Symmetry, Integrability and Geometry: Methods and Applications SIGMA 8 (2012), 101, 17 pages

Renormalization Method and Mirror Symmetry*

Si LI

Department of mathematics, Northwestern University,
2033 Sheridan Road, Evanston IL 60208, USA
E-mail: *sili@math.northwestern.edu*

Received May 07, 2012, in final form December 13, 2012; Published online December 18, 2012
http://dx.doi.org/10.3842/SIGMA.2012.101

Abstract. This is a brief summary of our works [arXiv:1112.4063, arXiv:1201.4501] on constructing higher genus B-model from perturbative quantization of BCOV theory. We analyze Givental's symplectic loop space formalism in the context of B-model geometry on Calabi–Yau manifolds, and explain the Fock space construction via the renormalization techniques of gauge theory. We also give a physics interpretation of the Virasoro constraints as the symmetry of the classical BCOV action functional, and discuss the Virasoro constraints in the quantum theory.

Key words: BCOV; Calabi–Yau; renormalization; mirror symmetry

2010 Mathematics Subject Classification: 14N35; 58A14; 81T15; 81T70

1 Introduction

Mirror symmetry is a remarkable duality between symplectic geometry and complex geometry. It originated from string theory as a duality between superconformal field theories, and became a celebrated idea in mathematics since the physics work [6] on the successful prediction of the number of rational curves on the quintic 3-fold.

The symplectic side of mirror symmetry, which is called the A-model, is established in mathematics as the Gromov–Witten theory [20, 27] counting the number of Riemann surfaces on smooth projective varieties. The complex side of mirror symmetry, which is called the B-model, is concerned with the deformation of complex structures. The B-model at genus 0 is described by the variation of Hodge structures on Calabi–Yau manifolds, which has been proven to be equivalent to the genus 0 Gromov–Witten theory on the mirror Calabi–Yau for a large class of examples [15, 24]. However, for compact Calabi–Yau manifolds, the mirror symmetry at higher genus is not very well formulated, due to the difficulty of B-model geometry at higher genus.

Bershadsky, Cecotti, Ooguri and Vafa [5] proposed a gauge theory interpretation of the B-model via polyvector fields, which they called the Kodaira–Spencer gauge theory of gravity. BCOV suggested that the higher genus B-model could be constructed from the quantum theory of Kodaira–Spencer gravity. This point of view has remarkable consequences in physics [5, 18, 29], but is much less appreciated in mathematics due to the difficulty of rigorous quantum field theory.

In the long paper [10], we initiated a mathematical analysis of the quantum geometry of perturbative BCOV theory based on the effective renormalization method developed in [8]. In [10, 21], we have shown that the quantum BCOV theory on elliptic curves are equivalent to Gromov–Witten theory on the mirror elliptic curves. This gives the first compact Calabi–Yau example where mirror symmetry is established at all genera. A related work on the finite-dimensional toy model of BCOV theory has been discussed by Losev, Shadrin and Shneiberg [25] to avoid the issue of renormalization.

*This paper is a contribution to the Special Issue "Mirror Symmetry and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/mirror_symmetry.html

In this paper we will outline the main techniques used in [10] toward the construction of higher genus B-model. We will focus on the Fock space formalism, and give an interpretation of Virasoro constraints in the classical and quantum BCOV theory.

2 Classical BCOV theory

2.1 Polyvector fields

Let X be a compact Calabi–Yau manifold of dimension d. Follow [4] we consider the space of polyvector fields on X

$$\mathrm{PV}(X)=\bigoplus_{0\leq i,j\leq d}\mathrm{PV}^{i,j}(X),\qquad \mathrm{PV}^{i,j}(X)=\mathcal{A}^{0,j}\left(X,\wedge^i T_X\right).$$

Here T_X is the holomorphic tangent bundle of X, and $\mathcal{A}^{0,j}(X,\wedge^i T_X)$ is the space of smooth $(0,j)$-forms valued in $\wedge^i T_X$. $\mathrm{PV}(X)$ is a differential bi-graded commutative algebra; the differential is the operator

$$\bar\partial:\ \mathrm{PV}^{i,j}(X)\to\mathrm{PV}^{i,j+1}(X),$$

and the algebra structure arises from wedging polyvector fields. The degree of elements of $\mathrm{PV}^{i,j}(X)$ is $i+j$. The graded-commutativity says that

$$\alpha\beta=(-1)^{|\alpha||\beta|}\beta\alpha,$$

where $|\alpha|$, $|\beta|$ denote the degree of α, β respectively.

The Calabi–Yau condition implies that there exists a nowhere vanishing holomorphic volume form

$$\Omega_X\in\Omega^{d,0}(X),$$

which is unique up to a multiplication by a constant. Let us fix a choice of Ω_X. It induces an isomorphism between the space of polyvector fields and differential forms

$$\mathrm{PV}^{i,j}(X)\overset{\vdash\Omega_X}{\cong}\mathcal{A}^{d-i,j}(X),\qquad \alpha\to\alpha\vdash\Omega_X,$$

where $\vdash$ is the contraction map.

The holomorphic de Rham differential ∂ on differential forms defines an operator on polyvector fields via the above isomorphism, which we still denote by

$$\partial:\ \mathrm{PV}^{i,j}(X)\to\mathrm{PV}^{i-1,j}(X),$$

i.e.

$$(\partial\alpha)\vdash\Omega_X\equiv\partial(\alpha\vdash\Omega_X),\qquad \alpha\in\mathrm{PV}(X).$$

Obviously, the definition of ∂ doesn't depend on the choice of Ω_X. It induces a bracket on polyvector fields

$$\{\alpha,\beta\}=\partial\left(\alpha\beta\right)-\left(\partial\alpha\right)\beta-(-1)^{|\alpha|}\alpha\partial\beta,$$

which associates $\mathrm{PV}(X)$ the structure of *Batalin–Vilkovisky algebra*.

We define the *trace map* $\mathrm{Tr}:\mathrm{PV}(X)\to\mathbb{C}$ by

$$\mathrm{Tr}(\alpha)=\int_X(\alpha\vdash\Omega_X)\wedge\Omega_X.$$

Let $\langle-,-\rangle$ be the induced pairing

$$\mathrm{PV}(X)\otimes\mathrm{PV}(X)\to\mathbb{C},\qquad \alpha\otimes\beta\to\langle\alpha,\beta\rangle\equiv\mathrm{Tr}\left(\alpha\beta\right).$$

It's easy to see that $\bar\partial$ is skew self-adjoint for this pairing and ∂ is self-adjoint.

2.2 BCOV theory and Givental formalism

Following Givental's symplectic formulation [16, 17] of Gromov–Witten theory in the A-model and the parallel Barannikov's works [1, 3] in the B-model, we consider the dg symplectic vector space

$$\mathcal{S}(X) = \mathrm{PV}(X)((t))[2],$$

which is the space of Laurent polynomials in t with coefficient in $\mathrm{PV}(X)$. Here t has cohomology degree two, and the differential is given by $Q = \bar{\partial} + t\partial$, [2] is the conventional shifting of degree by two such that $\mathrm{PV}^{1,1}(X)$, which describes fields associated to the classical deformation of complex structures as in BCOV theory [5], is of degree zero.

The symplectic pairing is

$$\omega(\alpha f(t), \beta g(t)) = \operatorname*{Res}_{t=0}(f(t)g(-t)dt)\,\langle \alpha, \beta\rangle\,, \qquad \alpha, \beta \in \mathrm{PV}(X).$$

Bershadsky, Cecotti, Ooguri and Vafa [5] introduce a gauge theory for polyvector fields on Calabi–Yau three-folds. This is further extended to arbitrary Calabi–Yau manifolds in [10]. The space of fields of the BCOV theory is

$$\mathcal{E}(X) \equiv \mathcal{S}_+(X) \equiv \mathrm{PV}(X)[[t]][2],$$

which is a Lagrangian subspace of $\mathcal{S}(X)$. The classical action functional of the BCOV theory can be constructed from the Lagrangian cone describing the period map of semi-infinite variation of Hodge structures [1]. Let

$$\mathcal{L}_X = \left\{ t\big(1 - e^{f/t}\big) \mid f \in \mathcal{S}_+(X) \right\} \subset \mathcal{S}(X).$$

The geometry of $\mathcal{L}_X$ can be described by the following

Lemma 1 ([10]). *$\mathcal{L}_X$ is a formal Lagrangian submanifold of $\mathcal{S}(X)$, preserved by the differential $Q = \bar{\partial} + t\partial$. Moreover, $\mathcal{L}_X - t$ is a Lagrangian cone preserved by the infinitesimal symplectomorphism of $\mathcal{S}(X)$ given by multiplying by t^{-1}.*

Remark 1. $\mathcal{L}_X - t$ is called the dilaton shift of $\mathcal{L}_X$ [16].

Consider the polarization

$$\mathcal{S}(X) = \mathcal{S}_+(X) \oplus \mathcal{S}_-(X),$$

where $\mathcal{S}_-(X) = t^{-1}\,\mathrm{PV}(X)\big[t^{-1}\big][2]$. It allows us to formally identify

$$\mathcal{S}(X) \cong T^*(\mathcal{S}_+(X)).$$

The generating functional $\mathbf{F}_{\mathcal{L}_X}$ is a formal function on $\mathcal{S}_+(X)$ such that

$$\mathcal{L}_X = \mathrm{Graph}(d\mathbf{F}_{\mathcal{L}_X}).$$

It's easy to see that $\mathbf{F}_{\mathcal{L}_X}$ has cohomology degree $6-2d$, where $d = \dim_{\mathbb{C}} X$. The explicit formula is worked out in [10].

Proposition 1 ([10]).

$$\mathbf{F}_{\mathcal{L}_X}(\mu) = \mathrm{Tr}\,\langle e^{\mu}\rangle_0\,,$$

where $\langle - \rangle_0 : \mathrm{Sym}(\mathrm{PV}(X)[[t]]) \to \mathrm{PV}(X)$ is the map given by intersection of ψ-classes over the moduli space of marked rational curves

$$\left\langle \alpha_1 t^{k_1}, \ldots, \alpha_n t^{k_n} \right\rangle_0 = \alpha_1 \cdots \alpha_n \int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} = \binom{n-3}{k_1, \ldots, k_n} \alpha_1 \cdots \alpha_n.$$

The homogeneous degree n part of the formal function $\mathbf{F}_{\mathcal{L}_X}$ will be denoted by

$$D_n\mathbf{F}_{\mathcal{L}_X}:\ \mathcal{E}(X)^{\otimes n}\to\mathbb{C},\qquad D_n\mathbf{F}_{\mathcal{L}_X}(\mu_1,\dots,\mu_n)=\left(\frac{\partial}{\partial\mu_1}\cdots\frac{\partial}{\partial\mu_n}\right)\mathbf{F}_{\mathcal{L}_X}(0).$$

Then $\mathbf{F}_{\mathcal{L}_X}(\mu)=\sum\limits_{n\geq 3}\frac{1}{n!}D_n\mathbf{F}_{\mathcal{L}_X}(\mu^{\otimes n})$ and

$$D_n\mathbf{F}_{\mathcal{L}_X}\left(\alpha_1t^{k_1},\dots,\alpha_nt^{k_n}\right)=\binom{n-3}{k_1,\dots,k_n}\operatorname{Tr}(\alpha_1\cdots\alpha_n).$$

The lowest component is cubic: it's non-zero only on $t^0\,\mathrm{PV}(X)\subset\mathcal{E}(X)$

$$D_3\mathbf{F}_{\mathcal{L}_X}(\alpha_1,\alpha_2,\alpha_3)=\operatorname{Tr}(\alpha_1\alpha_2\alpha_3),\qquad \alpha_i\in\mathrm{PV}(X),$$

which is literally called the *Yukawa-coupling*.

Definition 1 ([10])**.** The classical BCOV action functional is the formal local functional on polyvector fields $\mathcal{E}(X)$ given by $\mathbf{F}_{\mathcal{L}_X}$.

Our definition of BCOV action functional extends the original BCOV action in [5] by including the "gravitational descendants" t.

We can transfer the geometry of the Lagrangian $\mathcal{L}_X$ into properties of $\mathbf{F}_{\mathcal{L}_X}$.

Proposition 2 ([10])**.** *The BCOV action functional satisfies the classical master equation*

$$Q\mathbf{F}_{\mathcal{L}_X}+\frac{1}{2}\{\mathbf{F}_{\mathcal{L}_X},\mathbf{F}_{\mathcal{L}_X}\}=0,$$

where Q is the induced derivation on the functionals of $\mathcal{E}(X)$, and $\{-,-\}$ is the Poisson bracket on local functionals induced from the distribution representing the operator ∂ (see Remark 3).

This is equivalent to that $\mathcal{L}_X$ is preserved by Q. The appearance of the bracket term follows from the fact that the splitting $\mathcal{S}(X)=\mathcal{S}_+(X)\oplus\mathcal{S}_-(X)$ doesn't respect Q, or more precisely, $\mathcal{S}_-(X)$ is not preserved by Q. We refer to [10, 22] for more careful treatment.

The classical master equation implies that $Q+\{\mathbf{F}_{\mathcal{L}_X},-\}$ is a nilpotent operator acting on local functionals. In physics terminology, it generates the gauge symmetry, and defines the gauge theory in the Batalin–Vilkovisky formalism.

Proposition 3 ([10])**.** *The BCOV action functional satisfies the string equation*

$$D_{n+1}\mathbf{F}_{\mathcal{L}_X}\left(1,\alpha_1t^{k_1},\dots,\alpha_nt^{k_n}\right)=\sum_{k_i>0}D_n\mathbf{F}_{\mathcal{L}_X}\left(\alpha_1t^{k_1},\dots,\alpha_it^{k_i-1},\dots,\alpha_nt^{k_n}\right),\qquad n\geq 3,$$

and the dilaton equation

$$D_{n+1}\mathbf{F}_{\mathcal{L}_X}\left(t,t^{k_1},\dots,\alpha_nt^{k_n}\right)=(n-2)D_n\mathbf{F}_{\mathcal{L}_X}\left(t^{k_1},\dots,\alpha_nt^{k_n}\right)$$

for any $\alpha_1,\dots,\alpha_n\in\mathrm{PV}(X)$.

This follows from Givental's interpretation [16]: the dilaton equation is equivalent to the statement that $\mathcal{L}_X-t$ is a cone, and the string equation is that $\mathcal{L}_X-t$ is preserved by the infinitesimal symplectomorphism of $\mathcal{S}(X)$ given by multiplying by t^{-1}.

Remark 2. The BCOV action functional is equivalent to that used by Losev–Shadrin–Shneiberg [25] in the discussion of finite-dimensional models of Hodge field theory, where the dilaton equation and string equation have also been obtained. Lemma 1 is purely an algebraic property, which is essentially used in [28] in a different context.

3 Fock space and renormaliztion

According to Givental and Coates [7], the generating function for the Gromov–Witten invariants of a variety X is naturally viewed as a state in a Fock space constructed from the cohomology of X. We will describe the Fock space construction in the B-model from the renormalization and effective field theory method [8, 10].

3.1 Fock space and quantization

Let's recall the construction of the Fock module for a finite-dimensional dg symplectic vector space (V, ω, d), where ω is the symplectic form on V, and d is the differential which is skew self-adjoint with respect to ω.

Let $\mathcal{W}(V)$ be the Weyl algebra of V, which is the pro-free dg algebra generated by $V^\vee$ and a formal parameter $\hbar$, subject to the relation that

$$[a,b] = \hbar\omega^{-1}(a,b), \qquad \forall\, a,b \in V^\vee,$$

where $\omega^{-1} \in \wedge^2 V$ is the inverse of ω. Let L be a Lagrangian sub-complex of V, and $\mathrm{Ann}(L) \subset V^\vee$ is the annihilator of L. Then the Fock module $\mathrm{Fock}(L)$ is defined to be the quotient

$$\mathrm{Fock}(L) = \mathcal{W}(V)/\mathcal{W}(V)\,\mathrm{Ann}(L).$$

Since L is preserved by the differential, $\mathrm{Fock}(L)$ naturally inherits a dg structure from d. We will denote it by $\hat{d}$.

Let us choose a complementary Lagrangian $L' \subset V$, $V = L \oplus L'$. L' may not be preserved by the differential. It allows us to identify

$$V \cong T^*(L).$$

Let

$$\mathcal{O}(L) = \widehat{\mathrm{Sym}}^*(L^\vee)$$

be the space of formal functions on L. L' defines a splitting of the map $V^\vee \to L^\vee$, hence a map

$$\begin{array}{ccc} \mathcal{O}(L)((\hbar)) & \longrightarrow & \mathcal{W}(V) \\ & \searrow^{\cong} & \downarrow \\ & & \mathrm{Fock}(L) \end{array}$$

which identifies the Fock module with the algebra $\mathcal{O}(L)((\hbar))$. The differential $\hat{d}$ can be described as follows. Let $\pi : V \to L$ be the projection corresponding to the splitting $V = L \oplus L'$. Consider $(d \otimes 1)\omega^{-1}$, which is an element of $V \otimes V$. Let P be the projection

$$P = \pi \otimes \pi\big((d \otimes 1)\omega^{-1}\big) \in L \otimes L,$$

and it's easy to see that $P \in \mathrm{Sym}^2(L)$. Let $\partial_P : \mathcal{O}(L) \to \mathcal{O}(L)$ be the natural order two differential operator of contracting with P

$$\partial_P : \ \mathrm{Sym}^n\left(L^\vee\right) \to \mathrm{Sym}^{n-2}\left(L^\vee\right).$$

Then under the isomorphism $\mathrm{Fock}(L) \cong \mathcal{O}(L)[[\hbar]]$, $\hat{d}$ takes the form

$$\hat{d} = d_L + \hbar\partial_P,$$

where d_L is the induced differential on $\mathcal{O}(L)$ from that on L.

3.2 Renormalization and effective BV formalism

The dg symplectic vector space related to the BCOV theory is

$$\left(\mathcal{S}(X) = \mathrm{PV}(X)((t))[2], \omega, Q = \bar{\partial} + t\partial\right).$$

If we run the machine to construct the Fock space as in the previous section, we immediately run into trouble: $\mathrm{PV}(X)$ is infinite-dimensional! This is a well-known phenomenon in quantum field theory, which is related to the difficulty of ultra-violet divergence. The standard way of solving this is to use the renormalization technique. We'll follow the approach developed in [8].

3.2.1 Functionals on the fields

Let $\mathcal{E}(X)^{\otimes n}$ be the completed projective tensor product of n copies of $\mathcal{E}(X)$. It can be viewed as the space of smooth polyvector fields on X^n with a formal variable t for each factor. Let

$$\mathcal{O}^{(n)}(\mathcal{E}(X)) = \mathrm{Hom}\left(\mathcal{E}(X)^{\otimes n}, \mathbb{C}\right)_{S_n}$$

denote the space of continuous linear maps (distributions), and the subscript S_n denotes taking S_n coinvariants. $\mathcal{O}^{(n)}(\mathcal{E}(X))$ will be the space of homogeneous degree n functionals on the space of fields $\mathcal{E}(X)$, playing the role of $\mathrm{Sym}^n(V^\vee)$ in the case of finite-dimensional vector space V. We will also let

$$\mathcal{O}^{(n)}_{\mathrm{loc}}(\mathcal{E}(X)) \subset \mathcal{O}^{(n)}(\mathcal{E}(X))$$

be the subspace of local functionals, i.e. those of the form given by the integration of a Lagrangian density

$$\int_X \mathcal{L}(\mu), \qquad \mu \in \mathcal{E}(X).$$

Definition 2. The algebra of functionals $\mathcal{O}(\mathcal{E}(X))$ on $\mathcal{E}(X)$ is defined to be the product

$$\mathcal{O}(\mathcal{E}(X)) = \prod_{n\geq 0} \mathcal{O}^{(n)}(\mathcal{E}(X)),$$

and the space of local functionals is defined to be the subspace

$$\mathcal{O}_{\mathrm{loc}}(\mathcal{E}(X)) = \prod_{n\geq 0} \mathcal{O}^{(n)}_{\mathrm{loc}}(\mathcal{E}(X)).$$

3.2.2 Effective Fock space

Let g be a Kähler metric on X. Let

$$K^g_L \in \mathrm{PV}(X) \otimes \mathrm{PV}(X), \qquad L > 0$$

be the heat kernel for the operator $e^{-L[\bar{\partial}, \bar{\partial}^*]}$, where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ with respect to the metric g. It's a smooth polyvector field on $X \times X$ defined by the equation

$$\left(e^{-L[\bar{\partial},\bar{\partial}^*]}\alpha\right)(x) = \int_X \left(K^g_L(x,y)\alpha(y) \vdash \Omega_X(y)\right) \wedge \Omega_X(y), \qquad \forall\, \alpha \in \mathrm{PV}(X),$$

where we have chosen coordinates (x, y) on $X \times X$, and we integrate over the second copy of X using the trace map.

Definition 3. The effective inverse $\omega_{g,L}^{-1}$ for the symplectic form ω is defined to be the kernel

$$\omega_{g,L}^{-1} = \sum_{k\in\mathbb{Z}} K_L^g(-t)^k \otimes t^{-k-1} \in \mathcal{S}(X)\otimes\mathcal{S}(X), \qquad L>0.$$

Note that $\lim\limits_{L\to 0} K_L^g$ is the delta-function distribution, which is no longer a smooth polyvector field, hence not an element of $S(X)\otimes S(X)$. $\omega_{g,L}^{-1}$ can be viewed as the regularization of ω^{-1} in the infinite-dimensional setting.

Definition 4. The effective Weyl algebra $\mathcal{W}(\mathcal{S}(X),g,L)$ is the quotient of the completed tensor algebra

$$\left(\prod_{n\geq 0}\left(\mathcal{S}(X)^\vee\right)^{\otimes n}\right)\otimes\mathbb{C}((\hbar))$$

by the topological closure of the two-sided ideal generated by

$$[a,b]-\hbar\langle\omega_{g,L}^{-1}, a\otimes b\rangle, \qquad L>0$$

for $a,b\in\mathcal{S}(X)^\vee$. Here $\langle\ \rangle$ is the natural pairing between $S(X)$ and its dual.

Similarly, the Fock space can also be defined using the regularized kernel $\omega_{g,L}^{-1}$.

Definition 5. The Fock space $\mathrm{Fock}\,(\mathcal{S}_+(X),g,L)$ is the quotient of $\mathcal{W}(\mathcal{S}(X))$ by the left ideal generated topologically by the subspace

$$\mathrm{Ann}(\mathcal{S}_+(X),g,L)\subset\mathcal{S}(X)^\vee.$$

Similar to the finite-dimensional case, the polarization $\mathcal{S}(X)=\mathcal{S}_+(X)\oplus\mathcal{S}_-(X)$ gives the identification

$$\mathrm{Fock}\,(\mathcal{S}_+(X),g,L)\cong\mathcal{O}(\mathcal{E}(X))[[\hbar]].$$

We refer to [10] for detailed discussions.

3.2.3 Effective BV formalism

We would like to understand the quantized operator $\hat{Q}_L$ for Q acting on the Fock space represented by the above identification. This is completely similar to the finite-dimensional construction. Let

$$(\partial\otimes 1)K_L^g\in\mathrm{Sym}^2(\mathrm{PV}(X))$$

be the kernel for the operator $\partial e^{-L[\bar\partial,\bar\partial^*]}$. It can be viewed as the projection of $(Q\otimes 1)\omega_{L,g}^{-1}\in\mathrm{Sym}^2(\mathcal{S}(X))$ to $\mathrm{Sym}^2(\mathcal{E}(X))$.

Definition 6. The effective BV operator

$$\Delta_L:\ \mathcal{O}(\mathcal{E}(X))\to\mathcal{O}(\mathcal{E}(X))$$

is the operator of contracting with the smooth kernel $(\partial\otimes 1)K_L^g$.

Since $\Delta_L : \mathcal{O}^{(n)}(\mathcal{E}(X)) \to \mathcal{O}^{(n-2)}(\mathcal{E}(X))$, it could be viewed as an order two differential operator on the infinite-dimensional vector space $\mathcal{E}(X)$. Note that Δ_L has odd cohomology degree, therefore

$$(\Delta_L)^2 = 0.$$

It defines a Batalin–Vilkovisky structure on $\mathcal{O}(\mathcal{E}(X))$, with the Batalin–Vilkovisky bracket defined by

$$\{S_1, S_2\}_L = \Delta_L (S_1 S_2) - (\Delta_L S_1) S_2 - (-1)^{|S_1|} S_1 (\Delta_L S_2), \qquad L > 0.$$

Remark 3. If $S_1, S_2 \in \mathcal{O}_{\rm loc}(\mathcal{E}(X))$, then $\lim\limits_{L\to 0} \{S_1, S_2\}_L$ is well-defined, which is precisely the Poisson bracket in Proposition 2.

Proposition 4 ([10])**.** *Under the isomorphism* $\mathrm{Fock}\,(\mathcal{S}_+(X), g, L) \cong \mathcal{O}(\mathcal{E}(X))[[\hbar]]$*, the induced differential* $\hat{Q}_L$ *is given by*

$$\hat{Q}_L = Q + \hbar \Delta_L.$$

The proof is similar to the finite-dimensional case.

3.2.4 Renormalization group flow and homotopy equivalence

We need to specify a choice of the metric g and a positive number $L > 0$ to construct the Fock space $\mathrm{Fock}\,(\mathcal{S}_+(X), g, L)$. However, we are in a bit better situation. The general machinery of renormalization theory in [8] allows us to prove that the effective Fock spaces are independent of the choice of g and L up to homotopy. This is discussed in detail in [10]. We'll discuss here the homotopy between different choices of the scale L, which is related to the renormalization group flow in quantum field theory.

Definition 7. The effective propagator is defined to be the smooth kernel

$$P_\epsilon^L = \int_\epsilon^L du (\bar\partial^* \partial \otimes 1) K_u^g \in \mathrm{Sym}^2(\mathrm{PV}(X)), \qquad L > \epsilon > 0$$

representing the operator $\bar\partial^* \partial e^{-L[\bar\partial, \bar\partial^*]}$.

Lemma 2. *As an operator on* $\mathcal{O}(\mathcal{E}(X))[[\hbar]]$,

$$\hat{Q}_L = e^{\hbar \partial_{P_\epsilon^L}} \hat{Q}_\epsilon e^{-\hbar \partial_{P_\epsilon^L}},$$

where $\partial_{P_\epsilon^L} : \mathcal{O}(\mathcal{E}(X)) \to \mathcal{O}(\mathcal{E}(X))$ *is the operator of contraction by the smooth kernel* P_ϵ^L.

It follows from this lemma that $e^{\hbar \partial_{P_\epsilon^L}}$ defines the homotopy

$$e^{\hbar \partial_{P_\epsilon^L}} : \ (\mathcal{O}(\mathcal{E}(X))[[\hbar]], Q + \hbar \Delta_\epsilon) \to (\mathcal{O}(\mathcal{E}(X))[[\hbar]], Q + \hbar \Delta_L)$$

between Fock spaces defined at scales ϵ and L. It defines a flow on the space of functionals on the fields, which is called the renormalization group flow in [8] following the physics terminology.

Proposition 5 ([10])**.** *The cohomology* $H^*(\mathrm{Fock}(\mathcal{S}_+(X), g, L), \hat{Q}_L)$ *is independent of* g *and* L*. There are canonical isomorphisms*

$$H^*(\mathrm{Fock}(\mathcal{S}_+(X), g, L), \hat{Q}_L) \cong \mathrm{Fock}\,(H^* \mathcal{S}_+(X)),$$

where $\mathrm{Fock}\,(H^*\mathcal{S}_+(X))$ *is the Fock space for the Lagrangian subspace* $H^*(\mathcal{S}_+(X), Q)$ *of the symplectic space* $(H^*(\mathcal{S}(X), Q), \omega)$.

Remark 4. $\mathrm{Fock}\,(H^*\mathcal{S}_+(X))$ is the mirror of the Fock space of de Rham cohomology classes for Gromov–Witten theory discussed in [7].

4 Quantization and higher genus B-model

We have discussed the classical BCOV action functional $\mathbf{F}_{\mathcal{L}_X}$, and the effective Fock space in the B-model. In this section, we will make the connection between these two. We will discuss the perturbative quantization of BCOV theory from the effective field theory point of view [8], which is modeled on the usual Feynman graph integrals. Any such quantization will give rise to a state in the Fock space. In the case of elliptic curves, we find a canonical quantization, and the corresponding higher genus B-model is mirror to the Gromov–Witten theory.

4.1 Quantum BCOV theory

4.1.1 Perturbative quantization

Definition 8 ([10])**.** A perturbative quantization of the BCOV theory on X is given by a family of functionals

$$\mathbf{F}[L] = \sum_{g\geq 0} \hbar^g \mathbf{F}_g[L] \in \mathcal{O}(\mathcal{E}(X))[[\hbar]]$$

for each $L \in \mathbb{R}_{>0}$, with the following properties

1. The renormalization group flow equation
$$\mathbf{F}[L] = W\left(P_\epsilon^L, \mathbf{F}[\epsilon]\right)$$
for all $L, \epsilon > 0$. Here $W\left(P_\epsilon^L, \mathbf{F}[\epsilon]\right)$ is the connected Feynman graph integrals (connected graphs) with propagator P_ϵ^L and vertices $\mathbf{F}[\epsilon]$. This is equivalent to
$$e^{\mathbf{F}[L]/\hbar} = e^{\hbar\partial_{P_\epsilon^L}} e^{\mathbf{F}[\epsilon]/\hbar}.$$
2. The quantum master equation
$$Q\mathbf{F}[L] + \hbar\Delta_L \mathbf{F}[L] + \frac{1}{2}\{\mathbf{F}[L], \mathbf{F}[L]\}_L = 0, \qquad \forall\, L > 0.$$
This is equivalent to
$$\left(Q + \hbar\Delta_L\right) e^{\mathbf{F}[L]/\hbar} = 0.$$
3. The locality axiom, as in [8]. This says that $\mathbf{F}[L]$ has a small L asymptotic expansion in terms of local functionals.
4. The classical limit condition
$$\lim_{L\to 0}\lim_{\hbar\to 0} \mathbf{F}[L] \equiv \lim_{L\to 0} \mathbf{F}_0[L] = \mathbf{F}_{\mathcal{L}_X}.$$
5. Degree axiom. The functional $\mathbf{F}_g[L]$ is of cohomological degree
$$(\dim X - 3)(2g - 2).$$
6. Hodge weight axiom. We will give $\mathcal{E}(X)$ an additional grading, which we call Hodge weight, by saying that elements in
$$t^m \Omega^{0,*}\left(\wedge^k TX\right) = t^m\, \mathrm{PV}^{k,*}(X)$$
have Hodge weight $k + m - 1$. Then the functional $\mathbf{F}_g[L]$ must be of Hodge weight
$$(\dim X - 3)(g - 1).$$

Remark 5. The space of quantizations of the BCOV theory has a simplicial enrichment, where the above definition gives the 0-simplices. The higher simplices are given by families of metrics, and homotopies of theories, which allow us to identify the quantizations with different choices of the metric. We refer to [8, 10] for the general discussion.

4.1.2 Higher genus B-model

Given a quantization $\{\mathbf{F}[L]\}_{L>0}$ of the BCOV theory, we obtain a state $\left[e^{\mathbf{F}[L]/\hbar}\right]$ in the Fock space $\mathrm{Fock}\left(H^*(\mathcal{S}_+(X))\right)[\hbar^{-1}]$ by Proposition 5. We will denote it by $Z_{\mathbf{F}}$, the partition function.

Definition 9. A polarization of $H^*\mathcal{S}(X)$ is a Lagrangian subspace $\mathcal{L} \subset H^*\mathcal{S}(X)$ preserved by

$$t^{-1}: \ H^*\mathcal{S}(X) \to H^*\mathcal{S}(X),$$

such that

$$H^*\mathcal{S}(X) = H^*\mathcal{S}_+(X) \oplus \mathcal{L}.$$

This definition of polarization is used by Barannikov [2] in the study of semi-infinite variation of Hodge structures. We refer to [10] for more detailed discussion about the relation with BCOV theory.

There's a natural bijection between polarizations of $H^*\mathcal{S}(X)$ and splittings of the Hodge filtration on $H^*(X)$. Moreover, the choice of a polarization induces isomorphisms

$$H^*S(X) \cong H^*_{\bar\partial}(X, \wedge^* T_X)((t))[2], \qquad H^*S_+(X) \cong H^*_{\bar\partial}(X, \wedge^* T_X)[[t]][2].$$

In particular, it induces a natural identification

$$\Phi_{\mathcal{L}}: \ \mathrm{Fock}(H^*(\mathcal{S}(X))) \overset{\cong}{\to} \mathcal{O}(H^*_{\bar\partial}(X, \wedge^* T_X)[[t]][2])[[\hbar]].$$

Definition 10. Let $\mathbf{F}$ be a quantization of the BCOV theory on X, and $\mathcal{L}$ be a polarization of $H^*(\mathcal{S}(X))$. Let $\alpha_1, \ldots, \alpha_n \in H^*_{\bar\partial}(X, \wedge^* T_X)$. The *correlation functions* associated to $\mathbf{F}$, $\mathcal{L}$ is defined by

$$\mathbf{F}_X^{\mathrm{B},\mathcal{L}}\left(t^{k_1}\alpha_1, \ldots, t^{k_n}\alpha_n\right) = \left(\frac{\partial}{\partial t^{k_1}\alpha_1} \cdots \frac{\partial}{\partial t^{k_n}\alpha_n}\right) \hbar \log \Phi_{\mathcal{L}}\left(Z_{\mathbf{F}}\right)(0) \in \mathbb{C}[[\hbar]].$$

Here the superscript "B" refers to the B-model. We can further decompose $\mathbf{F}_X^{\mathrm{B},\mathcal{L}} = \sum_{g \geq 0} \hbar^g \mathbf{F}_{g,X}^{\mathrm{B},\mathcal{L}}$. Then $\mathbf{F}_{g,X}^{\mathrm{B},\mathcal{L}}$ will be the candidate for the higher B-model invariants on X. It's natural to conjecture that there's a unique quantization $\mathbf{F}$ (up to homotopy) of the BCOV theory on X satisfying natural symmetry constraints. Then Z_X will be the mirror of the Gromov–Witten invariants on the mirror Calabi–Yau manifold $X^\vee$. This proves to be the case for X being an elliptic curve.

4.1.3 The polarizations

There are two natural polarizations of $H^*(\mathcal{S}(X))$.

The first one is given by the complex conjugate splitting of the Hodge filtration, which we denote by $\mathcal{L}_{\bar X}$. In this case the correlation function $\mathbf{F}_X^{\mathrm{B},\mathcal{L}_{\bar X}}$ can be realized explicitly as follows. Consider the limit

$$\mathbf{F}[\infty] = \lim_{L\to\infty} \mathbf{F}[L],$$

which is well-defined since X is compact, hence P_L^∞ is smooth. The quantum master equation at $L = \infty$ says that

$$Q\mathbf{F}[\infty] = 0$$

as $\lim_{L\to\infty} \Delta_L = 0$. It follows that $\mathbf{F}[\infty]$ descends to a functional on $H^*(\mathcal{E}(X), Q)$

$$\mathbf{F}[\infty] \in H^*(\mathcal{O}(\mathcal{E}(X))[[\hbar]], Q) \cong \mathcal{O}(H^*(\mathcal{E}(X), Q))[[\hbar]].$$

On the other hand, the choice of the metric induces isomorphisms

$$H^*(\mathcal{S}(X),Q) \cong H^*_{\bar\partial}(X,\wedge^* T_X)((t))[2], \qquad H^*(\mathcal{S}_+(X),Q) \cong H^*_{\bar\partial}(X,\wedge^* T_X)[[t]][2]$$

via Hodge theory, hence defining a polarization

$$H^*(\mathcal{S}(X),Q) = H^*(\mathcal{S}_+(X),Q) \oplus t^{-1} H^*_{\bar\partial}(X,\wedge^* T_X)\left[t^{-1}\right][2],$$

which is easily seen to be precisely $\mathcal{L}_{\bar X}$. Therefore

$$\mathbf{F}_X^{\mathrm{B},\mathcal{L}_{\bar X}} = \mathbf{F}[\infty].$$

The second choice of the polarization is relevant for mirror symmetry, which is defined near a large complex limit in the moduli space of complex structures on X. Near any such large complex limit point, there's an associated limiting weight filtration $\mathcal{W}$ which splits the Hodge filtration. Then the correlation function

$$\mathbf{F}_{g,n,X}^{\mathrm{B},\mathcal{W}} : \ \mathrm{Sym}^n\left(H^*_{\bar\partial}(X,\wedge^* T_X)[[t]][2]\right) \to \mathbb{C}$$

will be the mirror of the descendant Gromov–Witten invariants on the mirror Calabi–Yau $X^\vee$

$$\langle - \rangle_{g,n,X^\vee}^{\mathrm{GW}} : \ \mathrm{Sym}^n\left(H^*(X^\vee,\mathbb{C})\right) \to \mathbb{C}$$

under the mirror map.

Note that $\mathbf{F}_X^{\mathrm{B},\mathcal{L}_{\bar X}}$ doesn't vary holomorphically due to the complex conjugate splitting $\mathcal{L}_{\bar X}$. This is the famous holomorphic anomaly discovered in [5]. Given a large complex limit point, the natural way to retain holomorphicity is to consider $\mathbf{F}_{g,X}^{\mathrm{B},\mathcal{W}}$, which is usually denoted in physics literature by

$$\mathbf{F}_{g,X}^{\mathrm{B},\mathcal{W}} \equiv \lim_{\bar\tau\to\infty} \mathbf{F}_X^{\mathrm{B},\mathcal{L}_{\bar X}}$$

as the "$\bar\tau \to \infty$-limit" [5] near the large complex limit.

4.2 Virasoro equations

The "Virasoro conjecture" was invented by Eguchi, Hori, Jinzenji, Xiong and Katz [12, 13] and axiomatized in the context of topological field theory by Dubrovin and Zhang [11]. Givental [17] gave a geometric formulation of Virasoro equations on Fock spaces from quantization of symplectic vector spaces. For Calabi–Yau manifolds, the Virasoro equations greatly simplify the calculation of Gromov–Witten invariants on elliptic curves [26], but do not provide much information if the dimension is bigger than two [14]. However, the effective version of the Virasoro equations as below is useful to analyze the obstruction of the quantization of BCOV theory.

We will follow Givental's approach and discuss the Virasoro equations in the context of BCOV theory at genus 0. The formalism generalizes in fact to the Landau–Ginzburg twisted version of BCOV theory.

4.2.1 Virasoro operators in Givental formalism

Let $H : \mathcal{S}(X) \to \mathcal{S}(X)$ be the grading operator

$$H\left(t^k\alpha\right) = \left(k + i - \frac{\dim_{\mathbb{C}} X - 1}{2}\right) t^k\alpha, \qquad \text{if } \ \alpha \in \mathrm{PV}^{i,*}(X),$$

and $\hat{t}: \mathcal{S}(X) \to \mathcal{S}(X)$ be the linear operator given by multiplication by t. The Virasoro operators are defined by the linear maps given by the compositions [17]

$$L_n = \big(H\hat{t}\big)^n H: \ \mathcal{S}(X) \to \mathcal{S}(X), \qquad n \geq -1,$$

where it's understood that $L_{-1} = \hat{t}^{-1}$. Since both H and $\hat{t}$ are skew self-adjoint for the symplectic pairing, L_n is also skew self-adjoint, hence defining linear symplectic vector field on $\mathcal{S}(X)$. They satisfy the Virasoro relations

$$[L_n, L_m] = (m-n)L_{n+m}, \qquad \forall\, n, m \geq -1,$$

which follows from $\big[H, \hat{t}\big] = \hat{t}$.

Lemma 3. *The Lagrangian cone $\mathcal{L}_X - t$ is perserved by L_n, $n \geq -1$.*

Proof. Given $\alpha = -te^{f/t} \in \mathcal{L}_X - t$, $f \in \mathcal{S}_+(X)$, the tangent space at α is given by

$$T_\alpha\left(\mathcal{L}_X - t\right) = \mathcal{S}_+(X)e^{f/t}.$$

The vector field of L_n at α is $L_n\alpha$. We observe that $H = U - \frac{\dim_{\mathbb{C}} X - 1}{2}$ where U respects the product structure

$$U(\alpha\beta) = U(\alpha)\beta + \alpha U(\beta), \qquad \forall\, \alpha, \beta \in \mathcal{S}(X).$$

It follows that

$$H\hat{t}: \ \mathcal{S}_+(X)e^{f/t} \to \mathcal{S}_+(X)e^{f/t}.$$

In fact, given any $g \in \mathcal{S}_+(X)$,

$$\begin{aligned} H\hat{t}\big(ge^{f/t}\big) &= U\big(tge^{f/t}\big) - \frac{\dim_{\mathbb{C}} X - 1}{2} tge^{f/t} \\ &= tgU(f/t)e^{f/t} + e^{f/t}U(tg) - \frac{\dim_{\mathbb{C}} X - 1}{2} tge^{f/t}, \end{aligned}$$

which lies in $\mathcal{S}_+(X)e^{f/t}$. Hence $L_n(\alpha) = -\big(H\hat{t}\big)^{n+1} e^{f/t} \in \mathcal{S}_+(X)e^{f/t}$.

This proves the lemma. ■

It follows that the Lagrangian $\mathcal{L}_X$ is invariant under the symplectomorphism generated by the Virasoro vector fields L_n. This geometric interpretation can be equivalently stated for the generating functional $\mathbf{F}_{\mathcal{L}_X}$. We have seen that the L_0-symmetry is equivalent to the dilaton equation (if $\dim_{\mathbb{C}} X \neq 3$), and the L_{-1} symmetry is equivalent to the string equation. For L_n, $n > 0$, we have

Proposition 6. *The classical BCOV action functional $\mathbf{F}_{\mathcal{L}_X}$ satisfies the classical Virasoro equations*

$$-\frac{\partial}{\partial L_n(t)} \mathbf{F}_{\mathcal{L}_X} + L_n \mathbf{F}_{\mathcal{L}_X} + \frac{1}{2}\left\{\mathbf{F}_{\mathcal{L}_X}, \mathbf{F}_{\mathcal{L}_X}\right\}_{L_n} = 0, \qquad n \geq 1.$$

Here,

$$\frac{\partial}{\partial L_n(t)}: \ \mathcal{O}^{(n)}(\mathcal{E}(X)) \to \mathcal{O}^{(n-1)}(\mathcal{E}(X))$$

is the operator of contracting with $L_n(t) = t^{n+1} \prod_{k=0}^{n} \left(k - \frac{\dim_{\mathbb{C}} X - 3}{2}\right)$, and L_n acting on $\mathcal{O}(\mathcal{E}(X))$ is the induced derivation from the linear map L_n on $\mathcal{E}(X)$. The classical Virasoro bracket $\{-,-\}_{L_n}$ is defined in the next section.

The proof of the proposition is purely a translation between Lagrangian submanifold and generating functional, similar to the classical master equation. The appearance of $\frac{\partial}{\partial L_n(t)}$ comes from the dilaton shift. Therefore the classical BCOV action functional $\mathbf{F}_{\mathcal{L}_X}$ has the Virasoro transformations L_n, $n \geq -1$ as classical symmetries. If the Virasoro symmetries are preserved in the quantization (see the discuss below), then the quantum correlation functions will also satisfy the Virasoro symmetries. This gives a natural geometric interpretation of "Virasoro constraints" [12] in the context of B-model.

4.2.2 Quantum Virasoro operators on Fock spaces

The classical Virasoro operators L_n are linear symplectic, and commute with the heat operator $e^{-L[\bar\partial, \bar\partial^*]}$. The usual method of Weyl quantization gives rise to operators acting on the Fock space, similar to the discussion in Section 3.1. We will describe the explicit result in this section. Let

$$(L_n \otimes 1)\, \omega_{L,g}^{-1} \in \mathrm{Sym}^2(\mathcal{S}(X))$$

be the kernel for the operator $L_n e^{-L[\bar\partial, \bar\partial^*]}$ acting on $\mathcal{S}(X)$ with respect to the symplectic pairing ω. Let

$$\pi_+ : \ \mathcal{S}(X) \to \mathcal{S}_+(X)$$

denote the projection associated with the polarization $\mathcal{S}(X) = \mathcal{S}_+(X) \oplus \mathcal{S}_-(X)$. Let

$$\left[\, (L_n \otimes 1)\, \omega_{L,g}^{-1} \right]_+ \in \mathrm{Sym}^2(\mathcal{S}_+(X))$$

be the image of $(L_n \otimes 1)\, \omega_{L,g}^{-1}$ under the projection $\pi_+ \otimes \pi_+$. We will denote by

$$V_{n,L} = \frac{\partial}{\partial \left[\, (L_n \otimes 1)\, \omega_{L,g}^{-1} \right]_+} : \ \mathcal{O}^{(n)}(\mathcal{E}(X)) \to \mathcal{O}^{(n-2)}(\mathcal{E}(X))$$

the operator of contracting with the kernel $\left[\, (L_n \otimes 1)\, \omega_{L,g}^{-1} \right]_+$. Note that $V_{n,L} = 0$ for $n = -1, 0$. Similar to the effective BV bracket, we can define the effective Virasoro bracket by

$$\{S_1, S_2\}_{L_n[L]} = V_{n,L}\,(S_1 S_2) - (V_{n,L} S_1)\, S_2 - S_1\, (V_{n,L} S_2)\,, \qquad S_1, S_2 \in \mathcal{O}(\mathcal{E}(X)),$$

and the classical Virasoro bracket for local functionals

$$\{S_1, S_2\}_{L_n} = \lim_{L \to 0} \{S_1, S_2\}_{L_n[L]}\,, \qquad S_1, S_2 \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}(X)),$$

which is used in Proposition 6 to describe the Virasoro symmetry of the classical BCOV action functional $\mathbf{F}_{\mathcal{L}_X}$.

Definition 11. For $n \geq 0$, $L > 0$, the effective Virasoro operator is defined by

$$L_n[L] = -\frac{\partial}{\partial L_n(t)} + L_n + \hbar V_{n,L}.$$

The effective operator for L_{-1} needs extra care since L_{-1} doesn't preserve $\mathcal{E}(X)$. Let's define a derivation $Y[L]:\mathcal{O}(\mathcal{E}(X))\to\mathcal{O}(\mathcal{E}(X))$ as the derivation associated to the linear map

$$t^k\alpha \to \begin{cases} 0 & \text{if } k>0,\\ \bar\partial^*\partial\int_0^L du e^{-u[\bar\partial,\bar\partial^*]}\alpha & \text{if } k=0,\end{cases}$$

and $\hat t^{-1}:\mathcal{O}(\mathcal{E}(X))\to\mathcal{O}(\mathcal{E}(X))$ as the derivation associated to the linear map

$$t^k\alpha \to \begin{cases} t^{k-1}\alpha & \text{if } k>0,\\ 0 & \text{if } k=0.\end{cases}$$

Definition 12. The effective Virasoro operator $L_{-1}[L]$ is defined by

$$L_{-1}[L]=\hat t^{-1}-\frac{\partial}{\partial(1)}+Y[L]+\frac{1}{\hbar}\left(\frac{\partial}{\partial(1)}D_3\mathbf{F}_{\mathcal{L}_X}\right),$$

where $\frac{\partial}{\partial(1)}D_3\mathbf{F}_{\mathcal{L}_X}\in\mathcal{O}^{(2)}_{\text{loc}}(\text{PV}(X))$ is seen to be the trace pairing.

Proposition 7. *The effective Virasoro operators satisfy the homotopic Virasoro relations*

$$[L_n[L],L_m[L]]=(n-m)L_{n+m}[L],\qquad \forall\, n,m\geq 0,$$

and

$$[L_n[L],L_{-1}[L]]=(n+1)L_{n-1}[L]+\left[\hat Q_L,U_n[L]\right],$$

where $U_0[L]=0$, and for $n\geq 1$, $U_n[L]$ is the derivation on $\mathcal{O}(\mathcal{E}(X))$ induced from the linear map on $\mathcal{E}(X)$

$$t^k\alpha \to \begin{cases} 0 & \textit{if } k>0,\\ \int_0^L du\bar\partial^* e^{-u[\bar\partial,\bar\partial^*]}L_n\big(t^{-1}\alpha\big) & \textit{if } k=0.\end{cases}$$

Moreover, they are compatible with $\hat Q_L$

$$\left[\hat Q_L,L_n[L]\right]=0,\qquad \forall\, n\geq -1$$

Proof. This is a straight-forward check. ∎

4.2.3 Quantization with Virasoro symmetries

Definition 13. We say that a quantization $\mathbf{F}[L]$ of the BCOV theory satisfies the L_n-Virasoro equation if there exists a family $\mathbf{G}[L]\in\hbar\mathcal{O}(\mathcal{E}(X))[[\hbar]]$ satisfying the quantum master equation

$$\big(\hat Q_L+\delta_n L_n[L]\big)e^{(\mathbf{F}[L]+\delta_n\mathbf{G}[L])/\hbar}=0,$$

the renormalization group flow equation

$$e^{(\mathbf{F}[L]+\delta_n\mathbf{G}[L])/\hbar}=e^{\hbar\partial_{P^L_{\epsilon,L_n}}}e^{(\mathbf{F}[\epsilon]+\delta_n\mathbf{G}[\epsilon])/\hbar},$$

and the locality axioms for $\mathbf{G}[L], L\to 0$. Here, δ_n is a parameter of cohomology degree $1-2n$ and square 0. The propagator P^L_{ϵ,L_n} is the kernel

$$P^L_{\epsilon,L_n}=\int_\epsilon^L du\big[\,\big(\bar\partial^*(Q+\delta_nL_n)\otimes 1\big)\,\omega^{-1}_{L,g}\big]_+\in\text{Sym}^2(\mathcal{E}(X)).$$

We can think of $\mathbf{G}[L]$ as giving a homotopy between the quantization $\mathbf{F}[L]$ and the infinitesimal nearby quantization generated by the Virasoro transformations.

If we expand the quantum master equation, we find

$$L_n[L]\mathbf{F}[L] + \frac{1}{2}\{\mathbf{F}[L], \mathbf{F}[L]\}_{L_n[L]} = \hat{Q}_L\mathbf{G}[L] + \{\mathbf{F}[L], \mathbf{G}[L]\}_L .$$

At the quantum limit $L = \infty$, it becomes

$$L_n[\infty]\mathbf{F}[\infty] + \frac{1}{2}\{\mathbf{F}[\infty], \mathbf{F}[\infty]\}_{L_n[\infty]} = 0$$

as an element of $\mathcal{O}(H^*(\mathcal{E}(X)))[[\hbar]]$. Note that at the quantum limit, $V_{n,\infty}$ consists of only harmonic elements. This is the Virasoro equations on cohomology classes in the B-model, mirror to the Virasoro equations for Gromov–Witten theory.

Remark 6. The reason we formulate the homotopic version of Virasoro equations is that this allows us to analyze the obstruction/anomaly of the quantization of BCOV theory preserving the Virasoro symmetry via homological algebra method. The obstructions are typical to the ultraviolet difficulty in quantum field theory due to the necessary introduction of counter-terms for Feynman integrals, which may break the classical symmetries at quantum level. The Virasoro equations above are described at the form level, which would help us to analyze the quantum corrections locally. See [10] for the details on the obstruction analysis.

4.3 Mirror symmetry for elliptic curves

4.3.1 Quantum BCOV theory on elliptic curves

We consider the simpliest example of Calabi–Yau manifolds: elliptic curves. Let E_τ be the elliptic curve

$$\mathrm{E}_\tau = \mathbb{C}/\Lambda_\tau,$$

where $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, and $\tau \in \mathbb{H}$ represents the complex structure.

Theorem 1 ([9, 10])**.** *There's a unique (up to homotopy) quantization of the BCOV theory on E_τ which satisfies the dilaton equation (L_0-Virasoro equation).*

We will denote such quantization by $\mathbf{F}_{\mathrm{E}_\tau}$. Note that in the one-dimensional case, the operators $V_{n,L} = 0$ for all $n \geq 0$. The effective Virasoro operators become derivations on the space of functionals.

Theorem 2 ([10])**.** *The quantization $\mathbf{F}_{E_\tau}$ satisfies the L_n-Virasoro equations for all $n \geq -1$.*

The proof of both theorems are based on the deformation-obstruction theory developed in [8] for the renormalization of gauge theories. This can be viewed as the mirror theorem of the Virasoro constraints for Gromov–Witten theory on elliptic curves established in [26].

4.3.2 Higher genus mirror symmetry

The mirror symmetry for elliptic curves is easy to describe. Let E represent an elliptic curve. In the A-model, we have the moduli of (complexified) Kähler class $[\omega] \in H^2(E, \mathbb{C})$ parametrized by the symplectic volume

$$q = \mathrm{Tr}\,\omega,$$

where the trace map in the A-model is given by the integration $\mathrm{Tr} = \int_E$. The mirror in the B-model is the elliptic curve E_τ, with complex structure τ related by the mirror map

$$q = e^{2\pi i \tau}.$$

Let

$$\Phi_\tau : \bigoplus_{i,j} H^i(E, \wedge^j T_E^*)[-i-j] \to \bigoplus_{i,j} H^i(\mathrm{E}_\tau, \wedge^j T_{\mathrm{E}_\tau})[-i-j]$$

be the unique isomorphism of commutative bigraded algebras which is compatible with the trace on both sides.

Theorem 3 ([21])**.** *For all $\alpha_1, \dots, \alpha_n \in H^*(E, \wedge^* T_E)[[t]]$, the* A*-model Gromov–Witten invariants on E can be identified with the* B*-model BCOV correlation functions on E_τ*

$$\sum_d q^d \langle \alpha_1, \dots, \alpha_n \rangle_{g,n,d}^{\mathrm{GW}(E)} = \lim_{\bar\tau \to \infty} \mathbf{F}_{E_\tau}^{\mathrm{B}, \mathcal{L}_{\bar{E_\tau}}} \left(\Phi_\tau(\alpha_1), \dots, \Phi_\tau(\alpha_n) \right),$$

where the large complex limit is taken to be $\operatorname{Im} \tau \to \infty$ on the upper half plane $\mathbb{H}$.

It's proved in [21, 23] that the correlation functions for $\mathbf{F}_{\mathrm{E}_\tau}^{\mathrm{B}, \mathcal{L}_{\bar{\mathrm{E}_\tau}}}$, before taking the $\bar\tau \to \infty$ limit, are almost holomorphic modular forms exhibiting mild anti-holomorphic dependence on $\bar\tau$. On the other hand, the correlation functions of Gromov–Witten theory are given by quasi-modular forms [26]. In this example, the $\bar\tau \to \infty$ limit is the well-known identification between almost holomorphic modular forms and quasi-modular forms [19].

References

[1] Barannikov S., Extended moduli spaces and mirror symmetry in dimensions $n > 3$, Ph.D. thesis, University of California, Berkeley, 1999, math.AG/9903124.

[2] Barannikov S., Non-commutative periods and mirror symmetry in higher dimensions, *Comm. Math. Phys.* **228** (2002), 281–325.

[3] Barannikov S., Quantum periods. I. Semi-infinite variations of Hodge structures, *Int. Math. Res. Not.* **2001** (2001), no. 23, 1243–1264, math.AG/0006193.

[4] Barannikov S., Kontsevich M., Frobenius manifolds and formality of Lie algebras of polyvector fields, *Int. Math. Res. Not.* **1998** (1998), no. 4, 201–215, alg-geom/9710032.

[5] Bershadsky M., Cecotti S., Ooguri H., Vafa C., Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes, *Comm. Math. Phys.* **165** (1994), 311–427, hep-th/9309140.

[6] Candelas P., de la Ossa X.C., Green P.S., Parkes L., A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory, *Nuclear Phys. B* **359** (1991), 21–74.

[7] Coates T., Givental A.B., Quantum Riemann–Roch, Lefschetz and Serre, *Ann. of Math. (2)* **165** (2007), 15–53, math.AG/0110142.

[8] Costello K.J., Renormalization and effective field theory, *Mathematical Surveys and Monographs*, Vol. 170, American Mathematical Society, Providence, RI, 2011.

[9] Costello K.J., Li S., Open-closed BCOV theory on Calabi–Yau manifolds, in preparation.

[10] Costello K.J., Li S., Quantum BCOV theory on Calabi–Yau manifolds and the higher genus B-model, arXiv:1201.4501.

[11] Dubrovin B., Zhang Y., Frobenius manifolds and Virasoro constraints, *Selecta Math. (N.S.)* **5** (1999), 423–466, math.AG/9808048.

[12] Eguchi T., Hori K., Xiong C.S., Quantum cohomology and Virasoro algebra, *Phys. Lett. B* **402** (1997), 71–80, hep-th/9703086.

[13] Eguchi T., Jinzenji M., Xiong C.S., Quantum cohomology and free-field representation, *Nuclear Phys. B* **510** (1998), 608–622, hep-th/9709152.

[14] Getzler E., The Virasoro conjecture for Gromov–Witten invariants, in Algebraic Geometry: Hirzebruch 70 (Warsaw, 1998), *Contemp. Math.*, Vol. 241, Amer. Math. Soc., Providence, RI, 1999, 147–176, math.AG/9812026.

[15] Givental A.B., A mirror theorem for toric complete intersections, in Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996), *Progr. Math.*, Vol. 160, Birkhäuser Boston, Boston, MA, 1998, 141–175, alg-geom/9701016.

[16] Givental A.B., Gromov–Witten invariants and quantization of quadratic Hamiltonians, *Mosc. Math. J.* **1** (2001), 551–568, math.AG/0108100.

[17] Givental A.B., Symplectic geometry of Frobenius structures, in Frobenius Manifolds, *Aspects Math.*, Vol. E36, Vieweg, Wiesbaden, 2004, 91–112, math.AG/0305409.

[18] Huang M.X., Klemm A., Quackenbush S., Topological string theory on compact Calabi–Yau: modularity and boundary conditions, in Homological Mirror Symmetry, *Lecture Notes in Phys.*, Vol. 757, Springer, Berlin, 2009, 45–102, hep-th/0612125.

[19] Kaneko M., Zagier D., A generalized Jacobi theta function and quasimodular forms, in The Moduli Space of Curves (Texel Island, 1994), *Progr. Math.*, Vol. 129, Birkhäuser Boston, Boston, MA, 1995, 165–172.

[20] Li J., Tian G., Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties, *J. Amer. Math. Soc.* **11** (1998), 119–174, alg-geom/9602007.

[21] Li S., BCOV theory on the elliptic curve and higher genus mirror symmetry, arXiv:1112.4063.

[22] Li S., Calabi–Yau geometry and higher genus mirror symmetry, Ph.D. thesis, Harvard University, 2011.

[23] Li S., Feynman graph integrals and almost modular forms, *Commun. Number Theory Phys.* **6** (2012), 129–157, arXiv:1112.4015.

[24] Lian B.H., Liu K., Yau S.T., Mirror principle. I, *Asian J. Math.* **1** (1997), 729–763, alg-geom/9712011.

[25] Losev A., Shadrin S., Shneiberg I., Tautological relations in Hodge field theory, *Nuclear Phys. B* **786** (2007), 267–296, arXiv:0704.1001.

[26] Okounkov A., Pandharipande R., Virasoro constraints for target curves, *Invent. Math.* **163** (2006), 47–108, math.AG/0308097.

[27] Ruan Y., Tian G., A mathematical theory of quantum cohomology, *J. Differential Geom.* **42** (1995), 259–367.

[28] Shadrin S., BCOV theory via Givental group action on cohomological fields theories, *Mosc. Math. J.* **9** (2009), 411–429, arXiv:0810.0725.

[29] Yamaguchi S., Yau S.T., Topological string partition functions as polynomials, *J. High Energy Phys.* **2004** (2004), no. 7, 047, 20 pages, hep-th/0406078.