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Restriction of Odd Degree Characters of $\mathfrak{S}_n$

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Abstract. Let n and k be natural numbers such that $2^k < n$. We study the restriction to $\mathfrak{S}_{n-2^k}$ of odd-degree irreducible characters of the symmetric group $\mathfrak{S}_n$. This analysis completes the study begun in [Ayyer A., Prasad A., Spallone S., *Sém. Lothar. Combin.* **75** (2015), Art. B75g, 13 pages] and recently developed in [Isaacs I.M., Navarro G., Olsson J.B., Tiep P.H., *J. Algebra* **478** (2017), 271–282].

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1 Introduction

Let n be a natural number, and let χ be an irreducible character of odd degree of the symmetric group $\mathfrak{S}_n$. Then there exists a unique odd-degree irreducible constituent of the restriction $\chi_{\mathfrak{S}_{n-1}}$. This interesting fact was discovered recently in [1]. The result had immediate applications in the study of natural correspondences of characters of finite groups (see for example [2]). In [3, Theorem A] the result mentioned above was generalized, by showing that given any $k \in \mathbb{N}$ such that $2^k < n$, there exists a unique odd-degree irreducible constituent $f_k^n(\chi)$ of $\chi_{\mathfrak{S}_{n-2^k}}$ appearing with odd multiplicity. The main goal of this article is to study for all $n, k \in \mathbb{N}$ the map

$$f_k^n\colon\ \mathrm{Irr}_{2'}(\mathfrak{S}_n) \longrightarrow \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^k}),$$

naturally defined by Theorem A of [3]. All our results are proved using a description of f_k^n in terms of the natural partition labels of the involved irreducible characters.

Before describing the main results of this paper, we introduce some vocabulary. If 2^k appears in the binary expansion of n we say that 2^k is a *binary digit* of n. Similarly we say that two natural numbers m and n are 2-*disjoint* if they do not have any common binary digit. On the other hand, if $m \leq n$ and all the binary digits of m appear in the binary expansion of n, then we say that m is a *binary subsum* of n. This will be denoted by $m \subseteq_2 n$. Let $\nu_2(n)$ be the exponent of the highest power of 2 dividing the integer n.

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A question raised in [3] may be phrased as: *For which n and k is f_k^n surjective?* The authors showed that f_k^n is surjective whenever 2^k is a binary digit of n, and they observed that otherwise f_k^n could be both surjective or not (see [3, Proposition 4.5 and Remark 4.6]). In this paper we answer the question of surjectivity completely with the following result.

Theorem A. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $d(n,k) = \nu_2\big(\lfloor \frac{n}{2^k} \rfloor\big)$.*

- *If $k = 0$ then f_k^n is surjective if and only $d(n,k) \leq 2$.*
- *If $k > 0$ then f_k^n is surjective if and only $d(n,k) \leq 1$.*

Theorem A is a consequence of Theorem 3.5 below, which describes the images of the maps f_k^n. For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $2^k < n$ and any $\psi \in \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^k})$ we define the set

$$\mathcal{E}\big(\psi, 2^k\big) = \big\{\chi \in \mathrm{Irr}_{2'}(\mathfrak{S}_n) \mid f_k^n(\chi) = \psi\big\},$$

and set $e\big(\psi, 2^k\big) = \big|\mathcal{E}\big(\psi, 2^k\big)\big|$. We show in Corollary 3.8 that the maps f_k^n are regular on their images. This means that for any ψ in the image of f_k^n, the number $e(\psi, 2^k)$ depends only on n and k and not on the specific ψ. We also give a complete description of those $\psi \in \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^k})$ such that $e(\psi, 2^k) = 0$, in Theorem 3.5.

In the final part of the paper we study commutativity. For convenience, we sometimes denote f_k^n just by f_k, when the natural number n is clear from the context. Then, for $k, \ell \in \mathbb{N}_0$, $k < \ell$, such that $2^k + 2^\ell \leq n$, we may ask: *when is $f_k f_\ell = f_\ell f_k$?* or more specifically: *when is $f_k^{n-2^\ell} f_\ell^n = f_\ell^{n-2^k} f_k^n$?* In [3, Proposition 4.3] it was proved that $f_k f_\ell = f_\ell f_k$ whenever $2^\ell < n < 2^{\ell+1}$. This is the case $\ell = t$ in our second main result, which answers the question completely.

Theorem B. *Let $n = 2^t + m$ where $0 \leq m < 2^t$. Suppose that k, ℓ satisfy $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. Then,* with the exception of the case *$n = 6$, $k = 0$, $\ell = 1$,*

$$f_k f_\ell = f_\ell f_k \text{ if and only if } 2^k > m \text{ or } \ell = t.$$

2 Notation and background

Let n be a natural number. We let $\mathrm{Irr}(\mathfrak{S}_n)$ denote the set of irreducible characters of $\mathfrak{S}_n$ and $\mathcal{P}(n)$ the set of partitions of n. The notation $\lambda \in \mathcal{P}(n)$ is sometimes replaced by $\lambda \vdash n$ and we write $|\lambda| = n$. There is a natural correspondence $\lambda \leftrightarrow \chi^\lambda$ between $\mathcal{P}(n)$ and $\mathrm{Irr}(\mathfrak{S}_n)$. We say then that λ labels χ^λ. We denote by $\mathrm{Irr}_{2'}(\mathfrak{S}_n)$ the set of irreducible characters of $\mathfrak{S}_n$ of odd degree. If $\chi^\lambda \in \mathrm{Irr}_{2'}(\mathfrak{S}_n)$ we say that χ^λ is an *odd character*, we call λ an *odd partition* of n and write $\lambda \vdash_o n$. Also the empty partition will be considered as an odd partition.

Remark 2.1. Let n, k be such that $2^k < n$. In [3, Theorem A and Proposition 4.2] it is shown that the map $f_k^n \colon \mathrm{Irr}_{2'}(\mathfrak{S}_n) \to \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^k})$ may be described in terms of the odd partitions labelling the odd characters as follows:

$$f_k^n(\chi^\lambda) = \chi^\mu \Leftrightarrow \mu \vdash_o n - 2^k \text{ can be obtained from } \lambda \vdash_o n \text{ by removing a } 2^k\text{-hook.}$$

Correspondingly we write (by abuse of notation) $f_k^n(\lambda) = \mu$. In fact when λ is odd, there is only one 2^k-hook of λ whose removal leads again to an odd partition; we will refer to such a hook as an *odd hook* of λ. This combinatorial description of f_k^n will be used throughout this paper, and we will regard f_k^n also as a map between the corresponding sets of odd partitions. Also, for $\mu \vdash_o n - 2^k$ we set $e\big(\mu, 2^k\big) = e\big(\chi^\mu, 2^k\big)$.

We need some concepts and basic facts concerning hooks in partitions. For any integer $e \in \mathbb{N}$ we denote by $C_e(\lambda)$ and $Q_e(\lambda)$ the e-core and the e-quotient of λ, respectively. Then $Q_e(\lambda) = (\lambda_0, \ldots, \lambda_{e-1})$ is an e-tuple of partitions satisfying $n = |C_e(\lambda)| + e \sum_{i=0}^{e-1} |\lambda_i|$. It is well-known that a partition is uniquely determined by its e-core and e-quotient (we refer the reader to [6] or [4, Chapter 2.7] for a detailed discussion on this topic).

Let $\mathcal{H}_e(\lambda)$ be the set of hooks of λ having length divisible by e, and let $\mathcal{H}(Q_e(\lambda)) = \cup_{i=0}^{e-1}\mathcal{H}(\lambda_i)$. As explained in [6, Theorem 3.3], there is a bijection between $\mathcal{H}_e(\lambda)$ and $\mathcal{H}(Q_e(\lambda))$ mapping hooks in λ of length ex to hooks in the quotient of length x. Moreover, the bijection respects the process of hook removal. Namely, the partition μ obtained by removing a ex-hook from λ is such that $C_e(\mu) = C_e(\lambda)$ and the e-quotient of μ is obtained by removing an x-hook from one of the partitions involved in $Q_e(\lambda)$.

For $e = 2$ we want to repeat the process of taking 2-cores and 2-quotients to obtain the 2-*quotient tower* $\mathcal{Q}_2(\lambda)$ and the 2-*core tower* $\mathcal{C}_2(\lambda)$ of λ. They have rows numbered by $k \geq 0$. The kth row $\mathcal{Q}_2^{(k)}(\lambda)$ of $\mathcal{Q}_2(\lambda)$ contains 2^k partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, and the kth row $\mathcal{C}_2^{(k)}(\lambda)$ of $\mathcal{C}_2(\lambda)$ contains the 2-cores of these partitions in the same order, i.e., $C_2\big(\lambda_i^{(k)}\big)$, $0 \leq i \leq 2^k - 1$. The 0th row of $\mathcal{Q}_2(\lambda)$ contains $\lambda = \lambda_0^{(0)}$ itself, row 1 contains the partitions $\lambda_0^{(1)}$, $\lambda_1^{(1)}$ occurring in the 2-quotient $Q_2(\lambda)$, row 2 contains the partitions occurring in the 2-quotients of partitions occurring in row 1, and so on. Specifically we have $Q_2\big(\lambda_i^{(k)}\big) = \big(\lambda_{2i}^{(k+1)}, \lambda_{2i+1}^{(k+1)}\big)$ for $i \in \big\{0, 1, \ldots, 2^k - 1\big\}$. We remark that the 2^k partitions in $\mathcal{Q}_2^{(k)}(\lambda)$ are the same as those in the 2^k-quotient $Q_{2^k}(\lambda)$ of λ, but in a different order for $k \geq 2$.

We also introduce the k-*data* $\mathcal{D}_2^{(k)}(\lambda)$ of λ. This is a table containing the following $k+1$ rows: the k rows $\mathcal{C}_2^{(j)}(\lambda)$, $j = 0, \ldots, k-1$, and in addition the row $\mathcal{Q}_2^{(k)}(\lambda)$.

Remark 2.2. A partition λ may be recovered from its 2-core tower. For $k > 0$, it may also be recovered from the knowledge of the k-data $\mathcal{D}_2^{(k)}(\lambda)$ of λ, because the rows $\mathcal{C}_2^{(l)}(\lambda)$ with $l \geq k$ of $\mathcal{C}_2(\lambda)$ consist of the 2-core towers of the partitions in $\mathcal{Q}_2^{(k)}(\lambda)$.

Lemma 2.3. *Suppose that $\lambda \vdash n - 2^k$ and $\mu \vdash n$. The following are equivalent.*

(i) λ is obtained from μ by removing a 2^k-hook.

(ii) The k-data $\mathcal{D}_2^{(k)}(\mu)$ and $\mathcal{D}_2^{(k)}(\lambda)$ coincide, except that for one $i \in \big\{0, \ldots, 2^k - 1\big\}$ $\lambda_i^{(k)}$ is obtained from $\mu_i^{(k)}$ by removing a 1-hook.

Proof. A 2^k-hook H_0 in μ corresponds in a canonical way to a 2^{k-1}-hook H_1 in a partition in $\mathcal{Q}_2^{(1)}(\mu)$, i.e., in row 1 of the 2-quotient tower $\mathcal{Q}_2(\mu)$. Continuing we see that H_0 corresponds in a canonical way to a 1-hook H_k in a partition $\mu_i^{(k)}$ in $\mathcal{Q}_2^{(k)}(\mu)$, row k of $\mathcal{Q}_2(\mu)$. If λ is obtained by removing H_0 from μ, this corresponds to $\lambda_i^{(k)}$ being obtained by removing the 1-hook H_k from $\mu_i^{(k)}$ (by repeated applications of [6, Theorem 3.3]). Apart from this the rows $\mathcal{Q}_2^{(k)}(\mu)$ and $\mathcal{Q}_2^{(k)}(\lambda)$ coincide. Note also that the rows $\mathcal{C}_2^{(j)}(\mu)$ and $\mathcal{C}_2^{(j)}(\lambda)$ coincide for $j = 0, \ldots, k-1$, since the removal of the hooks H_j of even length do not change the 2-cores. ■

Odd-degree characters of $\mathfrak{S}_n$ and thus odd partitions were completely described in [5]. We restate this result in a language which is convenient for our purposes. We let $c_2^{(k)}(\lambda)$ be the sum of the cardinalities of the partitions in the kth row $\mathcal{C}_2^{(k)}(\lambda)$ of $\mathcal{C}_2(\lambda)$.

Lemma 2.4 ([5])**.** *Let λ be a partition. Then λ is odd if and only if $c_2^{(k)}(\lambda) \leq 1$ for all $k \geq 0$.*

It may be decided from the k-data $\mathcal{D}_2^{(k)}(\lambda)$ whether λ is odd. The case $k = 1$ of the following result appeared in [3, Lemma 4.1] and also in [1, Lemma 6].

Theorem 2.5. *Let $\lambda \vdash n$, and let $k \geq 0$ be fixed. Consider $\mathcal{Q}_2^{(k)}(\lambda) = \big(\lambda_i^{(k)}\big)$. Then λ is odd if and only if the following conditions are all fulfilled:*

(i) $c_2^{(j)}(\lambda) \leq 1$ *for all* $j < k$.

(ii) The partitions $\lambda_i^{(k)}$, $0 \leq i \leq 2^k - 1$, *are all odd.*

(iii) The numbers $\big|\lambda_i^{(k)}\big|$, $0 \leq i \leq 2^k - 1$, *are pairwise 2-disjoint.*

In this case $\sum_{i \geq 0} \big|\lambda_i^{(k)}\big| = \lfloor \frac{n}{2^k} \rfloor$.

Proof. This is proved by induction on $k \geq 0$, using Remark 2.2 and Lemma 2.4. ∎

We illustrate the result above by giving an example.

Example 2.6. Let $n = 15$ and take $\lambda = \big(5, 4, 2^2, 1^2\big) \vdash 15$. To decide whether λ is odd, we choose $k = 2$ and compute the 2-data $\mathcal{D}_2^{(2)}(\lambda)$. The 2-core is $C_2(\lambda) = (1)$, giving $\mathcal{C}_2^{(0)}(\lambda) = ((1))$. Furthermore, the 2-quotient is $Q_2(\lambda) = \big((2^2, 1^2), (1)\big)$, and computing the 2-cores $C_2\big((2^2, 1^2)\big) = (0)$, $C_2((1)) = (1)$, we obtain the next row: $\mathcal{C}_2^{(1)}(\lambda) = ((0), (1))$. The 2-quotients are $Q_2\big((2^2, 1^2)\big) = \big((1^2), (1)\big)$, $Q_2((1)) = ((0), (0))$; hence the final row of the 2-data table is obtained as $\mathcal{Q}_2^{(2)}(\lambda) = \big((1^2), (1), (0), (0)\big)$.

We visualize $\mathcal{D}_2^{(2)}(\lambda)$ like this:

$$
\begin{array}{lcccccccc}
\mathcal{C}_2^{(0)}(\lambda): & & & & (1) & & & & \\
\mathcal{C}_2^{(1)}(\lambda): & & (0) & & & & (1) & & \\
\mathcal{Q}_2^{(2)}(\lambda): & (1^2) & & (1) & & (0) & & (0) &
\end{array}
$$

Theorem 2.5 shows that λ is odd and thus it contains a unique odd 4-hook. Again using the theorem, it is clear that removing this 4-hook corresponds to the second partition (1) in $\mathcal{Q}_2^{(2)}$ being replaced by (0). Thus, removing the corresponding 4-hook of λ we obtain the odd partition $\mu = \big(3, 2^3, 1^2\big) \vdash 11$ with the property that $\mathcal{D}_2^{(2)}(\lambda)$ and $\mathcal{D}_2^{(2)}(\mu)$ differ only in their final row.

Remark 2.7. Using the construction of partitions from their 2-cores and 2-quotients already mentioned, the criterion above can be applied to construct all odd partitions of n with a specific kth row in the 2-quotient tower. For this, let $n, k \in \mathbf{N}$, and take any sequence of odd partitions ν_i, $0 \leq i \leq 2^k - 1$, such that the numbers $|\nu_i|$ are pairwise 2-disjoint, and $\sum_{i \geq 0} |\nu_i| = \lfloor \frac{n}{2^k} \rfloor$. Then there are exactly $\prod_{\substack{m<k \\ 2^m \subseteq_2 n}} 2^m$ odd partitions λ of n with $\mathcal{Q}_2^{(k)}(\lambda) = (\nu_i)$, obtained by choosing one 2-core in row m of the k-data table to be (1), for each $m < k$ such that $2^m \subseteq_2 n$.

The following easy consequence of Theorem 2.5 will be used repeatedly.

Lemma 2.8. *Let 2^t be the largest binary digit of n. A partition λ of n is odd if and only if λ contains a unique 2^t-hook and the partition obtained from λ by removing this 2^t-hook is an odd partition of $n - 2^t$.*

3 Surjectivity and regularity

The aim of this section is to study the images of the maps f_k^n for all n, k such that $2^k \leq n$. For this purpose we introduce the concept of *d-good partitions* (see Definition 3.1 below). This will allow us to prove Theorem 3.5 (describing the images) and thus Theorem A (describing exactly when f_k^n is surjective) and to show that the maps f_k^n are always regular on their image (see Corollary 3.8).

Definition 3.1. Let $d \geq 0$. We call an odd partition λ *d-good*, if

(i) $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.

(ii) $C_{2^d}(\lambda)$ is a hook partition.

Let us remark that condition (i) may be reformulated as

(i*) $\nu_2(|\lambda| + 1) = d$.

In particular, if λ is d-good, then $|\lambda|$ is odd if and only if $d > 0$.

The relevance of d-good partitions in our context is illuminated by the following reformulation of [1, Theorem 2]:

Lemma 3.2. *Let $\lambda \vdash_o n$. Let $d = \nu_2(n+1)$. Then $e(\lambda, 1) \neq 0$ if and only if λ is d-good. In this case, $e(\lambda, 1) = 1$ if $d = 0$, and $e(\lambda, 1) = 2$ if $d > 0$.*

Lemma 3.3. *Let λ be an odd partition, and let $d \geq 0$. Then the following hold.*

(1) *For $d \leq 2$, λ is d-good if and only if $|\lambda| \equiv 2^d - 1 \mod 2^{d+1}$.*

(2) *If λ is d-good, then $C_{2^d}(\lambda)$ is a partition of $2^d - 1$.*

Proof. If the odd partition λ is d-good, then $|\lambda| = \left(2^d - 1\right) + m$ where the binary digits of m are at least 2^{d+1}. The hooks of λ corresponding to the binary digits of m may be decomposed into 2^d-hooks and thus do not contribute to $C_{2^d}(\lambda)$. Thus $|C_{2^d}(\lambda)| = 2^d - 1$. This shows (2). For $d = 0, 1, 2$ we have $|C_{2^d}(\lambda)| = 0$, 1 and 3, respectively. Since all partitions of 0, 1 and 3 are hook partitions, (1) follows. ∎

Definition 3.4. If $2^k \leq n$, we define $d(n,k) = \nu_2\left(\lfloor \frac{n}{2^k} \rfloor\right)$. Thus $d(n,k)$ is the smallest integer $d \geq 0$ satisfying the condition $2^{k+d} \subseteq_2 n$. In particular, $d(n,k) = 0$ if and only if $2^k \subseteq_2 n$. Moreover, we may write $\lfloor \frac{n}{2^k} \rfloor = 2^{d(n,k)} + m(n,k)$ where $2^{d(n,k)+1} \mid m(n,k)$.

As mentioned in the introduction, the results in [3] show that f_k^n is a surjective (2^k-to-1)-map whenever $2^k \subseteq_2 n$, i.e., $d(n,k) = 0$. In the spirit of [1, Theorem 2], we now give a characterization of the image of the map f_k^n for all n, k such that $2^k < n$.

Theorem 3.5. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$. Let $\lambda \vdash_o n - 2^k$. Then $e\left(\lambda, 2^k\right) \neq 0$ if and only if there exists a $d(n,k)$-good partition in the kth row of $\mathcal{Q}_2(\lambda)$. In this case, $e\left(\lambda, 2^k\right) = 2^k$ if $d(n,k) = 0$, and $e\left(\lambda, 2^k\right) = 2$ if $d(n,k) > 0$.*

Proof. If $k = 0$ then the statement follows from Lemma 3.2. Hence assume that $k \geq 1$. Let $d = d(n,k)$. By assumption $\lfloor \frac{n}{2^k} \rfloor = 2^d + m$, where the binary digits of m are at least 2^{d+1}. Thus $\lfloor \frac{n-2^k}{2^k} \rfloor = \left(2^d - 1\right) + m$.

Suppose first that $e\left(\lambda, 2^k\right) \neq 0$ and that $\mu \vdash_o n$ satisfies $f_k(\mu) = \lambda$. From Remark 2.1 and Lemma 2.3 we get that there exists an $i \in \{0, 1, \ldots, 2^k - 1\}$ such that $f_0\left(\mu_i^{(k)}\right) = \lambda_i^{(k)}$. Since $\mu_i^{(k)}$ and $\lambda_i^{(k)}$ are odd, we get $e\left(\lambda_i^{(k)}, 1\right) \neq 0$. We have that $\left|\lambda_i^{(k)}\right|$ and $\left|\mu_i^{(k)}\right|$ are both 2-disjoint with $m_1 := \sum_{j \neq i} \left|\lambda_j^{(k)}\right| = \sum_{j \neq i} \left|\mu_j^{(k)}\right| \subseteq_2 \lfloor \frac{n-2^k}{2^k} \rfloor$, by Theorem 2.5. Since $m_1 \subseteq_2 \lfloor \frac{n-2^k}{2^k} \rfloor$ and $m_1 \subseteq_2 \lfloor \frac{n}{2^k} \rfloor$, we get $m_1 \subseteq_2 m$. Thus $\left|\lambda_i^{(k)}\right| = (2^d - 1) + m_2$ and $\left|\mu_i^{(k)}\right| = 2^d + m_2$, where $m_2 = m - m_1 \subseteq_2 m$. In particular $\nu_2\left(\left|\lambda_i^{(k)}\right| + 1\right) = \nu_2\left(\left|\mu_i^{(k)}\right|\right) = d$. Then Lemma 3.2 shows that $\lambda_i^{(k)}$ is d-good.

Conversely, if $\lambda_i^{(k)}$ is a d-good partition for some $i \in \left\{0, 1, \ldots, 2^k - 1\right\}$, then there exists a $\mu^* \vdash_o \left|\lambda_i^{(k)}\right| + 1$ such that $f_0(\mu^*) = \lambda_i^{(k)}$, by Lemma 3.2. We let μ be the partition where the k-data $\mathcal{D}_2^{(k)}(\mu)$ and $\mathcal{D}_2^{(k)}(\lambda)$ coincide, except that $\mu_i^{(k)} = \mu^*$. Since λ is odd and $\lambda_i^{(k)}$ is d-good, we know that $|\lambda_i^{(k)}| = (2^d - 1) + m'$ where $m' \subseteq_2 m$, and $|\lambda_j^{(k)}| \subseteq_2 m - m'$ for all $j \neq i$. Hence $|\mu^*| = |\lambda_i^{(k)}| + 1 = 2^d + m'$ is 2-disjoint from all $|\lambda_j^{(k)}|$, $j \neq i$. Thus μ is an odd partition of n by Theorem 2.5, and $f_k(\mu) = \lambda$ by Lemma 2.3 and Remark 2.1.

We conclude that $e(\lambda, 2^k) = \sum_{\lambda_i^{(k)} d\text{−good}} e(\lambda_i^{(k)}, 1)$. If $d = 0$ then $\lfloor \frac{n-2^k}{2^k} \rfloor$ is even. This implies that all $\lambda_i^{(k)}$ are of even cardinality and thus d-good. Thus $e(\lambda_i^{(k)}, 1) = 1$ for all i, and we get $e(\lambda, 2^k) = 2^k$. If $d > 0$ there is exactly one $\lambda_i^{(k)}$ in $\mathcal{Q}_2^{(k)}(\lambda)$ of odd cardinality. Only this $\lambda_i^{(k)}$ may be d-good and then $e(\lambda, 2^k) = e(\lambda_i^{(k)}, 1) = 2$. Otherwise $e(\lambda, 2^k) = 0$. ■

Corollary 3.6. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$, and let $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_o n - 2^k$. Then $e(\lambda, 2^k) \neq 0$ if and only if there exists a partition $\lambda_i^{(k)}$ in the kth row of $\mathcal{Q}_2(\lambda)$ such that $|\lambda_i^{(k)}| \equiv 2^d - 1 \mod 2^{d+1}$, and $C_{2^d}(\lambda_i^{(k)})$ is a hook partition. In this case, $e(\lambda, 2^k) = 2^k$ if $d = 0$, and $e(\lambda, 2^k) = 2$ if $d > 0$.*

We are now ready to prove Theorem A. In fact, this is a consequence of Theorem 3.5 and it is stated here as the following corollary.

Corollary 3.7 (Theorem A)**.** *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$.*

- *If $k = 0$ then f_k^n is surjective if and only if $d(n, k) \leq 2$.*
- *If $k > 0$ then f_k^n is surjective if and only if $d(n, k) \leq 1$.*

Proof. By Theorem 3.5, f_k^n is surjective if and only if for all $\lambda \vdash_o n - 2^k$ we have that the kth row of $\mathcal{Q}_2(\lambda)$ contains a $d(n,k)$-good partition $\lambda_j^{(k)}$. By Theorem 2.5 and Definition 3.4, for any $\lambda \vdash_o n - 2^k$ we have $\sum_{j \geq 0} |\lambda_j^{(k)}| = \lfloor \frac{n-2^k}{2^k} \rfloor = (2^{d(n,k)} - 1) + m(n, k)$.

If $k = 0$ then $\mathcal{Q}_2^{(0)}(\lambda)$ contains only $\lambda = \lambda_0^{(0)}$. Hence f_0^n is surjective if and only all odd partitions of $n - 1$ are $d(n, 0)$-good. By Lemma 3.3(1), the latter condition holds when $d = d(n, 0) \leq 2$. On the other hand, if $d = \nu_2(n) > 2$, then $\lambda = (n - 5, 2, 2)$ is an odd partition of $n - 1$ by Theorem 2.5, but $C_8(\lambda) = (3, 2, 2)$ is not a hook, and hence $C_{2^d}(\lambda)$ is not a hook. So λ is not d-good, and thus f_0^n is not surjective.

Now assume $k \geq 1$. Then $\mathcal{Q}_2^{(k)}(\lambda)$ contains at least two odd partitions. If $d(n, k) \geq 2$ then any $d(n, k)$-good partition μ satisfies $3 \subseteq_2 2^{d(n,k)} - 1 \subseteq_2 |\mu|$. Write $\lfloor \frac{n-2^k}{2^k} \rfloor = 1 + m_1$ where m_1 is even. Applying Remark 2.7, take any $\lambda \vdash_o n - 2^k$ such that $|\lambda_0^{(k)}| = 1$ and $\lambda_1^{(k)}$ is an odd partition with $|\lambda_1^{(k)}| = m_1$. Then no partition in $\mathcal{Q}_2^{(k)}(\lambda)$ is $d(n, k)$-good. Thus f_k^n is not surjective. On the other hand, if $d(n, k) = 0$ then $2^k \subseteq_2 n$ and f_k^n is surjective [3, Proposition 4.5]. If $d(n, k) = 1$ then $\lfloor \frac{n-2^k}{2^k} \rfloor = 1 + m(n, k)$, where $4 \mid m(n, k)$. Thus any $\mathcal{Q}_2^{(k)}(\lambda)$ contains a partition with odd cardinality; this partition is 1-good, by Lemma 3.3. Again f_k^n is surjective. ■

It is an immediate consequence of Theorem 3.5 that f_k^n is regular on its image for all relevant choices of n, k such that $2^k < n$. We have:

Corollary 3.8. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ be such that $2^k < n$; set $d = \nu_2(\lfloor \frac{n}{2^k} \rfloor)$. Let $\lambda \vdash_o n - 2^k$. Then*

$$e(\lambda, 2^k) = \begin{cases} 2^k & \text{if } d = 0; \\ 2 & \text{if } d > 0, \text{ and the } k\text{th row of } \mathcal{Q}_2(\lambda) \text{ contains a } d\text{-good partition}; \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.9. For an illustration, we consider odd extensions of odd partitions by a 4-hook, i.e., we take $k = 2$ above. For $n > 2^2$ we first compute $d(n,k) = \nu_2\big(\lfloor \frac{n}{2^k} \rfloor\big)$, and then consider odd partitions of $n-4$ and their 4-extensions. For $n = 6$, $d(6,2) = 0$. Thus $e((2),4) = 4$. The odd 4-extensions of (2) are (6), (3^2), $(2^2,1^2)$, $(2,1^4)$. For $n = 10$, $d(10,2) = 1$. In this case, $e(\lambda,4) = 2$ for all odd partitions λ of 6. For instance, the odd 4-extensions of (6) are (10) and $(6,3,1)$. For $n = 19$, $d(19,2) = 2$. Example 2.6 shows that for $\lambda = (5,4,2^2,1^2) \vdash_o 15$ there is no 2-good partition in $\mathcal{Q}_2^{(2)}(\lambda)$, hence $e(\lambda,4) = 0$.

4 Deciding commutativity of the maps f_k and f_ℓ

Let $n \in \mathbb{N}$, and suppose that $0 \le k < \ell$ satisfy $2^k + 2^\ell \le n$. As stated in the introduction, we want to complete the discussion of the commutativity of the maps f_k and f_ℓ. Since the relevant n will always be apparent for the maps f_k^n in this section, we just write f_k.

We write $(n;k,\ell) \in \mathcal{T}$ if for all $\lambda \vdash_o n$ we have $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. Otherwise we write $(n;k,\ell) \in \mathcal{F}$.

In this section we will prove Theorem B, which may be reformulated as follows.

Theorem 4.1. *Let $n = 2^t + m$ where $0 \le m < 2^t$. Suppose that k, ℓ satisfy $0 \le k < \ell$ and $2^k + 2^\ell \le n$. Then with the exception of $(6;0,1)$*

$$(n;k,\ell) \in \mathcal{F} \text{ if and only if } \ell < t \text{ and } 2^k \le m.$$

The proof of Theorem 4.1 is based on a series of lemmas. The first lemmas concern two *extreme* cases, where f_k and f_ℓ commute.

In the case $\ell = t$ we have the following result as a reformulation of [3, Proposition 4.3].

Lemma 4.2. *Let $n = 2^t + m$ with $0 \le m < 2^t$. If $2^k \le m$, then $(n;k,t) \in \mathcal{T}$.*

It is also known that in the case where n is a power of 2, the maps f_k and f_ℓ commute [3, Remark 4.4], and we include a short proof here.

Lemma 4.3. *If $n = 2^t$ then $(n;k,\ell) \in \mathcal{T}$ for all k, ℓ.*

Proof. If $0 \le b \le a$ are integers then the binomial coefficient $\binom{a}{b}$ is odd if and only if $b \subseteq_2 a$, by Lucas' theorem. The odd partitions of 2^t are exactly the hook partitions $(2^t - b, 1^b)$, $0 \le b \le 2^t - 1$, of degree $\binom{2^t-1}{b}$. Hence for $k \in \{0,1,\dots,t-1\}$ we have

$$f_k(\lambda) = \begin{cases} (2^t - b - 2^k, 1^b) & \text{if } 2^k \not\subseteq_2 b, \\ (2^t - b, 1^{b-2^k}) & \text{if } 2^k \subseteq_2 b. \end{cases}$$

It follows that for any $k, \ell < t$ and odd partition λ of 2^t, we have $f_\ell f_k(\lambda) = f_k f_\ell(\lambda)$. ■

Lemma 4.4. *Let $n = 2^t + m$ with $0 \le m < 2^t$. Suppose that k, ℓ satisfy $0 \le k < \ell$ and $2^k + 2^\ell \le n$. If $m < 2^k$ then $(n;k,\ell) \in \mathcal{T}$.*

Proof. We use induction on $k \ge 0$. For $k = 0$ we have $m = 0$ and the claim follows from Lemma 4.3. Suppose that $k \ge 1$ and that the claim has been proved up to $k-1$. Let $\lambda \vdash_o n$. Odd hooks of length 2^k and 2^ℓ in λ correspond to odd hooks of length 2^{k-1} and $2^{\ell-1}$ in the 2-quotient $Q_2(\lambda) = (\lambda_0, \lambda_1)$ of λ. From Theorem 2.5 we deduce that $|\lambda_0|$ and $|\lambda_1|$ are 2-disjoint binary subsums of $\lfloor \frac{n}{2} \rfloor$, so one of them contains 2^{t-1}, say $|\lambda_0|$; then $|\lambda_1| \le \lfloor \frac{m}{2} \rfloor < 2^{k-1} < 2^{\ell-1}$. Thus the odd 2^{k-1}-hook in $Q_2(\lambda)$ has to be in λ_0. Therefore

$$Q_2(f_k(\lambda)) = (f_{k-1}(\lambda_0), \lambda_1).$$

Applying f_ℓ, the odd $2^{\ell-1}$-hook cannot be in λ_1, hence

$$Q_2(f_\ell f_k(\lambda)) = (f_{\ell-1}f_{k-1}(\lambda_0), \lambda_1)).$$

In particular, we know that $|\lambda_0| \geq 2^{\ell-1} + 2^{k-1}$. Also $|\lambda_0| + |\lambda_1| = \lfloor \frac{n}{2} \rfloor = 2^{t-1} + \lfloor \frac{m}{2} \rfloor$. We have already seen that 2^{t-1} is the largest binary digit of $|\lambda_0|$; furthermore $|\lambda_0| - 2^{t-1}$ is a binary subsum of $\lfloor \frac{m}{2} \rfloor < 2^{k-1}$. We may therefore apply the inductive hypothesis to λ_0 to get $f_{\ell-1}f_{k-1}(\lambda_0) = f_{k-1}f_{\ell-1}(\lambda_0)$. This implies that $Q_2(f_k f_\ell(\lambda)) = Q_2(f_\ell f_k(\lambda))$ and thus $f_k f_\ell(\lambda) = f_\ell f_k(\lambda)$. ∎

Lemmas 4.2 and 4.4 show that the *only if* part of the theorem is true. We now turn to the *if* part. We start by proving the statement for $k = 0$ and use this as part of an inductive argument.

Lemma 4.5. *Let $n = 2^t + m$ with $0 < m < 2^t$. If $0 < \ell < t$ then $(n; 0, \ell) \in \mathcal{F}$, with the exception of $(6; 0, 1)$.*

Proof. The result is easily checked for $n \leq 8$, which includes the exception $(6; 0, 1)$. So we assume that $t \geq 3$.

Case 1: $2^\ell < m$. Then $m \geq 3$, since $\ell > 0$. Consider the partition $\lambda = \big(m, m, 1^a\big) \vdash n$ where $a = n - 2m = 2^t - m$. The (1,1)-hook length of λ is $2^t + 1$. The (2,1)-hook length of λ is 2^t. Removing the (2,1)-hook hook we get the odd partition (m), so λ is odd, by Lemma 2.8. We claim that

$$f_0(\lambda) = \big(m, m, 1^{a-1}\big).$$

Indeed we cannot have $f_0(\lambda) = \big(m, m-1, 1^a\big)$ because this partition does not have a hook of length 2^t, and thus it is not odd. Now

$$f_\ell(f_0(\lambda)) = f_\ell\big(m, m, 1^{a-1}\big) = \big(m, m - 2^\ell, 1^{a-1}\big)$$

since $\big(m, m, 1^{a-1-2^\ell}\big)$ and $\big(m-1, m-2^\ell+1, 1^{a-1}\big)$ both do not have a hook of length 2^t and thus are not odd (again by Lemma 2.8).

On the other hand,

$$f_\ell(\lambda) = \big(m-1, m - \big(2^\ell - 1\big), 1^a\big).$$

Indeed, the other candidates for $f_\ell(\lambda)$, which are $\big(m, m-2^\ell, 1^a\big)$ and $\big(m, m, 1^{a-2^\ell}\big)$, do not have hooks of length 2^t. Then

$$f_0(f_\ell(\lambda)) = f_0\big(m-1, m-\big(2^\ell-1\big), 1^a\big) = \big(m-1, m-2^\ell, 1^a\big).$$

This follows (again) by observing that all the other partitions of $n - 2^\ell - 1$ obtained from $\big(m-1, m-\big(2^\ell-1\big), 1^a\big)$ by removing a node do not have hooks of length 2^t. Thus $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 2: $m < 2^\ell$. Consider the partition $\lambda = \big(n - 2^\ell, m+1, 1^a\big)$, where $a = 2^\ell - (m+1)$. Note that $n - 2^\ell \geq m + 1$ since $\ell < t$ by assumption, and that $a \geq 0$. The (1,1)-hook length of λ is $n - m = 2^t$. Removing this hook we get the odd partition (m), so λ is odd. The (2,1)-hook length of λ is 2^ℓ. Now

$$f_0(\lambda) = \big(n - 2^\ell, m, 1^a\big)$$

since the other candidates do not have hooks of length 2^t. Then

$$f_\ell(f_0(\lambda)) = f_\ell\big(n-2^\ell, m, 1^a\big) = \mu,$$

where μ is obtained from $f_0(\lambda)$ by removing a 2^ℓ-hook in the first row. (There are only hooks of length $< 2^\ell$ in the other rows.) In fact, $\mu = \big(n - 2^{\ell+1}, m, 1^a\big)$ since $n - 2^{\ell+1} \geq n - 2^t = m$. Thus $f_\ell(f_0(\lambda))$ has at least 2 parts. On the other hand

$$f_\ell(\lambda) = \big(n - 2^\ell\big)$$

since this odd partition is obtained from the odd partition λ by removing a 2^ℓ-hook (the one in $(2,1)$). It follows that

$$f_0(f_\ell(\lambda)) = \big(n - 2^\ell - 1\big)$$

and again $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

Case 3: $m = 2^\ell$. Then $n = 2^t + 2^\ell$. If $\ell \geq 2$ then choose $\lambda = \big(2^t, 2^\ell - 1, 1\big)$. The $(1,2)$-hook length of λ is 2^t; thus λ is an odd partition since removing this 2^t-hook gives an odd partition $\big(2^\ell - 2, 1, 1\big)$ of 2^ℓ. We have $f_0(\lambda) = \big(2^t, 2^\ell - 2, 1\big)$ since the other candidates are not odd. Then

$$f_\ell(f_0(\lambda)) = \big(2^t - 2^\ell, 2^\ell - 2, 1\big).$$

The $(2,1)$-hook length of λ is 2^ℓ, so $f_\ell(\lambda) = \big(2^t\big)$ and

$$f_0(f_\ell(\lambda)) = \big(2^t - 1\big),$$

showing $f_0(f_\ell(\lambda)) \neq f_\ell(f_0(\lambda))$.

On the other hand, if $\ell = 1$ then choose $\lambda = \big(2^t - 2, 2, 2\big) \vdash_o 2^t + 2 = n$. Since $t \geq 3$, it is now easy to show that $f_1(f_0(\lambda)) = \big(2^t - 4, 2, 1\big)$. On the other hand we see that $f_0(f_1(\lambda))$ is a hook partition of $2^t - 1 = n - 3$ and therefore is not equal to $f_1(f_0(\lambda))$. ■

Lemma 4.6. *If $(n; k, \ell) \in \mathcal{F}$ then also $(2n; k+1, \ell+1) \in \mathcal{F}$ and $(2n+1; k+1, \ell+1) \in \mathcal{F}$.*

Proof. Let the odd partition μ of n satisfy $f_k f_\ell(\mu) \neq f_\ell f_k(\mu)$. Let λ be a partition of $2n$ or $2n+1$ having 2-quotient $Q_2(\lambda) = (\mu, (0))$. Then λ is odd, by Theorem 2.5. We have

$$Q_2(f_{k+1}f_{\ell+1}(\lambda)) = (f_k f_\ell(\mu), (0)) \neq (f_\ell f_k(\mu), (0)) = Q_2(f_{\ell+1}f_{k+1}(\lambda)),$$

so that $f_{k+1}f_{\ell+1}(\lambda) \neq f_{\ell+1}f_{k+1}(\lambda)$. ■

We are now ready to conclude this section with the proof of Theorem B.

Proof of Theorem 4.1. The *only if* part follows from Lemmas 4.2 and 4.4. To prove the *if* part we use induction on $k \geq 0$. If $k = 0$, then the statement follows from Lemma 4.5. Let $k > 1$ and suppose that the assertion is true up to and including $k - 1$. To show that $(n; k, \ell) \in \mathcal{F}$ it suffices to prove $(\lfloor \frac{n}{2} \rfloor; k-1, \ell-1) \in \mathcal{F}$, by Lemma 4.6. We are assuming $n = 2^t + m$, $0 \leq m < 2^t$, $0 \leq k < \ell \leq t$ and $2^k + 2^\ell \leq n$. This implies $\lfloor \frac{n}{2} \rfloor = 2^{t-1} + \lfloor \frac{m}{2} \rfloor$, $0 \leq \lfloor \frac{m}{2} \rfloor < 2^{t-1}$ and $2^{k-1} + 2^{\ell-1} \leq \lfloor \frac{n}{2} \rfloor$. We may apply the inductive hypothesis to get $\big(\lfloor \frac{n}{2} \rfloor; k-1, \ell-1\big) \in \mathcal{F}$, and then $(n; k, \ell) \in \mathcal{F}$ except when $\big(\lfloor \frac{n}{2} \rfloor; k-1, \ell-1\big) = (6; 0, 1)$. In that case we are considering (12;1,2) or (13;1,2) which are both in $\mathcal{F}$, by direct computation (consider for example $(6,4,2) \vdash_o 12$ and $(6,4,3) \vdash_o 13$, respectively). ■

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