Hidden Symmetry from Supersymmetry in One-Dimensional Quantum Mechanics*

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Abstract. When several inequivalent supercharges form a closed superalgebra in Quantum Mechanics it entails the appearance of hidden symmetries of a Super-Hamiltonian. We examine this problem in one-dimensional QM for the case of periodic potentials and potentials with finite number of bound states. After the survey of the results existing in the subject the algebraic and analytic properties of hidden-symmetry differential operators are rigorously elaborated in the Theorems and illuminated by several examples.

Key words: supersymmetric quantum mechanics; periodic potentials; hidden symmetry

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1 Introduction

The interest to hidden symmetries in quantum dynamical systems accompanies Quantum Mechanics (QM) nearly since its very formulation [1]. If existing they play an exceptional role in unraveling the energy spectra and, especially, in construction of integrable systems related to the soliton dynamics [2] where hidden symmetry operators represent key ingredients of the Lax pair method. One should also mention the related study of conditional (formal) symmetries of the Shrödinger equation realized by differential operators of finite order which was undertaken in [3].

Recently a particular class of hidden symmetries in one-dimensional QM has been explored with the help of Non-linear Supersymmetric QM [4]. The idea that a hidden QM symmetry is accounted for by the existence of several supercharges was outlined in [5]. Accordingly such a connection leads to the realization of $\mathcal{N} = 4$ Non-linear SUSY QM with a central charge. In [4] the representation of this algebra was found among the quantum systems with reflectionless potentials.

A similar class of SUSY-induced hidden symmetries exists among periodic systems, its possibility for periodic potentials was guessed in [6]. In [7, 8] it was found that a conventional supersymmetric extension of a periodic quantum system may give an isospectral pair with a zero energy doublet of the ground states. For non-periodic systems with real scalar potentials it can happen only[1] for non-linear SUSY with supercharges of higher order in derivatives [11]. It was also established [8] that among isospectral supersymmetric periodic systems there are some

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[1] There are more options for matrix [9] and complex potentials [10].

self-isospectral samples for which the partner potentials are identical in shape but related by translation for a half-period and/or reflection. Later on, different aspects of isospectral and self-isospectral supersymmetric periodic systems were examined in [12, 13, 14]. Recently [15] a property of the self-isospectrality was revealed in some periodic finite-gap systems based on a nonlinear supersymmetry of the second order. In the papers [16], the superextension of quantum periodic systems has been studied with a parity-even finite-gap potential of general form, and it was shown that it is characterized by a tri-supersymmetric structure. This supersymmetric structure originates from the higher order differential operator of the Lax pair.

After this concise review of the state-of-art let us formulate the aim of our paper and its novel results. The main goal is to fill the missing points of previous studies of hidden symmetry operators as elements of a SUSY algebra, in other words, when they interplay with the Darboux–Crum construction. We pay more attention to the spectral problem and therefore investigate the kernels of both intertwining and symmetry operators simultaneously. The novel element is in extension of present and previously known facts to complex potentials [10, 17, 18, 19, 20, 21, 22, 23, 24] for Hamiltonians which are not necessarily diagonalizable [10, 18]. We examine also the possibility of a non-Hermitian Hamiltonian to be PT symmetric [21]. Section 2 is devoted to the summary of algebraic and differential properties of Non-linear SUSY in one-dimensional QM (in particular, of the matrix $\mathbf{S}$ construction) and to the definitions related to minimization of this algebra while keeping invariant the spectrum and the structure of the Super-Hamiltonian. In Section 3 Extended SUSY is introduced and its optimal structure is elucidated. When the extension is nontrivial its central charge is built of symmetry operators which can be chosen to be antisymmetric under transposition (t-antisymmetric) by a suitable redefinition of elements of supercharges. When the Hamiltonians are PT-symmetric the corresponding t-antisymmetric symmetry operators become PT-antisymmetric. Theorem 3 culminating for the paper is formulated and proven in Section 4. It summarizes all about t-antisymmetric non-minimizable symmetry operators for periodic potentials, including the smoothness of their coefficients, the content of their kernels based on the characteristic polynomials, relationship of zero-modes and the borders of forbidden bands, their factorization into the product of elementary intertwining operators and generation of the ladder of intermediate Hamiltonians together with corresponding symmetry operators intertwined with the basic ones. Separately, in Section 5 we re-analyze the Hamiltonians with bound states and extend the previous results to complex potentials for Hamiltonians which may possess not only eigenfunctions but also normalizable associated functions. Conventionally they are not diagonalizable but can be reduced to a Jordan form. In this section, mainly the potentials with constant asymptotics are within the scope although the basic results are valid for a large class of potentials. Again the content of the symmetry operator kernels based on the characteristic polynomials, relationship of their zero-modes and bound-states, their factorization into the product of elementary intertwining operators and generation of the ladder of intermediate Hamiltonians together with their symmetry operators are investigated. A number of examples illuminating the general statements are elaborated throughout the paper. In Conclusions possible generalizations of the obtained results onto the Hamiltonians with quasiperiodic potentials and the Hamiltonians with reflectionless potentials against the background of finite-zone potentials are briefly discussed.

2 Basic definitions and notations

We start with the definition of a SUSY algebra in Quantum Mechanics [25, 26, 27, 28, 29, 30, 31, 32, 33, 34] and notations for its components. Consider two one-dimensional Hamiltonians of the Schrödinger type defined on the entire axis and having sufficiently smooth and in general complex-valued potentials $V_1(x)$ and $V_2(x)$. The Hamiltonians are assembled into the Super-

Hamiltonian,

$$H = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix}, \qquad h^+ = -\partial^2 + V_1(x), \quad h^- = -\partial^2 + V_2(x), \qquad \partial \equiv d/dx.$$

Assume that the Hamiltonians h^+ and h^- have (almost) equal energy spectra of bound states and equal spectral densities of the continuous spectrum part. Let it be provided by the Darboux–Crum operators $q_N^\pm$ with the help of intertwining:

$$h^+ q_N^+ = q_N^+ h^-, \qquad q_N^- h^+ = h^- q_N^-. \tag{2.1}$$

Further on, we restrict ourselves to differential Darboux–Crum operators of a finite order N,

$$q_N^\pm = \sum_{k=0}^{N} w_k^\pm(x)\partial^k, \qquad w_N^\pm \equiv (\mp 1)^N,$$

with sufficiently smooth and in general complex-valued coefficients $w_k^\pm(x)$. In this case, in the fermion number representation, the nonlinear $\mathcal{N} = 1$ SUSY QM [11, 35, 36, 37, 38] is formed by means of the nilpotent supercharges:

$$Q_N = \begin{pmatrix} 0 & q_N^+ \\ 0 & 0 \end{pmatrix}, \qquad \bar{Q}_N = \begin{pmatrix} 0 & 0 \\ q_N^- & 0 \end{pmatrix}, \qquad Q_N^2 = \bar{Q}_N^2 = 0.$$

Obviously, the intertwining relations (2.1) lead to the supersymmetry of the Hamiltonian H:

$$[H, Q_N] = [H, \bar{Q}_N] = 0.$$

In view of intertwining (2.1) the kernel of $q_N^\pm$ is an invariant subspace with respect to the Hamiltonian $h^\mp$,

$$h^\mp \ker q_N^\pm \subset \ker q_N^\pm.$$

Hence, there is a constant $N \times N$ matrix $\mathbf{S}^\mp \equiv \|S_{ij}^\mp\|$ for an arbitrary basis $\phi_1^\pm(x), \ldots, \phi_N^\pm(x)$ in $\ker q_N^\pm$ such that

$$h^\mp \phi_i^\pm = \sum_{j=1}^{N} S_{ij}^\mp \phi_j^\pm. \tag{2.2}$$

In what follows, for an intertwining operator, its *matrix* $\mathbf{S}$ is defined as the matrix which is related to the operator in the same way as the matrices $\mathbf{S}^\mp$ are related to $q_N^\pm$. In this case, we do not specify the basis in the kernel of the intertwining operator in which the matrix $\mathbf{S}$ is chosen if we concern ourselves only with spectral characteristics of the matrix, or, what is the same, spectral characteristics of the restriction of the corresponding Hamiltonian to the kernel of the intertwining operator considered (cf. (2.2)).

The introduced above, nonlinear SUSY algebra is closed by the following relation between the supercharges and Super-Hamiltonian:

$$\{Q_N, \bar{Q}_N\} = \mathcal{P}_N(H),$$

where $\mathcal{P}_N(H)$ is a differential operator of $2N$th order commuting with the Super-Hamiltonian. Depending on a relation between the supercharges Q_N and $\bar{Q}_N$ (the intertwining operators $q_N^\pm$), the operator $P_N(H)$ can be either a polynomial of the Super-Hamiltonian, if the supercharges are connected by the operation of transposition:

$$\bar{Q}_N = Q_N^t, \qquad q_N^- = (q_N^+)^t \equiv \sum_{k=0}^{N} (-\partial)^k w_k^+(x), \tag{2.3}$$

or in general a function of both the Super-Hamiltonian and a differential symmetry operator of odd order in derivatives (see a detailed analysis and references in [4]). In the present paper, we confine ourselves to the first case in which the conjugated supercharge is produced by transposition (2.3). A relevant theorem on the structure of such a SUSY [4, 10] reads.

Theorem 1 (on SUSY algebra with transposition symmetry). *The closure of the supersymmetry algebra with $\bar{Q}_N = Q_N^t$ takes a polynomial form:*

$$\{Q_N, Q_N^t\} = \det[E\mathbf{I} - \mathbf{S}^+]_{E=H} = \det[E\mathbf{I} - \mathbf{S}^-]_{E=H} \equiv \mathcal{P}_N(H),$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{S}^\pm$ is the matrix $\mathbf{S}$ of the intertwining operator $q_N^\mp$.

Corollary 1. *The spectra of the matrices $\mathbf{S}^+$ and $\mathbf{S}^-$ are equal.*

A basis in the kernel of an intertwining operator in which the matrix $\mathbf{S}$ of this operator has a Jordan form is called *canonical*; elements of a canonical basis are called *transformation functions*.

If a Jordan form of the matrix $\mathbf{S}$ of an intertwining operator has cells of a size higher than one, then the corresponding canonical basis contains not only formal solutions of the Schrödinger equation but also formal associated functions, which are defined as follows.

A function $\psi_{n,i}(x)$ is called *a formal associated function of i-th order* of the Hamiltonian h for a spectral value λ_n if

$$(h - \lambda_n)^{i+1}\psi_{n,i} \equiv 0 \qquad \text{and} \qquad (h - \lambda_n)^i \psi_{n,i} \not\equiv 0.$$

The term "formal" emphasizes that this function is not necessarily normalizable (not necessarily belongs to $L_2(\mathbb{R})$). In particular, an associated function $\psi_{n,0}(x)$ of zero order is a formal eigenfunction of h (not necessarily a normalizable solution of the homogeneous Schrödinger equation).

Assume that the intertwining operator $q_N^\pm$ is represented as a product of intertwining operators $k_{N-M}^\pm$ and $p_M^\pm$, $0 < M < N$, so that

$$q_N^+ = p_M^+ k_{N-M}^+, \qquad q_N^- = k_{N-M}^- p_M^-, \qquad p_M^+ h_M = h^+ p_M^+, \qquad p_M^- h^+ = h_M p_M^-,$$
$$k_{N-M}^+ h^- = h_M k_{N-M}^+, \qquad k_{N-M}^- h_M = h^- k_{N-M}^-, \qquad \text{and} \qquad h_M = -\partial^2 + v_M(x),$$

where the coefficients $k_{N-M}^\pm$ and $p_M^\pm$ as well as the potential $v_M(x)$ may be complex-valued and/or containing pole-like singularities. The Hamiltonian h_M is called *intermediate with respect to h^+ and h^-*. In this case, by Theorem 1, the spectrum of the matrix $\mathbf{S}$ of the operator $q_N^\pm$ is the union of the spectra of the matrices $\mathbf{S}$ for the operators $k_{N-M}^\pm$ and $p_M^\pm$.

The potentials $V_1(x)$ and $V_2(x)$ of the Hamiltonians h^+ and h^- are interrelated by the equation

$$V_2(x) = V_1(x) - 2[\ln W(x)]'', \tag{2.4}$$

where $W(x)$ is the Wronskian of elements of an arbitrary (a canonical as well) basis in $\ker q_N^-$. The validity of equation (2.4) follows from the Liouville–Ostrogradsky relation and the equality of coefficients at ∂^N in $q_N^- h^+$ and $h^- q_N^-$ (see the intertwining in (2.1)).

With the help of a basis $\phi_1^\pm(x)$, ..., $\phi_N^\pm(x)$ in $\ker q_N^\pm$ the operator $q_N^\pm$ can be presented in the form

$$q_N^\pm = \frac{1}{W_\pm(x)} \begin{vmatrix} \phi_1^\pm(x) & \phi_1^{\pm\prime}(x) & \ldots & \phi_1^{\pm(N)}(x) \\ \multicolumn{4}{c}{\dotfill} \\ \phi_N^\pm(x) & \phi_N^{\pm\prime}(x) & \ldots & \phi_N^{\pm(N)}(x) \\ 1 & \partial & \ldots & \partial^N \end{vmatrix}, \tag{2.5}$$

where $W_\pm(x)$ is the Wronskian of these basis elements and the differential operators must be placed on the right-hand side when calculating the determinant elements.

An intertwining operator $q_N^\pm$ is called *minimizable* (*$q_N^\pm$ can be "stripped off"*) if this operator can be represented in the form[2]

$$q_N^\pm = \mathcal{P}(h^\pm)p_M^\pm = p_M^\pm \mathcal{P}(h^\mp),$$

where $p_M^\pm$ is an operator of order M which intertwines the same Hamiltonians as $q_N^\pm$ (i.e., $p_M^\pm h^\mp = h^\pm p_M^\pm$), and $\mathcal{P}(h)$ is a polynomial of degree $(N-M)/2 > 0$. Otherwise, the intertwining operator $q_N^\pm$ is called *non-minimizable* (*$q_N^\pm$ cannot be "stripped off"*). The following theorem [4, 10] contains the necessary and sufficient conditions controlling whether an intertwining operator is minimizable or not.

Theorem 2 (on minimization of an intertwining operator). *An intertwining operator $q_N^\pm$ can be represented in the form*

$$q_N^\pm = p_M^\pm \prod_{l=1}^{m} (\lambda_l - h^\mp)^{\delta k_l},$$

where $p_M^\pm$ is a non-minimizable operator intertwining the same Hamiltonians as $q_N^\pm$ (so that $p_M^\pm h^\mp = h^\pm p_M^\pm$), if and only if a Jordan form of the matrix $\mathbf{S}$ of the operator $q_N^\pm$ has m pairs (and no more) of Jordan cells with equal eigenvalues λ_l such that, for the l-th pair, δk_l is an order of the smallest cell and $k_l + \delta k_l$ is an order of the largest cell. In this case, $M = N - 2\sum_{l=1}^m \delta k_l = \sum_{l=1}^n k_l$ (where the k_l, $m+1 \leqslant l \leqslant n$, are orders of the remaining unpaired Jordan cells).

Remark 1. A Jordan form of the matrix $\mathbf{S}$ of the intertwining operator $q_N^\pm$ cannot have more than two cells with the same eigenvalue λ; otherwise, $\ker(\lambda - h^\mp)$ includes more than two linearly independent elements.

Corollary 2. *Jordan forms of the matrices $\mathbf{S}$ of the operators q_N^+ and q_N^- coincide up to permutation of Jordan cells.*

3 Several supercharges and extended SUSY

Let us examine the case when for a Super-Hamiltonian H there are two different supercharges K and P of the type Q and of the order N and M respectively,

$$K = \begin{pmatrix} 0 & k_N^+ \\ 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & p_M^+ \\ 0 & 0 \end{pmatrix}.$$

In particular, if a complex supercharge Q exists for the Hermitian Super-Hamiltonian H, we can choose K and P as $(Q + Q^*)/2$ and $(Q - Q^*)/(2i)$ respectively, where * denotes complex

[2] The possibility of existence of a cofactor polynomial in the Hamiltonian for an intertwining operator was mentioned in [35].

conjugation of coefficient functions. Let us assume that $N > M$ (in the case $N = M$, instead of p_M^+ we can use a suitable linear combination of k_N^+ and p_M^+ , the order of which is less than N). Each of supercharges K and P generates a unique supercharge of the type $\bar{Q}$:

$$\bar{K} = K^t = \begin{pmatrix} 0 & 0 \\ k_N^- & 0 \end{pmatrix}, \qquad k_N^- = (k_N^+)^t; \qquad \bar{P} = P^t = \begin{pmatrix} 0 & 0 \\ p_M^- & 0 \end{pmatrix}, \qquad p_M^- = (p_M^+)^t.$$

The existence of two supercharges of the type Q (i.e. K and P) conventionally signifies the extension of SUSY algebra. To close the algebra one has to include all anti-commutators between supercharges. Two supercharges K and P generate two Polynomial SUSY,

$$\left\{K, K^t\right\} = \tilde{\mathcal{P}}_N(H), \qquad \left\{P, P^t\right\} = \tilde{\mathcal{P}}_M(H),$$

which have to be embedded into a $\mathcal{N} = 4$ SUSY algebra. The closure of the extended, $\mathcal{N} = 4$ SUSY algebra is given by

$$\left\{P, K^t\right\} \equiv \mathcal{R} = \begin{pmatrix} p_M^+ k_N^- & 0 \\ 0 & k_N^- p_M^+ \end{pmatrix}, \qquad \left\{K, P^t\right\} \equiv \bar{\mathcal{R}} = \begin{pmatrix} k_N^+ p_M^- & 0 \\ 0 & p_M^- k_N^+ \end{pmatrix}.$$

Evidently the components of operators $\mathcal{R}$, $\bar{\mathcal{R}} = \mathcal{R}^t$ are differential operators of $N + M$ order commuting with the Hamiltonians $h^\pm$, hence they form symmetry operators $\mathcal{R}$, $\bar{\mathcal{R}}$ for the Super-Hamiltonian. However, in general, they are not polynomials of the Hamiltonians $h^\pm$ and these symmetries impose certain constraints on potentials.

Let us find the formal relation between the symmetry operators $\mathcal{R}$, $\bar{\mathcal{R}}$ and the Super-Hamiltonian. These operators can be decomposed into t-symmetric and t-antisymmetric parts,

$$\mathcal{B} \equiv \tfrac{1}{2}(\mathcal{R} + \bar{\mathcal{R}}) \equiv \begin{pmatrix} b^+ & 0 \\ 0 & b^- \end{pmatrix}, \qquad \mathcal{E} \equiv \tfrac{1}{2}(\mathcal{R} - \bar{\mathcal{R}}) \equiv \begin{pmatrix} e^+ & 0 \\ 0 & e^- \end{pmatrix}.$$

The operator $\mathcal{B}$ plays essential role in the one-parameter non-uniqueness of the SUSY algebra. Indeed, one can always redefine the higher-order supercharge as follows,

$$K_\zeta = K + \zeta P, \qquad \left\{K_\zeta, K_\zeta^t\right\} = \tilde{\mathcal{P}}_{\zeta,N}(H) \tag{3.1}$$

keeping the same order N of Polynomial SUSY for arbitrary complex parameter ζ. From (3.1) one gets,

$$2\zeta\mathcal{B}(H) = \tilde{\mathcal{P}}_{\zeta,N}(H) - \tilde{\mathcal{P}}_N(H) - \zeta^2\tilde{\mathcal{P}}_M(H),$$

thereby t-symmetric operator $\mathcal{B}$ is a polynomial of the Super-Hamiltonian of the order $N_b \leqslant (N + M)/2$.

If the second t-antisymmetric symmetry operator $\mathcal{E}$ does not vanish identically, then it is a differential operator of *odd* order and cannot be realized by a polynomial in H. But at the same time

$$\mathcal{E}^2(H) \equiv \tfrac{1}{4}\left[(\mathcal{R} + \bar{\mathcal{R}})^2 - 2(\mathcal{R}\bar{\mathcal{R}} + \bar{\mathcal{R}}\mathcal{R})\right] = \mathcal{B}^2(H) - \tilde{\mathcal{P}}_N(H)\tilde{\mathcal{P}}_M(H) \equiv -\mathcal{P}_e(H) \tag{3.2}$$

is a polynomial in H. Thus the nontrivial operator $\mathcal{E}(H)$ is a non-polynomial function of H – the square root of (3.2) in an operator sense. This operator is certainly non-trivial if the sum of orders $N + M$ of the operators $k_N^\pm$ and $p_M^\pm$ is odd and therefore the order of $\mathcal{E}(H)$ amounts to $N_e = N + M$.

As it is known [4, 10] the case $\mathcal{E}(H) = 0$ implies relationship between the supercharges K and P, namely, their identity after minimization. The case $\mathcal{B}(H) = 0$ entails the equalities

$$\mathcal{E} = \mathcal{R} \;\Leftrightarrow\; \mathcal{R} = -\bar{\mathcal{R}},$$

accordingly the Hamiltonian h^- (h^+) is intermediate with respect to some factorization of the nontrivial t-antisymmetric symmetry operator e^+ (e^-) of the Hamiltonian h^+ (h^-),

$$e^+ = p_M^+ k_N^- (= -k_N^+ p_M^-) \qquad \big(e^- = k_N^- p_M^+ (= -p_M^- k_N^+)\big).$$

If the supercharges K and P are independent (i.e. if $\mathcal{E}(H) \neq 0$) one can achieve vanishing of $\mathcal{B}(H)$-type symmetry operator taking instead of K and P the new pair of independent supercharges:

$$\tilde{K} = \mathcal{P}_M(H)K - \mathcal{B}(H)P = \begin{pmatrix} 0 & \mathcal{P}_M(h^+)k_N^+ - b^+ p_M^+ \\ 0 & 0 \end{pmatrix}, \qquad P,$$

wherefrom it follows that

$$\begin{aligned}\tilde{\mathcal{B}}(H) &= \tfrac{1}{2}\big(\{P,\tilde{K}^t\} + \{\tilde{K},P^t\}\big)\\ &= \tfrac{1}{2}\big(\mathcal{P}_M(H)\{P,K^t\} - \mathcal{B}(H)\{P,P^t\} + \mathcal{P}_M(H)\{K,P^t\} - \mathcal{B}(H)\{P,P^t\}\big)\\ &= \mathcal{P}_M(H)\mathcal{B}(H) - \mathcal{B}(H)\mathcal{P}_M(H) = 0.\end{aligned}$$

If the supercharges K and P are dependent (i.e. if $\mathcal{E}(H) = 0$) then obviously the polynomial $\mathcal{B}(H)$ cannot vanish and its order is $(N+M)/2$.

Let us assume that $\mathcal{E}(H) \neq 0$ and minimize the symmetry operators e^+ and e^-,

$$e^+ = \mathcal{P}_+(h^+)\tilde{e}^+, \qquad e^- = \mathcal{P}_-(h^-)\tilde{e}^-,$$

where $\tilde{e}^+$ and $\tilde{e}^-$ are non-minimizable symmetry operators for h^+ and h^- respectively. The operators $\tilde{e}^+$ and $\tilde{e}^-$ are t-antisymmetric as well, because in the opposite case

$$e^\pm + (e^\pm)^t = \mathcal{P}_\pm(h^\pm)\tilde{e}^\pm + (\tilde{e}^\pm)^t\mathcal{P}_\pm(h^\pm) = \mathcal{P}_\pm(h^\pm)[\tilde{e}^\pm + (\tilde{e}^\pm)^t] \neq 0.$$

It is known (see the example in [4]) that the polynomials $\mathcal{P}_+$ and $\mathcal{P}_-$ are in general different. Thus evidently the symmetry operator $\mathcal{E}(H)$ of the SUSY algebra can be minimized only by separation of the polynomial in the Super-Hamiltonian H which is the greatest common polynomial divisor of the polynomials $\mathcal{P}_+$ and $\mathcal{P}_-$. In view of Theorem 1 and t-antisymmetry of $\tilde{\mathcal{E}}(H)$, the spectra of the matrices $\mathbf{S}$ for elements of the minimized $\mathcal{E}(H)$ (we shall denote it by $\tilde{\mathcal{E}}(H)$) are identical among themselves and to the set of zeros of the polynomial $\tilde{\mathcal{E}}^2(H)$. Any element of these spectra obviously belong either to the spectrum of the matrix $\mathbf{S}$ of $\tilde{e}^+$ or to the spectrum of the matrix $\mathbf{S}$ of $\tilde{e}^-$.

In the following sections it will be shown that the spectrum of the matrix $\mathbf{S}$ of a non-minimizable t-antisymmetric operator e for a Hamiltonian h consists of energies of all bound states of h and of all boundaries of continuous spectrum of h as well as (in the case of non-Hermitian h) of other characteristic points of the h spectrum (herein under h and e we imply any of Hamiltonians $h^\pm$ and of a related, properly minimized symmetry operator $\tilde{e}^\pm$). Hence, all zeros of the polynomial $\tilde{\mathcal{E}}^2(H)$ possess a physical meaning and represent characteristic points of H spectrum, in particular, all energies of bound states, all boundaries of continuous spectrum etc.

It has been established [4, 10] that any nonzero t-antisymmetric symmetry operator can be presented in the form $\mathcal{P}(h)e$, where $\mathcal{P}$ is a polynomial of the Hamiltonian and e is a unique t-antisymmetric non-minimizable symmetry operator with unit coefficient at the highest-order derivative. Moreover, if the potential $V(x)$ is real-valued then all coefficients of e are obviously real-valued as well. The two following sections are devoted to investigation of properties of the operator e.

For the characteristic polynomial of the matrix $\mathbf{S}$ for e (with the help of which the e squared is expressed through the Hamiltonian h in virtue of Theorem 1), we shall use the following notation

$$\mathcal{P}_e(h) \equiv ee^t = -e^2 = e^t e. \tag{3.3}$$

It is evident that the degree of this polynomial is equal to the order of e and that e is an algebraic function (square root of polynomial) of the Hamiltonian h.

We shall proceed in investigation of e in two cases: in the case of periodic $V(x)$ and in the case, when there are bound states for h.

Remark 2. One can easily check, that in the case of PT-symmetric potential $V(x)$ the operator $PTePT$ is t-antisymmetric differential symmetry operator for h of the same order as e. Thus, in view of the uniqueness of e, its odd order and of the equality $PT\partial = -\partial PT$ the following relations hold:

$$PTePT = -e \;\Leftrightarrow\; PTe = -ePT. \tag{3.4}$$

It follows obviously from (3.4) and from the equality $PTh = hPT$ (which is equivalent to PT-symmetry of $V(x)$), that:

(1) the part of a canonical basis in the $\ker e$, corresponding to real eigenvalues of the matrix $\mathbf{S}$ of e, can be constructed from PT-symmetric functions;

(2) if there is a non-real eigenvalue of the matrix $\mathbf{S}$ of e, then there is also the complex conjugated eigenvalue of the same algebraic multiplicity for this matrix and the elements of a canonical basis in $\ker e$ corresponding to these eigenvalues can be constructed from mutually PT conjugated functions.

As it is shown in the following sections, any eigenvalue of the matrix $\mathbf{S}$ of e is a characteristic point of h spectrum. Thus in the case of unbroken PT-symmetry all elements of a canonical basis in $\ker e$ can be chosen PT-symmetric.

4 t-antisymmetric symmetry operators: Hamiltonians with periodic potential

Properties of t-antisymmetric symmetry operator in this case are elucidated in the following

Theorem 3. *Assume that:*

(1) *the potential $V(x)$ of the Hamiltonian $h = -\partial^2 + V(x)$ is a real-valued periodic function belonging to $C^\infty_{\mathbb{R}}$ and $X_0 > 0$ is a period of $V(x)$;*

(2) *there is a t-antisymmetric non-minimizable symmetry operator*

$$e = \partial^N + \alpha_{N-1}(x)\partial^{N-1} + \cdots + \alpha_1(x)\partial + \alpha_0(x)$$

for the Hamiltonian h,

$$eh = he, \qquad e^t = -e,$$

and $\alpha_l(x)$ belongs to $C^l_{\mathbb{R}} \cap C^2_{\mathbb{R}}$, $l = 0, \dots, N-1$;

(3) *$\psi_j(x)$ is a real-valued periodic or antiperiodic wave function of h corresponding to the boundary E_j ($j+1$th from below) between forbidden and allowed bands of the h spectrum, $j = 0, 1, 2, \dots$;*

(4) *$\mathcal{P}_e(h) = e^t e$.*

Then:

(1) $\alpha_l(x)$ *is a real-valued periodic* (*with the period* X_0) *function belonging to* $C^\infty_{\mathbb{R}}$, $l = 0, \ldots, N-2$ *and* $\alpha_{N-1}(x) \equiv 0$*;*

(2) *the following equalities hold,*

$$e\psi_j = 0, \qquad \mathcal{P}_e(E_j) = 0 \Leftrightarrow \mathcal{P}_e(E_j)\psi_j = \mathcal{P}_e(h)\psi_j = -e^2\psi_j = 0, \qquad j = 0, 1, 2, \ldots, \tag{4.1}$$

and moreover:

(*a*) *the set of functions* $\psi_j(x)$, $j = 0, 1, 2, \ldots$ *is a canonical basis in* $\ker e$*;*

(*b*) *any of the numbers* E_j, $j = 0, 1, 2, \ldots$ *is an eigenvalue of algebraic multiplicity* 1 *for the matrix* $\mathbf{S}$ *of the operator* e *and there are no other eigenvalues of this matrix;*

(3) *there are* $((N+1)/2)!$ (*and no more*) *different nonsingular factorizations of* e *into product of one intertwining operator of the first order and* $(N-1)/2$ *intertwining operators of the second order; moreover:*

(*a*) *all intermediate Hamiltonians of these factorizations possess the same spectrum as* h *and potentials of all these Hamiltonians are real-valued periodic* (*with the period* X_0) *functions belonging to* $C^\infty_{\mathbb{R}}$*;*

(*b*) *the coefficient at the highest-order derivative in any intertwining operator of the first or the second orders is* 1 *and all other coefficients of these operators are real-valued periodic* (*with the period* X_0) *functions belonging to* $C^\infty_{\mathbb{R}}$*;*

(*c*) *the spectrum of the matrix* $\mathbf{S}$ *of an intertwining operator of the first order consists of* E_0 *and the spectrum of the matrix* $\mathbf{S}$ *of an intertwining operator of the second order consists of borders of a forbidden band so that every forbidden band corresponds to only one of the second order operators;*

(*d*) *if*

$$e = r_{(N+1)/2} \cdots r_1 \tag{4.2}$$

is one of the possible factorizations of e *and* h_i, $i = 1, \ldots, (N-1)/2$ *are intermediate Hamiltonians corresponding to this factorization,*

$$r_i h_{i-1} = h_i r_i, \qquad r_i^t h_i = h_{i-1} r_i^t, \qquad i = 1, \ldots, (N+1)/2, \qquad h_0 \equiv h_{(N+1)/2} \equiv h,$$

then a canonical basis in kernel of r_i *consists of those band edge wave functions of* h_{i-1}*, energies of which form the spectrum of the matrix* $\mathbf{S}$ *for* r_i*, and*

$$r_i \cdots r_1 \cdot r_{(N+1)/2} \cdots r_{i+1} \tag{4.3}$$

is a t*-antisymmetric non-minimizable symmetry operator of* N*-th order for* h_i, $i = 1, \ldots, (N-1)/2$.

Proof. Inclusion of the coefficients of e into $C^\infty_{\mathbb{R}}$ can be proved on the same way as in Lemma 1 in [39]. Reality of these coefficients is obvious. The identity $\alpha_{N-1}(x) \equiv 0$ holds in view of t-antisymmetry of e. Periodicity of the coefficients with the period X_0 follows from the uniqueness of a normalized non-minimizable t-antisymmetric symmetry operator and from the fact that operator different from e only by shift of all coefficient's arguments by X_0 is a normalized non-minimizable t-antisymmetric symmetry operator as well, by virtue of the periodicity of the potential $V(x)$.

Let us now verify that the equalities (4.1) take place for any j. As all coefficients of e are periodic and there is the only (up to a constant cofactor) periodic or antiperiodic eigenfunction of h for a border between forbidden and allowed energy bands so $\psi_j(x)$ is eigenfunction of e,

$$e\psi_j = \mu_j \psi_j, \qquad j = 0, 1, 2, \ldots,$$

where μ_j is corresponding eigenvalue. In view of periodicity of e coefficients, t-antisymmetry of e and periodicity or anti-periodicity of $\psi_j(x)$ the equalities holds,

$$\mu_j \int_0^{X_0} \psi_j^2(x)\,dx = \int_0^{X_0} [e\psi_j](x)\psi_j(x)\,dx = \int_0^{X_0} \psi_j(x)[e^t\psi_j](x)\,dx = -\mu_j \int_0^{X_0} \psi_j^2(x)\,dx,$$

wherefrom it follows in view of reality of $\psi_j(x)$ that all numbers μ_j, $j = 0, 1, 2, \dots$ are equal to zero and thus the equalities (4.1) are valid for any j. It follows from (4.1) for any j and from Theorem 1 that all numbers E_j, $j = 0, 1, 2, \dots$ belong to the spectrum of the matrix $\mathbf{S}$ of e.

Let us show that the spectrum of the matrix $\mathbf{S}$ for e contains the values E_j, $j = 0, 1, 2, \dots$ only. Suppose that the spectrum contains a value λ, located either inside of an allowed band or inside of a forbidden band or outside of real axis. Then this λ in accordance with Theorem 1 is a zero of the polynomial $\mathcal{P}_e$. By virtue of periodicity of e coefficients Bloch solutions of the equation $(h - \lambda)\psi = 0$ are formal eigenfunctions of the symmetry operator e and moreover the corresponding eigenvalues in view of the equalities $e^2 = -\mathcal{P}_e(h)$ and $\mathcal{P}_e(\lambda) = 0$ are zeros. Thus, the kernel of e contains two linearly independent solutions of the equation $(h - \lambda)\psi = 0$ that contradicts (see Theorem 2) to non-minimizability of e. Hence, the spectrum of the matrix $\mathbf{S}$ of e cannot contain a value situated inside of an allowed or a forbidden band or outside of real axis.

Now suppose that the spectrum of the matrix $\mathbf{S}$ of e contains a value λ, located on a border between two allowed bands. In this case λ is a zero of $\mathcal{P}_e$ again. In addition, in the case under consideration any two linearly independent solutions of the equation $(h - \lambda)\psi = 0$ are simultaneously periodic or antiperiodic functions. It is evident that acting of the operator ie on elements of the kernel $h - \lambda$ in some orthogonal basis with respect to scalar product

$$(f_1, f_2) = \int_0^{X_0} f_1(x) f_2^*(x)\,dx$$

is described by a Hermitian matrix. Consequently, a basis in the kernel of $h - \lambda$ can be chosen from eigenfunctions of e. Together with the condition $\mathcal{P}_e(\lambda) = 0$ the latter leads to contradiction as before. Thus, the spectrum of the matrix $\mathbf{S}$ of e cannot contain a value situated on a border between two allowed bands and this spectrum consists of the values E_j, $j = 0, 1, 2, \dots$ only.

Next we check that the algebraic multiplicity of any eigenvalue of the matrix $\mathbf{S}$ of e is one. After this check it will be obvious that the functions $\psi_j(x)$, $j = 0, 1, 2, \dots$ form a canonical basis in the kernel of e. Suppose that the algebraic multiplicity of an eigenvalue E_j of the matrix $\mathbf{S}$ of e is greater than one. It was shown in [8, 12, 13, 15] that h can be intertwined with some Hamiltonian $\tilde{h}$, having real-valued periodic potential, with the help of intertwining operator r of the first order (if $j = 0$) or the second order (if $j > 0$) whose kernel consists of $\psi_0(x)$ (if $j = 0$) or $\psi_j(x)$ and $\psi_{j+1}(x)$ (if j is odd) or $\psi_j(x)$ and $\psi_{j-1}(x)$ (if $j > 0$ is even). Moreover, the Wronskian of transformation functions in any of these cases has no zeros. Therefrom as well as from (2.4), (2.5) and the condition $V(x) \in C^\infty_{\mathbb{R}}$ it follows that the potential of $\tilde{h}$ and coefficients of r are real-valued and belong to $C^\infty_{\mathbb{R}}$.

Now consider the operator rer^t. It is obvious that this operator is t-antisymmetric symmetry operator for $\tilde{h}$ and all its coefficients belong to $C^\infty_{\mathbb{R}}$. As e is t-antisymmetric non-minimizable symmetry operator, $\psi_j(x)$ belongs to a canonical bases in $\ker e$ and $\ker r$ and the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of e is greater than one, so that with the help of Lemma 1 from [4] one can separate from the right-hand side of r the intertwining operator $\partial - \psi_j'/\psi_j$ and from the right-hand side of e the same intertwining operator $\partial - \psi_j'/\psi_j$ and simultaneously from the left-hand side of e the intertwining operator $(\partial - \psi_j'/\psi_j)^t$. Thus, in view of Theorem 1 the operator rer^t is minimizable and the polynomial which can be separated from rer^t contains the binomial $E_j - \tilde{h}$ as a cofactor in the power which is greater than or equal to two. Therefrom as well as from Theorems 1 and 2 and from uniqueness of the normalized non-minimizable

t-antisymmetric symmetry operator $\tilde{e}$ for the Hamiltonian $\tilde{h}$ it follows that algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of $\tilde{e}$ is less than the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of e, at least, by two. Moreover, all coefficients of $\tilde{e}$ are real-valued and belong to $C^\infty_{\mathbb{R}}$, because in the opposite case coefficients of rer^t obviously cannot be from $C^\infty_{\mathbb{R}}$.

It can be verified that a canonical basis in $\ker r^t$ consists of periodic or antiperiodic wave functions of the Hamiltonian $\tilde{h}$, namely: from $1/\psi_0(x)$ corresponding to the energy E_0 (if $j = 0$) or from $\psi_{j+1}(x)/[\psi'_j\psi_{j+1} - \psi_j\psi'_{j+1}]$ and $\psi_j(x)/[\psi'_j\psi_{j+1} - \psi_j\psi'_{j+1}]$ corresponding to the energies E_j and E_{j+1} respectively (if j is odd) or from $\psi_{j-1}(x)/[\psi'_j\psi_{j-1} - \psi_j\psi'_{j-1}]$ and $\psi_j(x)/[\psi'_j\psi_{j-1} - \psi_j\psi'_{j-1}]$ corresponding to the energies E_j and E_{j-1} respectively (if $j > 0$ is even). Hence, the Hamiltonians h and $\tilde{h}$ as well as the symmetry operators e and $\tilde{e}$ can be equally employed in the previous argumentation. Thus, one finds that if the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of $\tilde{e}$ is greater than one, then the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of e as compared to itself is less, at least, by four. As well if the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of $\tilde{e}$ is equal to one, then it is evident that one can separate from the symmetry operator $r^t\tilde{e}r$ the binomial $E_j - h$ in the power which is equal to one, wherefrom it follows that the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of e as compared to itself is less by two at least. From these contradictions it follows that the algebraic multiplicity of E_j in the spectrum of the matrix $\mathbf{S}$ of e is equal to one for any j. Thus, the statements 1 and 2 of Theorem 3 are proved.

The statement 3 of Theorem 3 can be related to the Corollary of Lemma 1 from [4] if to take into account the following:

(1) one should separate intertwining operators of the first order from e on its right-hand side so that for any odd j the intertwining operator, whose matrix $\mathbf{S}$ spectrum consists of E_j, has as its neighbor the intertwining operator whose matrix $\mathbf{S}$ spectrum consists of E_{j+1}; in addition one must consider these pairs of neighbors as joined operators of the second order;

(2) the properties of coefficients in e factorization cofactors and of corresponding intermediate Hamiltonians are easily verifiable by induction from the right to the left with the help of (2.4), (2.5) and the facts that (i) Wronskian of wave functions corresponding to borders of a forbidden band has no zeros [13, 15], (ii) the wave function of a Hamiltonian with periodic potential corresponding to the lower bound of the spectrum has no zeros and (iii) an intertwining operator with periodic coefficients obviously maps a periodic or antiperiodic wave function (with exception for transformation functions) to a periodic or antiperiodic wave function respectively and increasing, decreasing or bounded Bloch eigenfunction to increasing, decreasing or bounded Bloch eigenfunction accordingly;

(3) there is no an intertwining operator of the second order with smooth coefficients with the canonical basis of its kernel consisting of wave functions corresponding to borders of different forbidden bands, this is true because of (2.5) and due to the fact that the Wronskian of functions under consideration cannot be nodeless in view of different numbers of zeros of these functions per period;

(4) the operator (4.3) is a t-antisymmetric symmetry operator for h_i by virtue of the construction of Section 3 with $h^+ = h$, $h^- = h_i$, $k^+_N = r_{(N+1)/2}\cdots r_{i+1}$ and $p^+_M = r^t_1\cdots r^t_i$ or $k^+_N = r^t_1\cdots r^t_i$ and $p^+_M = r_{(N+1)/2}\cdots r_{i+1}$ depending on the relation between orders $r_{(N+1)/2}\cdots r_{i+1}$ and $r^t_1\cdots r^t_i$ (here N is not the order of e); non-minimizability of the operator (4.3) follows from Theorem 2.

Theorem 3 is proved. ■

Corollary 3. *Under the conditions of Theorem 3, in view of Theorem 1 the equality holds,*

$$\mathcal{P}_e(h) \equiv -e^2 = \prod_{j=0}^{N-1}(h - E_j), \tag{4.4}$$

and there are $(N+1)/2$ *(and not more) forbidden energy bands for the Hamiltonian* h*. Thus, as there is a one-to-one correspondence between forbidden energy bands of* h *and the cofactors of an* e *factorization* (4.2) (*see the statement* 3.c *of Theorem* 3), *then the* $((N+1)/2)!$ *factorizations of* e *described in Theorem* 3 *correspond in one-to-one to all possible permutations of* h *forbidden energy bands.*

Remark 3. The formula (4.4) and the statement of Corollary 3 about the number of forbidden energy bands for the periodic solutions of stationary higher-order Korteweg – de Vries equations were derived in [2]. The statements 1 and 2 of Theorem 3 and the partial case of the statement 3 of this theorem, corresponding to increasing of eigenvalues of the matrices $\mathbf{S}$ for cofactors in (4.2) from the right to the left (without formula (4.3)) were proved in [40]. The facts that the borders between allowed and forbidden bands of an arbitrary Hamiltonian h with periodic potential having t-antisymmetric symmetry operator e correspond to certain zeros of the polynomial $\mathcal{P}_e(h) \equiv e^t e$ and the related wave functions belong to $\ker e$ were mentioned in [6]. Special factorizations of t-antisymmetric non-minimizable symmetry operator of a Hamiltonian with parity-even finite-gap periodic potential can be found in [16].

Remark 4. In the case of complex periodic potential the spectrum of the matrix $\mathbf{S}$ of e can contain values located inside of the continuous spectrum of the corresponding Hamiltonian h and moreover algebraic multiplicity of these values can be greater than one. The following example[3] illustrates this situation,

$$h = -\partial^2 + \frac{2k_0^2}{\cos^2[k_0(x-z)]}, \qquad k_0 > 0, \qquad \operatorname{Im} z \neq 0,$$
$$e = -p_1^- \partial p_1^+, \qquad p_1^{\mp} = \pm\partial + k_0 \operatorname{tg}[k_0(x-z)], \qquad p_1^- = (p_1^+)^t,$$
$$h_0 = -\partial^2, \qquad p_1^- h_0 = h p_1^-, \qquad p_1^+ h = h_0 p_1^+, \tag{4.5}$$

the eigenfunctions of h continuous spectrum $\psi_k(x)$ and the eigenfunction of h at the bottom of this spectrum $\psi_0(x)$ take the form,

$$\psi_k(x) = \{ik + k_0 \operatorname{tg}[k_0(x-z)]\}e^{ikx}, \qquad h\psi_k = k^2\psi_k, \qquad k \in \mathbb{R},$$
$$\psi_0(x) = k_0 \operatorname{tg}[k_0(x-z)], \qquad h\psi_0 = 0.$$

It is interesting that there is a unique (up to a constant cofactor) bound eigenfunction of h on the level $E = k_0^2$,

$$\psi_{0,k_0}(x) = \frac{1}{\cos[k_0(x-z)]} \equiv -\frac{i}{k_0} e^{-ik_0 z}\psi_{k_0}(x) \equiv \frac{i}{k_0} e^{ik_0 z}\psi_{-k_0}(x),$$
$$h\psi_{0,k_0} = k_0^2 \psi_{0,k_0}, \tag{4.6}$$

and there is a bound associated function for this eigenfunction,

$$\psi_{1,k_0}(x) = \frac{1}{2k_0^2}\cos[k_0(x-z)], \qquad (h-k_0^2)\psi_{1,k_0} = \psi_{0,k_0}. \tag{4.7}$$

The functions $\psi_0(x)$, $\psi_{1,k_0}(x)$ and $\psi_{0,k_0}(x)$ form a canonical basis in the $\ker e$ by virtue of (4.5)–(4.7) and

$$p_1^+\psi_{0,k_0} = 0, \qquad \partial p_1^+ \psi_0 = 0, \qquad e\psi_{1,k_0} \equiv -p_1^- \partial p_1^+ \psi_{1,k_0} = 0.$$

Thus, in view of Theorem 1,

$$\mathcal{P}_e(h) = h(h-k_0^2)^2, \qquad \mathbf{S}_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k_0^2 & 1 \\ 0 & 0 & k_0^2 \end{pmatrix}.$$

[3]See similar examples in [41].

5 t-antisymmetric symmetry operators: Hamiltonians with bound state(s)

5.1 General properties

Let us assume that geometric multiplicity of any eigenvalue of the Hamiltonian h is 1, its algebraic multiplicity is finite and functions $\psi_{l,j}(x)$ form the complete set of normalized eigenfunctions and associated functions of h for the point spectrum (without eigenvalues inside or on boundaries of continuous spectrum),

$$h\psi_{l,0} = E_l\psi_{l,0}, \qquad (h - E_l)\psi_{l,j} = \psi_{l,j-1}, \qquad \int_{-\infty}^{+\infty} \psi_{l,j}(x)\psi_{l',k_{l'}-j'-1}(x)\,dx = \delta_{ll'}\delta_{jj'},$$
$$l, l' = 0, 1, 2, \ldots, \qquad j = 0, \ldots, k_l - 1, \qquad j' = 0, \ldots, k_{l'} - 1, \tag{5.1}$$

where k_l is an algebraic multiplicity of an eigenvalue E_l, $l = 0, 1, 2, \ldots$. It is known that in the case of Hermitian Hamiltonian h any multiplicity $k_l = 1$, $l = 0, 1, 2, \ldots$ and it was shown in [4] that in this case $e\psi_{l,0} = 0$ for any $l = 0, 1, 2, \ldots$.

Now we derive that, in general,

$$e\psi_{l,j} = 0, \qquad l = 0, 1, 2, \ldots, \quad j = 0, \ldots, k_l - 1. \tag{5.2}$$

Suppose that for some l there is a number j_0 such that $0 \leqslant j_0 \leqslant k_l - 1$, $e\psi_{l,j} = 0$, $j = 0, \ldots, j_0 - 1$ and $e\psi_{l,j_0} \neq 0$. Then, in view of the equalities

$$he\psi_{l,j_0} = eh\psi_{l,j_0} = E_l e\psi_{l,j_0}$$

the function $e\psi_{l,j_0}$ is an eigenfunction of h for the eigenvalue E_l. Hence, there is a constant $C \neq 0$ such that $e\psi_{l,j_0} = C\psi_{l,0}$. The latter leads to contradiction by virtue of the following chain,

$$\begin{aligned} C &= \int_{-\infty}^{+\infty} C\psi_{l,0}(x)\psi_{l,k_l-1}(x)\,dx = \int_{-\infty}^{+\infty} [e\psi_{l,j_0}](x)\psi_{l,k_l-1}(x)\,dx \\ &= -\int_{-\infty}^{+\infty} \psi_{l,j_0}(x)[e\psi_{l,k_l-1}](x)\,dx = -\int_{-\infty}^{+\infty} [(h - E_l)^{k_l-1-j_0}\psi_{l,k_l-1}](x)[e\psi_{l,k_l-1}](x)\,dx \\ &= -\int_{-\infty}^{+\infty} \psi_{l,k_l-1}(x)[e(h - E_l)^{k_l-1-j_0}\psi_{l,k_l-1}](x)\,dx \\ &= -\int_{-\infty}^{+\infty} \psi_{l,k_l-1}(x)[e\psi_{l,j_0}](x)\,dx = -C, \end{aligned}$$

where (5.1) is used. Therefore, the equalities (5.2) are valid.

It follows from (5.2) that the algebraic multiplicity of E_l in the spectrum of the matrix $\mathbf{S}$ of e is greater than or equal to k_l, $l = 0, 1, 2, \ldots$ and

$$\mathcal{P}_e(E_l) = 0 \Leftrightarrow \mathcal{P}_e(E_l)\psi_{l,0} = \mathcal{P}_e(h)\psi_{l,0} = -e^2\psi_{l,0} = 0, \qquad l = 0, 1, 2, \ldots. \tag{5.3}$$

Thus, if there is nonzero t-antisymmetric non-minimizable symmetry operator e for a Hamiltonian h, then the energies of all its bound states satisfy the algebraic equation (5.3).

5.2 Potentials with constant asymptotics

The number of bound states of a Hamiltonian h with nonzero t-antisymmetric nonminimizable symmetry operator e in view of (5.2) and (5.3) is finite. Consequently, such a Hamiltonian cannot have, for example, a real-valued potential infinitely increasing for $|x| \to +\infty$, because the Hamiltonians with potentials of this type possess (see [42]) infinite numbers of bound states irrespectively of the rate of increasing. Taking this into account, we restrict our consideration in this subsection by the subcase, when the potential $V(x)$ of a Hamiltonian h with a nonzero t-antisymmetric nonminimizable symmetry operator e tends to a constant E_c on one of infinities and either grows unboundly ($\operatorname{Re} V(x) \to +\infty$, $\operatorname{Im} V(x)/\operatorname{Re} V(x) = o(1)$) or tends to a constant different, in general, from E_c on another infinity. We assume for definiteness that $V(x) \to E_c$ for $x \to -\infty$, and denote the total number of energy levels E_l as N_b. We shall show that under some additional assumptions of technical character the following statements are valid.

(1) The potential $V(x)$ of the Hamiltonian h is reflectionless and tends to E_c for $x \to +\infty$ as well.

(2) The algebraic multiplicity of E_l in the spectrum of the matrix $\mathbf{S}$ of e is equal to $2k_l$, $l = 0, \ldots, N_b - 1$.

(3) The Hamiltonian h is intertwined with the Hamiltonian of a free particle $-\partial^2 + E_c$.

(4) A wave function of h, corresponding to the lower boundary of h continuous spectrum, belongs to $\ker e$ and the energy E_c, corresponding to this boundary, is contained in the spectrum of the matrix $\mathbf{S}$ of e with some odd algebraic multiplicity k_c. Moreover, for a real-valued potential $V(x)$ the algebraic multiplicity of E_c in the spectrum of the matrix $\mathbf{S}$ of e is 1 ($k_c = 1$).

(5) The spectrum of the matrix $\mathbf{S}$ of e contains only E_l, $l = 0, \ldots, N_b - 1$ and E_c.

(6) If the order of e is equal to N, then the number of bound states of the Hamiltonian h is less than or equal to $(N-1)/2$. Moreover, for a real-valued potential $V(x)$ the number of bound states of the Hamiltonian h is equal to $(N-1)/2$.

(7) For the squared symmetry operator e the following representation holds,

$$\mathcal{P}_e(h) \equiv -e^2 = (h - E_c)^{k_c} \prod_{l=0}^{N_b-1} (h - E_l)^{2k_l}. \tag{5.4}$$

(8) The operator e can be represented as a product of intertwining operators so that:

(a)

$$e = (-1)^{(N-1)/2} r_0^t \cdots r_{N_b}^t \, \partial \, r_{N_b} \cdots r_0, \tag{5.5}$$

$$r_l \cdots r_0 \psi_{l,j} = 0, \qquad l = 0, \ldots, N_b - 1, \quad j = 0, \ldots, k_l - 1, \tag{5.6}$$

$$r_{N_b} \cdots r_0 \psi_{c,j} = 0, \qquad j = 0, \ldots, \frac{k_c - 3}{2}, \qquad \partial \, r_{N_b} \cdots r_0 \psi_{c,(k_c-1)/2} = 0, \tag{5.7}$$

where $\{\psi_{c,j}(x)\}_{j=0}^{k_c-1}$ is a part of the canonical basis in $\ker e$, corresponding to the eigenvalue E_c:

$$h\psi_{c,0} = E_c \psi_{c,0}, \qquad (h - E_c)\psi_{c,j} = \psi_{c,j-1}, \qquad j = 0, \ldots, k_c - 1.$$

In addition, all operators $r_0, \ldots, r_{N_b}$ have unity coefficients at highest derivatives and other coefficients of these operators can have, in general, poles. Moreover, for a real-valued potential $V(x)$,

$$r_l = \partial + \chi_l(x), \qquad r_l \cdots r_0 \psi_{l,0} = 0 \;\Leftrightarrow$$
$$\Leftrightarrow\; \chi_l(x) = -\frac{(r_{l-1} \cdots r_0 \psi_{l,0})'}{r_{l-1} \cdots r_0 \psi_{l,0}}, \qquad l = 0, \ldots, N_b - 1, \qquad r_{N_b} = 1,$$

all superpotentials $\chi_l(x)$, $l = 0, \ldots, N_b - 1$ are real-valued functions and if $V(x) \in C^\infty_{\mathbb{R}}$ and the energies E_l, $l = 0, \ldots, N_b - 1$ are numbered in the order of increasing, then all these superpotentials belong to $C^\infty_{\mathbb{R}}$.

(b) The intermediate Hamiltonians h_l, $l = 1, \ldots, N_b + 1$, corresponding to the factorization (5.5), satisfy the following intertwinings,

$$h_l r_{l-1} = r_{l-1} h_{l-1}, \qquad r_{l-1}^t h_l = h_{l-1} r_{l-1}^t, \qquad l = 1, \ldots, N_b + 1, \qquad h_0 \equiv h \tag{5.8}$$

and take the Schrödinger form,

$$\begin{aligned}
&h_l = -\partial^2 + v_l(x), \qquad l = 0, \ldots, N_b + 1,\\
&v_0(x) = V(x), \qquad v_{l+1}(x) = V(x) - 2[\ln W_l(x)]'',\\
&W_l(x) = \begin{vmatrix} \psi_{0,0}(x) & \psi'_{0,0}(x) & \ldots & \psi_{0,0}^{(k_0+\ldots+k_l-1)}(x)\\ \psi_{0,1}(x) & \psi'_{0,1}(x) & \ldots & \psi_{0,1}^{(k_0+\ldots+k_l-1)}(x)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_{l,k_l-2}(x) & \psi'_{l,k_l-2}(x) & \ldots & \psi_{l,k_l-2}^{(k_0+\ldots+k_l-1)}(x)\\ \psi_{l,k_l-1}(x) & \psi'_{l,k_l-1}(x) & \ldots & \psi_{l,k_l-1}^{(k_0+\ldots+k_l-1)}(x) \end{vmatrix}, \qquad l = 0, \ldots, N_b - 1,\\
&W_{N_b}(x) = \begin{vmatrix} \psi_{0,0}(x) & \psi'_{0,0}(x) & \ldots & \psi_{0,0}^{(k_0+\ldots+k_{N_b-1}+(k_c-1)/2-1)}(x)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_{c,(k_c-3)/2}(x) & \psi'_{c,(k_c-3)/2}(x) & \ldots & \psi_{c,(k_c-3)/2}^{(k_0+\ldots+k_{N-b-1}+(k_c-1)/2-1)}(x) \end{vmatrix}.
\end{aligned} \tag{5.9}$$

All potentials of these Hamiltonians tend to E_c for $|x| \to +\infty$ and, in general, have poles. Moreover, in the case of real-valued potential $V(x)$ the following chain relations take place,

$$\begin{aligned}
&h_l = r_l^t r_l + E_l = r_{l-1} r_{l-1}^t + E_{l-1}, \qquad l = 1, \ldots, N_b - 1,\\
&h_0 \equiv h = r_0^t r_0 + E_0, \qquad h_{N_b+1} = h_{N_b} = r_{N_b-1} r_{N_b-1}^t + E_{N_b-1},\\
&v_l(x) = \chi_l^2(x) - \chi_l'(x) + E_l = \chi_{l-1}^2(x) + \chi_{l-1}(x) + E_{l-1},
\end{aligned}$$

all potentials $v_l(x)$, $l = 0, \ldots, N_b + 1$ are real-valued functions and if $V(x) \in C_{\mathbb{R}}^\infty$ and the energies E_l, $l = 0, \ldots, N_b - 1$ are numbered in the order of increasing, then all these potentials belong to $C_{\mathbb{R}}^\infty$.

(c) For any intermediate Hamiltonian h_l there is a nonzero t-antisymmetric nonminimizable symmetry operator e_l with the unity coefficient at highest derivative such that:

$$\begin{aligned}
&e_l h_l = h_l e_l, \qquad e_l^t = -e_l, \qquad l = 0, \ldots, N_b + 1, \qquad e_0 \equiv e,\\
&e_l = (-1)^{k_l+\cdots+k_{N_b-1}+(k_c-1)/2} r_l^t \cdots r_{N_b}^t \,\partial\, r_{N_b} \cdots r_l, \qquad l = 0, \ldots, N_b, \qquad e_{N_b+1} = \partial.
\end{aligned}$$

(d) The Hamiltonian h_{N_b+1} is a Hamiltonian of a free particle:

$$h_{N_b+1} = -\partial^2 + E_c.$$

(9) The operator e acts on an eigenfunction of h continuous spectrum

$$\psi_k(x) = (-1)^{(N-1)/2} r_0^t \cdots r_{N_b}^t e^{ikx} \tag{5.10}$$

as follows,

$$e\psi_k = [(ik)^{k_c} \prod_{l=0}^{N_b-1} (E_l - E_c - k^2)^{k_l}]\psi_k, \qquad k \in \mathbb{R}. \tag{5.11}$$

(10) The transmission coefficient $T(k)$ for h (we assume as usually that $T(k)$ is the ratio of the coefficient at e^{ikx} in the main term of $\psi_k(x)$ asymptotics for $x \to +\infty$ to one for $x \to -\infty$) takes the form[4]

$$T(k) = \prod_{l=0}^{N_b-1} \left(\frac{k + i\sqrt{E_c - E_l}}{k - i\sqrt{E_c - E_l}} \right)^{k_l}, \qquad \operatorname{Re} \sqrt{E_c - E_l} > 0, \qquad l = 0, \dots, N_b - 1. \tag{5.12}$$

In the case under consideration one can conjecture that the coefficients of the operator e for $x \to -\infty$ tend to constants. If all derivatives of the potential $V(x)$ of the Hamiltonian h behave as $O(1/|x|^{1+\varepsilon})$, $\varepsilon > 0$ for $x \to -\infty$, then the validity of assumption on the behavior of coefficients can be easily checked with the help of the system of equations

$$\alpha_j(x) = \alpha_j(0) - \frac{1}{2}[\alpha'_{j+1}(x) - \alpha'_{j+1}(0)] - \frac{1}{2} \int_0^x \sum_{l=j+2}^{N} C_l^{j+2} \alpha_l(t) V^{(l-j-1)}(t)\, dt,$$
$$j = N - 2, \dots, 0$$

with respect to the coefficients $\alpha_j(x)$, $j = 0, \dots, N$ of the operator[5]

$$e = \sum_{j=0}^{N} \alpha_j(x) \partial^j, \qquad \alpha_N(x) \equiv 1, \qquad \alpha_{N-1}(x) \equiv 0.$$

This system follows from the condition $eh = he$.

First, let us show that the spectrum of the matrix $\mathbf{S}$ of e contains only the values E_l, $l = 0, \dots, N_b-1$, E_c and (in the case of existence of a finite limit E'_c of $V(x)$ for $x \to +\infty$) the value E'_c. Belonging of E_l, $l = 0, \dots, N_b$ to this spectrum was derived before. For the values of spectral parameter λ such that $\lambda - E_c \geqslant 0$ there are formal eigenfunctions of h, which for $x \to -\infty$ are proportional to e^{ikx} and to e^{-ikx}, $k = \sqrt{\lambda - E_c}$. The formal eigenfunctions of h, proportional to e^{ikx} and to e^{-ikx} for $x \to -\infty$, are formal eigenfunctions of e by virtue of constant asymptotics of e coefficients. In view of t-antisymmetry of e , the related eigenvalues take the form $kf(k^2)$ and $-kf(k^2)$ correspondingly where $f(k^2)$ is a certain function. In addition, $f(k^2)$ cannot have zeros for real $k \neq 0$, since in the opposite case $\ker e$ contains two linearly independent formal eigenfunctions of h for the same value of a spectral parameter, that contradicts to non-minimizability of e in view of Theorem 2. Thus, the spectrum of the matrix $\mathbf{S}$ of e contains E_c and cannot include λ, which satisfy $\lambda - E_c > 0$. An analogous statement is valid also for finite E'_c. For any spectral value λ, which does not satisfy $\lambda - E_c \geqslant 0$ and $\lambda - E'_c \geqslant 0$ and is different from energies E_l, $l = 0, \dots, N_b - 1$, there is [10, 39] a formal eigenfunction of h, which tends to zero for $x \to -\infty$, and a formal eigenfunction of h, which tends to zero for $x \to +\infty$. These eigenfunctions are evidently linearly independent formal eigenfunctions of e and squared corresponding eigenvalues by virtue of (3.3) are equal to $-\mathcal{P}_e(\lambda)$. If a considered λ belongs to the spectrum of the matrix $\mathbf{S}$ of e, then $\mathcal{P}_e(\lambda) = 0$ and the eigenfunctions mentioned above belong to $\ker e$, then it contradicts to non-minimizability of e in view of Theorem 2. Thus, the spectrum of the matrix $\mathbf{S}$ of e contains only the values E_l, $l = 0, \dots, N_b - 1$, E_c and (in the case of existence of a finite limit E'_c of $V(x)$ for $x \to +\infty$) the value E'_c.

Now we derive that the algebraic multiplicity of any energy E_l, $l = 0, \dots, N_b - 1$ in the spectrum of the matrix $\mathbf{S}$ of e is equal $2k_l$. With the help of Lemma 1 from [4] one can represent the operator e so that

$$e = \hat{e} r_0, \qquad r_0 \psi_{0,j} = 0, \qquad j = 0, \dots, k_0 - 1,$$

[4]The partial case of the formula (5.12), corresponding to a real-valued $V(x)$, is described in [2].

[5]The identity $\alpha_{N-1}(x) \equiv 0$ is a consequence of the identity $\alpha_N(x) \equiv 1$ and of t-antisymmetry of the operator e.

$$\hat{e}h_1 = h\hat{e}, \qquad r_0 h = h_1 r_0, \qquad h r_0^t = r_0^t h_1,$$

where r_0 and h_1 are defined in (5.6), (5.9) and the coefficients of r_0, $\hat{e}$ and the potential of h_1 have poles in general. In accordance to Corollary 2 from [39] the main terms of asymptotics for $x \to +\infty$ ($x \to -\infty$) of the potentials in the Hamiltonians $h \equiv h_0$ and h_1 are identical.

One can continue h, h_1, r_0 and $\hat{e}$ for x, to a some path in complex plain, which avoids all above mentioned poles and can be identified to real axis for sufficiently large $|x|$ (absence of real poles for large x follows from invariance of the class $\mathcal{K}$ with respect to intertwinings proved in [39]). Using this conjecture and arguments analogous to ones in the proof of the index theorem in [39], we can derive that this theorem is valid for the case under consideration and thereby the Hamiltonian h_1 does not possess normalizable eigenfunctions and associated functions for the spectral value E_0. Hence, the functions $\psi_{0,j}$, $j = 0, \ldots, k_0 - 1$ belong to $\ker \hat{e}^t$ since in the opposite case the intertwining operator $\hat{e}^t$ maps these functions into the chain of eigenfunction and associated functions of h_1 for the eigenvalue E_0.

With the help of Lemma 1 from [4] we can represent the operator $\hat{e}^t$ in the form

$$\hat{e}^t = (-1)^{k_0} e_1^t r_0, \qquad e_1 h_1 = h_1 e_1, \qquad e_1^t h_1 = h_1 e_1^t,$$

where the coefficient at the highest derivative in e_1 is equal to 1. Thus, the symmetry operator e can be obviously factorized in the form

$$e = (-1)^{k_0} r_0^t e_1 r_0$$

and the symmetry operator e_1 is non-minimizable and t-antisymmetric, because otherwise the operator e is minimizable and/or $e + e^t = (-1)^{k_0} r_0^t (e_1 + e_1^t) r_0 \neq 0$. Taking into account that in view of Theorem 1 the spectrum of a product of intertwining operators is equal to a union of the spectra of the matrices $\mathbf{S}$ of the cofactors (with regard to algebraic multiplicities) and that the spectrum of the matrix $\mathbf{S}$ of e_1 does not contain E_0, we derive that the algebraic multiplicity of E_0 for the spectrum of the matrix $\mathbf{S}$ of e is equal to $2k_0$.

Using the fact, that the eigenfunctions and associated functions of h_1 and the corresponding eigenvalues take the form $r_0\psi_{l,j}(x)$, $j = 0, \ldots, k_l - 1$ and E_l respectively, $l = 1, \ldots, N_b - 1$, and inductive reasoning as well, we conclude that the algebraic multiplicity of E_l in the spectrum of the matrix $\mathbf{S}$ of e is equal to $2k_l$, $l = 0, \ldots, N_b - 1$ and that the corresponding part of the statement (8) is valid. The related part of the statement (8) for the case of real-valued $V(x)$ and for the subcase $V(x) \in C^\infty_{\mathbb{R}}$ is evidently valid in virtue of Lemma 1 from [4] and due to the fact that eigenfunctions of h in this case can be chosen real-valued and that the eigenfunction of a Hermitian Hamiltonian for its ground state does not have zeros.

Let us check now that the potential of h tend to E_c for $x \to +\infty$ as well as the potentials of all intermediate Hamiltonians. In so far as the order of the operator e is odd, the sum of algebraic multiplicities of E_c and (if $V(x) \to E'_c$, $x \to +\infty$) E'_c in the spectrum of the matrix $\mathbf{S}$ of e is obviously odd as well. Hence the algebraic multiplicity of either E_c or E'_c is odd. We shall restrict ourselves by the case, when the multiplicity of E_c is odd, because the examination of the opposite case is analogous.

With the help of Lemma 1 from [4] one can factorize the symmetry operator e_{N_b} in the product of intertwining operators of the first order so that first $k'_c/2$ and last $k'_c/2$ operators in this factorization correspond to the eigenvalue E'_c of the matrix $\mathbf{S}$ of e_{N_b}, where k'_c is an algebraic multiplicity of E'_c. This factorization is unique, since for any step of this factorization only the unique (up to constant cofactor) eigenfunction of corresponding intermediate Hamiltonian from the kernel of factorized operator can form a basis in the kernel of a separated intertwining operator of the first order. From uniqueness of considered factorization and from t-antisymmetry of e_{N_b} it follows that the central place in this factorization (i.e. the $(N - 2k_0 - \cdots - 2k_{N_b-1} + 1)/2 \equiv$

$(k_c + k'_c + 1)/2$-th position from the right or from the left) is occupied by ∂ and that the operator e_{N_b} can be represented in the form

$$e_{N_b} = (-1)^{(k_c+k'_c-1)/2} r^t\,\partial\, r,$$

where r is intertwining operator of the $(k_c + k'_c - 1)/2$-th order. In addition, according to Lemma 1 from [4], the operator r intertwines the Hamiltonian h_{N_b} with the Hamiltonian

$$h_{N_b+1} = \partial^t\partial + E_c \equiv -\partial^2 + E_c,$$

i.e. with the Hamiltonian of a free particle.

If $V(x) \to E_c$ for $x \to +\infty$, obviously r is identical to r_{N_b}, defined in (5.7), and the potentials of all intermediate Hamiltonians tend to E_c for $|x| \to +\infty$. If $V(x)$ infinitely increases ($\operatorname{Re} V(x) \to +\infty$, $\operatorname{Im} V(x)/\operatorname{Re} V(x) = o(1)$) for $x \to +\infty$, the operator r is equal to r_{N_b} as well and the potentials of the intermediate Hamiltonians $h_1, \ldots, h_{N_b}$ infinitely increase for $x \to +\infty$ as well. On the other hand, a canonical basis in $\ker r^t = \ker r^t_{N_b}$ consists of the chain of an eigenfunction and associated functions of h_{N_b+1} for the spectral value E_c and all these functions are evidently polynomials. Hence the Wronskian of these functions is a polynomial as well and in view of (2.4) the potential in h_{N_b} tends to E_c for $x \to +\infty$, that contradicts to what has been stated above. Consequently the potential $V(x)$ cannot increase unboundly for $x \to +\infty$.

Now we analyze the case, when $V(x)$ tends to a finite constant $E'_c \neq E_c$ for $x \to +\infty$. In this case the potentials of the intermediate Hamiltonians $h_1, \ldots, h_{N_b}$ tend to E'_c for $x \to +\infty$ as well. In accordance to the factorization mentioned above the operator r can be represented as follows,

$$r = r^{(a)} r^{(b)},$$

where $r^{(a)}$ and $r^{(b)}$ are intertwining operators of the orders $(k_c - 1)/2$ and $k'_c/2$ respectively, all eigenvalues of the matrix $\mathbf{S}$ of $r^{(a)}$ are equal E_c and all eigenvalues of the matrix $\mathbf{S}$ of $r^{(b)}$ are equal E'_c. If $E'_c - E_c \in \mathbb{C} \setminus [0, +\infty)$ the potential of the Hamiltonian h', intertwined by $r^{(a)}$ with h_{N_b+1} tends to E_c for $x \to +\infty$ and the potential of the Hamiltonian h_{N_b}, intertwined by $r^{(b)}$ with h', tends to E_c for $x \to +\infty$ by virtue of Corollary 2 from [39]. The latter contradicts to what has been written above and therefore $E'_c - E_c > 0$.

Let us re-factorize the operator r with the help of Lemma 1 from [4] in the form

$$r = r^{(c)} r^{(d)},$$

where $r^{(c)}$ and $r^{(d)}$ are intertwining operators of the orders $k'_c/2$ and $(k_c - 1)/2$ respectively, all eigenvalues of the matrix $\mathbf{S}$ of $r^{(c)}$ are equal E'_c and all eigenvalues of the matrix $\mathbf{S}$ of $r^{(d)}$ are equal E_c. The potential of the intermediate Hamiltonian h'', intertwined by $r^{(d)}$ with h_{N_b}, tends to E'_c for $x \to +\infty$ according to Corollary 2 from [39]. Thus, the Wronskian $W(x)$ of elements of a basis in $\ker(r^{(c)})^t$, in view of (2.4), can be estimated in the following way:

$$\begin{aligned} &[\ln W(x)]'' = -\tfrac{1}{2}\, {k'}^2 + o(1) \Rightarrow \ln W(x) = -\tfrac{1}{4}\, {k'}^2 x^2 + o(x^2) \Rightarrow W(x) = e^{-{k'}^2 x^2/4 + o(x^2)}, \\ &x \to +\infty, \qquad k' = \sqrt{E'_c - E_c} > 0. \end{aligned} \tag{5.13}$$

On the other hand, a canonical basis in $\ker(r^{(c)})^t$ consists of a chain of eigenfunction and associated functions of h_{N_b+1} for the spectral value E'_c and all these functions are linear combinations of $e^{ik'x}$ and $e^{-ik'x}$ with polynomial coefficients. Their Wronskian obviously cannot be of the form (5.13). Thus, the inequality $E'_c - E_c > 0$ cannot be realized also and the potential $V(x)$ as well as the potentials of all intermediate Hamiltonians tend to E_c for $|x| \to +\infty$.

It was noticed above, that the formal eigenfunctions of h proportional to e^{ikx} and to e^{-ikx} for $x \to -\infty$ are formal eigenfunctions of e, and corresponding eigenvalues take the form $kf(k^2)$ and $-kf(k^2)$ respectively, where $f(k^2) \neq 0$ for real $k \neq 0$. The same obviously takes place for $x \to +\infty$ as well. Moreover, it is evident, that the linear combination of these functions with nonzero coefficients cannot be an eigenfunction of e. Hence, the potential $V(x)$ is reflectionless, unless an eigenfunction of h is proportional to e^{ikx} for $x \to +\infty$ and to e^{-ikx} for $x \to -\infty$ (or respectively to e^{-ikx} and to e^{ikx}). The latter is impossible in view of (5.10) and of constant asymptotics of $r_0, \dots, r_{N_b}$ coefficients, which follows from the fact, that:

(1) the operator r_l for $x \to \pm\infty$ is asymptotically equal to

$$(\partial \pm \sqrt{E_c - E_l}\,)^{k_l}, \qquad \operatorname{Re}\sqrt{E_c - E_l} > 0, \qquad l = 0, \dots, N_b - 1, \tag{5.14}$$

because an element of the kernel of any cofactor of r_l factorization, obtained in accordance to Lemma 1 from [4], is proportional to $e^{\mp\sqrt{E_c - E_l}\,x}$ for $x \to \pm\infty$ (being an eigenfunction of the corresponding intermediate Hamiltonian);

(2) the operator r_{N_b} for $x \to \pm\infty$ is asymptotically equal to

$$\partial^{(k_c-1)/2}, \tag{5.15}$$

because, as was noticed above, the canonical basis in $\ker r^t_{N_b}$ consists of polynomials and consequently an element of the kernel of any cofactor of $r^t_{N_b}$ factorization, obtained in accordance to Lemma 1 from [4], is a rational function.

The formula (5.11) is a consequence of (5.5), (5.8), (5.10) and Theorem 1. The representation (5.12) for $T(k)$ follows from (5.10), (5.14) and (5.15).

At last we derive that $r_{N_b} = 1$ for real-valued $V(x)$. Let us assume the opposite and demonstrate, that this tends to a contradiction. For this purpose we show at first, that the Wronskians of elements of a canonical basis $\varphi_1(x), \dots, \varphi_{(k_c-1)/2}$ in $\ker r^t_{N_b}$ satisfy the following system,

$$\begin{gathered} \Bigl(\frac{\hat W_l(x)}{\hat W_{l-2}(x)}\Bigr)' = -\Bigl(\frac{\hat W_{l-1}(x)}{\hat W_{l-2}(x)}\Bigr)^2, \qquad l = 2, \dots, \frac{k_c - 1}{2}, \\ \hat W_0(x) \equiv 1, \qquad \hat W_l(x) = \begin{vmatrix} \varphi_1(x) & \varphi'_1(x) & \dots & \varphi_1^{(l-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_l(x) & \varphi'_l(x) & \dots & \varphi_l^{(l-1)}(x) \end{vmatrix}, \qquad l = 1, \dots, \frac{k_c - 1}{2}. \end{gathered} \tag{5.16}$$

This system arises by virtue of Lemma 1 from [4], when using the factorization $r^t_{N_b}$ in the product of intertwining operators of the first order and owing to (2.5),

$$\begin{gathered} r^t_{N_b} = (-1)^{(k_c-1)/2} \hat r_{N_b} \cdots \hat r_1, \qquad \hat r_l \cdots \hat r_1 \varphi_l = 0, \qquad \hat r_l = \partial + \hat\chi_l(x), \\ \hat\chi_l(x) = -\frac{(\hat r_{l-1} \cdots \hat r_1 \varphi_l(x))'}{\hat r_{l-1} \cdots \hat r_1 \varphi_l(x)} = -\frac{(\hat W_l(x)/\hat W_{l-1}(x))'}{\hat W_l(x)/\hat W_{l-1}(x)} = \frac{\hat W'_{l-1}(x)}{\hat W_{l-1}(x)} - \frac{\hat W'_l(x)}{\hat W_l(x)}, \\ l = 1, \dots, \frac{k_c - 1}{2} \end{gathered}$$

and employing the chain:

$$\begin{aligned} \left(\frac{\hat W_{l-1}(x)}{\hat W_{l-2}(x)}\right)^2 &= \frac{\hat W_{l-1}(x)}{\hat W_{l-2}(x)} \hat r_{l-2} \cdots \hat r_1 \varphi_{l-1} = \frac{\hat W_{l-1}(x)}{\hat W_{l-2}(x)} \hat r_{l-2} \cdots \hat r_1 (h_{N_b+1} - E_c)\varphi_l \\ &= \frac{\hat W_{l-1}(x)}{\hat W_{l-2}(x)} (\hat h_{l-2} - E_c) \hat r_{l-2} \cdots \hat r_1 \varphi_l = \frac{\hat W_{l-1}(x)}{\hat W_{l-2}(x)} \hat r^t_{l-1} \hat r_{l-1} \hat r_{l-2} \cdots \hat r_1 \varphi_l \end{aligned}$$

$$= \frac{\hat{W}_{l-1}(x)}{\hat{W}_{l-2}(x)}\left(-\partial + \frac{\hat{W}'_{l-2}(x)}{\hat{W}_{l-2}(x)} - \frac{\hat{W}'_{l-1}(x)}{\hat{W}_{l-1}(x)}\right)\frac{\hat{W}_{l}(x)}{\hat{W}_{l-1}(x)}$$
$$= -\left(\frac{\hat{W}_{l}(x)}{\hat{W}_{l-2}(x)}\right)', \quad l = 2,\dots,\frac{k_c-1}{2},$$

where $\hat{h}_1,\dots,\hat{h}_{(k_c-5)/2}$ are the corresponding intermediate Hamiltonians and $\hat{h}_0 = h_{N_b+1}$.

With the help of the system (5.16) we shall demonstrate, that it is possible to separate from the right-hand side of $r^t_{N_b}$ the intertwining operator of the first or of the second order with smooth coefficients. For this purpose, in view of (2.4) and (2.5), it is sufficient to derive, that there are no zeros either for $\hat{W}_2(x)$ or for $\hat{W}_{(k_c-1)/2-2l}(x)$, $l = 1,2,3,\dots$. Assume, that there are zeros for both $\hat{W}_2(x)$ and $\hat{W}_{(k_c-1)/2-2}(x)$, and show, that this assumption is contradictory. Let us notice, that the Wronskian $\hat{W}_{(k_c-1)/2}(x)$ has no zeros by virtue of (2.4) and of infinite smoothness of the potentials of h_{N_b} and h_{N_b+1} and that all functions $\varphi_1(x),\dots,\varphi_{(k_c-1)/2}(x)$ being polynomials possess finite numbers of zeros. Assume also without loss of generality that all these functions are real-valued.

In view of (5.16) the ratio $\hat{W}_{(k_c-1)/2}(x)/\hat{W}_{(k_c-2)/2-2}(x)$ decreases monotonically from $+\infty$ starting from the utmost right zero of $\hat{W}_{(k_c-2)/2-2}(x)$ and tends to a nonnegative limit for $x \to +\infty$. Using equation (5.16) for two successive l, one can obtain the system

$$\left(-\frac{\hat{W}_{l}(x)}{\hat{W}_{l-2}(x)}\right)'\left(\frac{\hat{W}_{l-3}(x)}{\hat{W}_{l-1}(x)}\right)' = 1, \qquad l = 3,\dots,\frac{k_c-1}{2}. \tag{5.17}$$

With the help of the equation from this system for $l = (k_c-1)/2$ we conclude, that the ratio $\hat{W}_{(k_c-1)/2-3}(x)/\hat{W}_{(k_c-1)/2-1}(x)$ monotonically increases towards the right side, starting from the utmost right zero of $\hat{W}_{(k_c-1)/2-1}(x)$, $\hat{W}_{(k_c-1)/2-2}(x)$ and $\hat{W}_{(k_c-1)/2-3}(x)$. As well, in view of equation (5.17) and the Bunyakovsky inequality, the following estimate holds,

$$(x-x_0)^2 \leqslant \left(\frac{\hat{W}_{(k_c-1)/2}(x_0)}{\hat{W}_{(k_c-1)/2-2}(x_0)} - \frac{\hat{W}_{(k_c-1)/2}(x)}{\hat{W}_{(k_c-1)/2-2}(x)}\right)\left(\frac{\hat{W}_{(k_c-1)/2-3}(x)}{\hat{W}_{(k_c-1)/2-1}(x)} - \frac{\hat{W}_{(k_c-1)/2-3}(x_0)}{\hat{W}_{(k_c-1)/2-1}(x_0)}\right),$$
$$x > x_0, \tag{5.18}$$

where x_0 is a fixed point on the right-hand side of the utmost right zero of $\hat{W}_{(k_c-1)/2-1}(x)$, $\hat{W}_{(k_c-1)/2-2}(x)$ and $\hat{W}_{(k_c-1)/2-3}(x)$. The left-hand side of (5.18) tends to $+\infty$ for $x \to +\infty$ and the first cofactor on the right-hand side of (5.18) approaches to a positive constant for $x \to +\infty$. Hence, the ratio $\hat{W}_{(k_c-1)/2-3}(x)/\hat{W}_{(k_c-1)/2-1}(x)$ tends to $+\infty$ for $x \to +\infty$ and $\hat{W}_{(k_c-1)/2-1}(x)/\hat{W}_{(k_c-1)/2-3}(x)$ monotonically decreases for $x > x_0$ and tends to zero for $x \to +\infty$. Arguing in the same way by induction, one obtains that the ratios $\hat{W}_{(k_c-1)/2-l-3}(x)/\hat{W}_{(k_c-1)/2-l-1}(x)$, $l = 1,\dots,(k_c-1)/2-3$ also tend to $+\infty$ for $x \to +\infty$, but the latter contradicts to the fact, that $\hat{W}_2(x) \equiv \hat{W}_2(x)/\hat{W}_0(x)$ monotonically decreases on the whole axis in view of (5.16) and is negative on the right from the unique zero of $\hat{W}_2(x)$. Thus, there are no zeros, at least, for one of the Wronskians $\hat{W}_2(x)$ and $\hat{W}_{(k_c-1)/2-2}(x)$. Hence, in view of (2.4) and (2.5) one can separate an intertwining operator of the second order with infinitely smooth coefficients from one of the sides of $r^t_{N_b}$. Using induction again, we conclude that it is possible to separate the intertwining operator of the first or of the second order with infinitely smooth coefficients from the right-hand side of $r^t_{N_b}$.

Finally let us demonstrate, that the latter result tends to a contradiction. The function $\varphi_1(x)$ as a formal eigenfunction of h_{N_b+1} for the spectral value E_c takes either the form $\varphi_1(x) = C$, $C \in \mathbb{R}$, $C \neq 0$ or the form $\varphi_1(x) = C_1x+C_2$, $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, $C_1 \neq 0$. The first one is impossible, because in this case $\hat{r}_1 = \partial$, that contradicts to non-minimizability of e ($\hat{r}_1\partial = \partial^2 = E_c - h_{N_b+1}$). For the second one the separation in $r^t_{N_b}$ on its right-hand side of the intertwining operator of the first order with infinitely smooth coefficients is impossible because the coefficient of $\hat{r}_1 = \partial - C_1/(C_1x+C_2)$ at ∂^0 possess the pole at $x = -C_2/C_1$. The separation of the intertwining operator of the second order in $r^t_{N_b}$ on the right-hand side is impossible as well, because one can easily check that $\hat{W}_2(x)$ cannot be nodeless. Therefore, $r_{N_b} = 1$.

Thus, all statements (1)–(9) from the beginning of this subsection are validated.

Remark 5. It follows from (5.12), that in the presence of non-real energy(-ies) of h bound state(s) the value $|T(k)|$ is different from identical unity. But if all non-real energies among E_l, $l = 0, \ldots, N_b - 1$ can be divided into pairs of mutually complex conjugated energies with equal (inside a pair) algebraic multiplicities, then obviously $|T(k)| \equiv 1$.

6 Examples

We present here three examples[6], illustrating results of the previous section.

Example 1. Non-Hermitian (in general) Hamiltonian with one bound state

$$
\begin{aligned}
& h = -\partial^2 - \frac{2\alpha^2}{\operatorname{ch}^2\alpha x}, \qquad \operatorname{Re}\alpha > 0, \\
& \psi_{0,0}(x) = \frac{1}{\operatorname{ch}\alpha x}, \qquad h\psi_{0,0} = E_0\psi_{0,0}, \qquad E_0 = -\alpha^2, \\
& \psi_c(x) = \operatorname{th}\alpha x, \qquad h\psi_c = E_c\psi_c, \qquad E_c = 0, \\
& e = -r_0^t\,\partial\, r_0, \\
& r_0 = \partial - \frac{\psi'_{0,0}(x)}{\psi_{0,0}(x)} \equiv \partial + \alpha\operatorname{th}\alpha x, \qquad r_0\psi_{0,0} = 0, \qquad \partial r_0\psi_c = 0, \\
& \mathbf{S}_e = \begin{pmatrix} E_c & 0 & 0 \\ 0 & E_0 & 1 \\ 0 & 0 & E_0 \end{pmatrix}, \\
& \mathcal{P}_e(h) = (h - E_c)(h - E_0)^2, \\
& \psi_k(x) = (ik - \alpha\operatorname{th}\alpha x)e^{ikx}, \qquad h\psi_k = k^2\psi_k, \qquad T(k) = \frac{k + i\alpha}{k - i\alpha}, \qquad k \in \mathbb{R}.
\end{aligned}
$$

Example 2. Non-Hermitian Hamiltonian with one Jordan cell of the 2nd order

$$
\begin{aligned}
& h = -\partial^2 - 16\alpha^2\frac{\alpha(x - z)\operatorname{sh}2\alpha x - 2\operatorname{ch}^2\alpha x}{[\operatorname{sh}2\alpha x + 2\alpha(x - z)]^2}, \qquad \alpha > 0, \qquad \operatorname{Im} z \neq 0, \\
& \psi_{0,0}(x) = \frac{\operatorname{ch}\alpha x}{\operatorname{sh}2\alpha x + 2\alpha(x - z)}, \qquad \psi_{0,1}(x) = \frac{2\alpha(x - z)\operatorname{sh}2\alpha x - \operatorname{ch}\alpha x}{(2\alpha)^2[\operatorname{sh}2\alpha x + 2\alpha(x - z)]^2}, \\
& h\psi_{0,0} = E_0\psi_{0,0}, \qquad (h - E_0)\psi_{0,1} = \psi_{0,0}, \qquad E_0 = -\alpha^2, \\
& \psi_c(x) = \frac{\operatorname{sh}2\alpha x - 2\alpha(x - z)}{\operatorname{sh}2\alpha x + 2\alpha(x - z)}, \qquad h\psi_c = E_c\psi_c, \qquad E_c = 0, \\
& e = r_0^t\,\partial\, r_0, \qquad r_0 = r_{0,1}r_{0,0}, \\
& r_{0,0} = \partial - \frac{\psi'_{0,0}(x)}{\psi_{0,0}(x)}, \qquad r_{0,1} = \partial - \frac{(r_{0,0}\psi_{0,1})'(x)}{r_{0,0}\psi_{0,1}(x)}, \\
& r_{0,0}\psi_{0,0} = 0, \qquad r_{0,1}r_{0,0}\psi_{0,1} = 0, \qquad \partial r_0\psi_c = 0,
\end{aligned}
$$

[6]Other relevant examples can found in [41].

$$\mathbf{S}_e = \begin{pmatrix} E_c & 0 & 0 & 0 & 0 \\ 0 & E_0 & 1 & 0 & 0 \\ 0 & 0 & E_0 & 1 & 0 \\ 0 & 0 & 0 & E_0 & 1 \\ 0 & 0 & 0 & 0 & E_0 \end{pmatrix},$$

$$\mathcal{P}_e(h) = (h - E_c)(h - E_0)^4,$$

$$\psi_k(x) = \frac{(\alpha^2 - k^2)\,\mathrm{sh}\,2\alpha x - 2i\alpha k(1 + \mathrm{ch}\,2\alpha x) - 2\alpha(\alpha^2 + k^2)(x - z)}{\mathrm{sh}\,2\alpha x + 2\alpha(x - z)} e^{ikx},$$

$$h\psi_k = k^2\psi_k, \qquad T(k) = \left(\frac{k + i\alpha}{k - i\alpha}\right)^2, \qquad k \in \mathbb{R}.$$

Example 3. Non-Hermitian Hamiltonian with one bound state at the bottom of continuous spectrum

$$h = -\partial^2 + \frac{2}{(x - z)^2}, \qquad \mathrm{Im}\, z \neq 0,$$

$$\psi_{c,0}(x) = \frac{1}{(x - z)} \in L_2(\mathbb{R}), \qquad h\psi_{c,0} = E_c\psi_{c,0}, \qquad E_c = 0,$$

$$\psi_{c,1}(x) = \frac{1}{2}(x - z), \qquad (h - E_c)\psi_{c,1} = \psi_{c,0},$$

$$e = -r_0^t\,\partial\, r_0,$$

$$r_0 = \partial - \frac{\psi'_{c,0}(x)}{\psi_{c,0}(x)} \equiv \partial + \frac{1}{x - z}, \qquad r_0\psi_{c,0} = 0, \qquad \partial r_0\psi_{c,1} = 0,$$

$$\mathbf{S}_e = \begin{pmatrix} E_c & 1 & 0 \\ 0 & E_c & 1 \\ 0 & 0 & E_c \end{pmatrix},$$

$$\mathcal{P}_e(h) = (h - E_c)^3, \tag{6.1}$$

$$\psi_k(x) = \left(ik - \frac{1}{x - z}\right)e^{ikx}, \qquad h\psi_k = k^2\psi_k, \qquad T(k) = 1, \qquad k \in \mathbb{R}.$$

7 Concluding remarks and generalizations

(1) Let us examine the situation when the Hamiltonian $h^+ = -\partial^2 + V_1(x)$ with a smooth real-valued periodic potential $V_1(x)$ is transformed into the Hamiltonian $h^- = -\partial^2 + V_2(x)$ with a smooth real-valued potential $V_2(x)$, whose spectrum is different from the spectrum of h^+ only by presence of an eigenvalue λ_1 or two eigenvalues λ_1 and λ_2. The former can be done [13, 15] with the help of one nodeless real-valued transformation function $\phi_1(x)$, which is non-Bloch formal eigenfunction of h^+ for a real spectral value λ_1, situated below continuous spectrum of h^+. The latter can be realized with the help of two real-valued transformation functions $\phi_1(x)$ and $\phi_2(x)$ with nodeless Wronskian $W_-(x) = \phi_1^-(x)\phi_2^{-\prime}(x) - \phi_1^{-\prime}(x)\phi_2^-(x)$, which are non-Bloch formal eigenfunctions of h^+ for a real spectral values λ_1 and $\lambda_2 \neq \lambda_1$ respectively, situated inside a forbidden energy band (the same for both values) of h^+.

Let us suppose that for h^+ there is a non-minimizable t-antisymmetric symmetry operator e^+ with unity coefficient at the derivative of the highest order,

$$h^+e^+ = e^+h^+, \qquad (e^+)^t = -e^+,$$

and that $q_1^\pm$ ($q_2^\pm$) are corresponding intertwining operators for the first (second) case mentioned above,

$$h^\pm q_1^\pm = q_1^\pm h^\mp, \qquad (q_1^+)^t = q_1^-, \qquad q_1^-\phi_1^- = 0,$$

$$h^{\pm}q_2^{\pm} = q_2^{\pm}h^{\mp}, \qquad (q_2^{+})^t = q_2^{-}, \qquad q_2^{-}\phi_1^{-} = q_2^{-}\phi_2^{-} = 0.$$

Then, for h^- there is obviously a nonzero t-antisymmetric symmetry operator e^- with the unity coefficient at the highest derivative:

$$h^-e^- = e^-h^-, \qquad (e^-)^t = -e^-, \qquad e^- = (-1)^j(q_j^+)^t e^+ q_j^+, \qquad j = 1, 2. \tag{7.1}$$

Moreover, in view of (5.2) and Theorems 2 and 3 the operator e^- is non-minimizable and the canonical basis in $\ker q_1^+$ ($\ker q_2^+$) consists of eigenfunction(s) of h^- for the eigenvalue(s) λ_1 (λ_1 and λ_2). As well by virtue of (2.4), (2.5) and Theorem 3 one can conclude that if $V_1(x) \in C^\infty_{\mathbb{R}}$, $l = 1, 2$ the potential $V_2(x)$ and coefficients of $q_j^{\pm}$ and e^- are infinitely smooth too.

It follows from Theorem 1 and (7.1), that

$$\mathcal{P}_{e^-}(h^-) \equiv -(e^-)^2 = \mathcal{P}_{e^+}(h^-)\prod_{l=1}^{j}(h^- - \lambda_l)^2, \qquad j = 1, 2, \qquad \mathcal{P}_{e^+}(h^+) \equiv -(e^+)^2, \tag{7.2}$$

where properties of the polynomial $\mathcal{P}_{e^+}(\lambda)$ are described in Theorem 3. Thus, the algebraic multiplicity of the energy of a bound state of h^- in the spectrum of the matrix $\mathbf{S}$ of e^- is equal to 2, i.e. to doubled algebraic multiplicity of this energy in the spectrum of h^- (cf. with (5.4)).

(2) As it is known [2], the Hamiltonians with finite-zone periodic potentials represent a partial case of Hamiltonians with quasiperiodic and, in general, complex potentials for which there are nonzero t-antisymmetric symmetry operators. As well the Hamiltonians of the type h^- considered above belong to the case of Hamiltonians with potentials, which are called "reflectionless potentials against the background of finite-zone potentials", and for which there are nonzero t-antisymmetric symmetry operators too.

One could generalize the results of the previous and present sections onto the Hamiltonians with quasiperiodic potentials and the Hamiltonians with reflectionless potentials against the background of finite-zone potentials. In particular, one can conjecture, that in the latter case the algebraic multiplicity of the energy of any bound state of h in the spectrum of the matrix $\mathbf{S}$ of e is equal to doubled algebraic multiplicity of this energy in the spectrum of h (we suppose, that the energy is not located inside or on a border of h continuous spectrum). This hypothesis is natural[7] in view of (4.4), (5.4), (7.2) and the fact, that reflectionless potentials against the background of finite-zone potentials are limiting cases [2] of quasiperiodic potentials, when some of its periods tend to infinity and some allowed energy bands shrink into points, being energies of bound states. If this hypothesis is valid we can derive the factorization for e analogous to (5.5) and (7.1). But the central position in the factorization will be occupied by a nonzero non-minimizable t-antisymmetric symmetry operator for the corresponding intermediate Hamiltonian with finite-zone potential without bound states. One can surmise also that the algebraic multiplicity of any border of the continuous spectrum of h (but not a border between allowed energy bands, see Remark 4), in the spectrum of the matrix $\mathbf{S}$ of e is odd (cf. with (4.4), (5.4) and (6.1)).

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[7]The appearance of second powers in (5.4) for cofactors corresponding to bound states is explained in [16] for a case with real-valued potentials through shrinking forbidden energy bands.

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