above theorem using the relation with the tau function as in [20].

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