Classical and Quantum Superintegrability of Stäckel Systems

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Abstract. In this paper we discuss maximal superintegrability of both classical and quantum Stäckel systems. We prove a sufficient condition for a flat or constant curvature Stäckel system to be maximally superintegrable. Further, we prove a sufficient condition for a Stäckel transform to preserve maximal superintegrability and we apply this condition to our class of Stäckel systems, which yields new maximally superintegrable systems as conformal deformations of the original systems. Further, we demonstrate how to perform the procedure of minimal quantization to considered systems in order to produce quantum superintegrable and quantum separable systems.

Key words: Hamiltonian systems; classical and quantum superintegrable systems; Stäckel systems; Hamilton–Jacobi theory; Stäckel transform

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1 Introduction

A real-valued function h_1 on a $2n$-dimensional manifold (phase space) $M = T^*Q$ is called a classical maximally superintegrable Hamiltonian if it belongs to a set of n Poisson-commuting functions $h_1, \ldots, h_n$ (constants of motion, so that $\{h_i, h_j\} = 0$ for all $i, j = 1, \ldots, n$) and for which there exist $n-1$ additional functions $h_{n+1}, \ldots, h_{2n-1}$ on M that Poisson-commute with the Hamiltonian h_1 and such that all the functions $h_1, \ldots, h_{2n-1}$ constitute a functionally independent set of functions. Analogously, a quantum maximally superintegrable Hamiltonian is a self-adjoint differential operator $\hat{h}_1$ acting in an appropriate Hilbert space of functions on the configuration space Q (square integrable with respect to some metric) belonging to a set of n commuting self-adjoint differential operators $\hat{h}_1, \ldots, \hat{h}_n$ acting in the same Hilbert space (so that $[\hat{h}_i, \hat{h}_j] = 0$ for all $i, j = 1, \ldots, n$) and such that it also commutes with an additional set of $n-1$ differential operators $\hat{h}_{n+1}, \ldots, \hat{h}_{2n-1}$ of finite order. Besides, in analogy with the classical case, it is required that all the operators $\hat{h}_1, \ldots, \hat{h}_{2n-1}$ are algebraically independent [19]. Throughout the paper it is tacitly assumed that $n > 1$ as the case $n = 1$ is not interesting from the point of view of our theory.

This paper is devoted to n-dimensional maximally superintegrable classical and quantum Stäckel systems with all constants of motion quadratic in momenta. Although superintegrable systems of second order, both classical and quantum, have been intensively studied (see for example [1, 2, 11, 14, 16, 17] and the review paper [19]), nevertheless all the results about superintegrable Stäckel systems (including the important classification results) were mainly restricted to two or three dimensions or focused on the situation when the Hamiltonian is a sum of one degree of freedom terms and therefore itself separates in the original coordinate

system (see for example [3, 12] or [15]). Here we present some general results concerning n-dimensional classical separable superintegrable systems in flat spaces, constant curvature spaces and conformally flat spaces. We also present how to separately quantize all considered classical systems. We stress, however, that we do not develop spectral theory of the obtained quantum systems, as it requires a separate investigation.

The paper is organized as follows. In Section 2 we briefly describe – following previous references, for example [18] and [5] – flat and constant curvature Stäckel systems that we consider in this paper. In Section 3 we prove (Theorem 3.3) a sufficient condition for this class of Stäckel system to be maximally superintegrable by finding a linear in momenta function $P = \sum_{s=1}^{n} y^s p_s$ on M such that $\{h_1, P\} = c$ (it also means that the vector field $Y = \sum_{s=1}^{n} y^s \frac{\partial}{\partial q_s}$ in P is a Killing vector for the metric generated by h_1) which yields additional $n-1$ functions $h_{n+i} = \{h_{i+1}, P\}$ commuting with h_1 and thus turning h_1 into a maximally superintegrable Hamiltonian. In Section 4 we briefly remind the notion of Stäckel transform (a functional transform that preserves integrability) and prove (Theorem 4.2) conditions that guarantee that a Stäckel transform transforms maximally superintegrable system into another maximally superintegrable system (i.e., preserves maximal superintegrability). In Section 5 we apply this result to our class of maximally superintegrable Stäckel systems, obtaining Theorem 5.2 stating when the Stäckel transform applied to the considered class of systems yields a Stäckel system that is flat, of constant curvature or conformally flat. We also demonstrate (Theorem 5.4) that the additional integrals h_{n+i} of systems after Stäckel transform can be obtained in two equivalent ways. Section 6 is devoted to the procedure of minimal quantization of considered Stäckel systems. As the procedure of minimal quantization depends on the choice of the metric on the configurational space, we remind first the result obtained in [5] explaining how to choose the metric in which a minimal quantization is performed so that the integrability of the quantized system is preserved (Theorem 6.1) and then apply Lemma 6.3 to obtain Corollary 6.4 stating under which conditions the procedure of minimal quantization of a classical Stäckel system, considered in previous sections, yields a quantum superintegrable and quantum separable system. The paper is furnished with several examples that continue throughout sections. The examples are all 3-dimensional in order to make the formulas readable but our theory works in arbitrary dimension.

2   A class of flat and constant curvature Stäckel systems

Let us first introduce the class of Hamiltonian systems that we will consider in this paper. Consider a $2n$-dimensional manifold $M = T^*Q$ (we remind the reader that $n > 1$) equipped with a set of (smooth) coordinates $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ defined on an open dense set of M and such that λ are the coordinates on the base manifold Q while μ are fibre coordinates. Define the bivector

$$\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}. \tag{2.1}$$

Then the bivector Π satisfies the Jacobi identity so it becomes a Poisson operator (Poisson tensor), our manifold M becomes Poisson manifold and the coordinates (λ, μ) become Darboux (canonical) coordinates for the Poisson tensor (2.1). Consider also a set of n algebraic equations on M

$$\sigma(\lambda_i) + \sum_{j=1}^{n} h_j \lambda_i^{\gamma_j} = \frac{1}{2} f(\lambda_i) \mu_i^2, \qquad i = 1, \ldots, n, \qquad \gamma_i \in \mathbf{N}, \tag{2.2}$$

where we normalize $\gamma_n = 0$ and where σ and f are arbitrary functions of one variable. The relations (2.2) constitute a system of n equations linear in the unknowns h_j. Solving these equations with respect to h_j we obtain n functions $h_j = h_j(\lambda, \mu)$ on M of the form

$$h_j = \frac{1}{2}\mu^T A_j(\lambda)\mu + U_j(\lambda), \qquad j = 1, \ldots, n, \tag{2.3}$$

where we denote $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ and $\mu = (\mu_1, \ldots, \mu_n)^T$. The functions h_j can be interpreted as n quadratic in momenta μ Hamiltonians on the manifold $M = T^*Q$ while the $n \times n$ symmetric matrices $A_j(\lambda)$ can be interpreted as n twice contravariant symmetric tensors on Q. The Hamiltonians h_j commute with respect to Π

$$\{h_i, h_j\} \equiv \Pi(dh_i, dh_j) = 0 \qquad \text{for all} \quad i, j = 1, \ldots, n,$$

since the right-hand sides of relations (2.2) commute. Thus, the Hamiltonians in (2.3) constitute a Liouville integrable Hamiltonian system (as they are moreover functionally independent). The Hamiltonians (2.3) constitute a wide class of the so called Stäckel systems [24] on M while the relations (2.2) are called separation relations [23] of this system. This is the class we will consider throughout our paper. Note that by the very construction of h_i the variables (λ, μ) are separation variables for all the Hamiltonians in (2.3) in the sense that the Hamilton–Jacobi equations associated with h_j admit a common additively separable solution.

Let us now treat the matrix A_1 as a contravariant form of a metric tensor on Q: $A_1 = G$, which turns Q into a Riemannian space. The covariant form of G will be denoted by g (so that $g = G^{-1}$). It turns out that the $(1,1)$-tensors K_j defined by

$$K_j = A_j g, \qquad j = 1, \ldots, n \tag{2.4}$$

(so that $A_j = K_j G$ and $K_1 = I$) are Killing tensors of the metric g.

In this article we will focus on a particular subclass of systems (2.2) that is given by the separation relations

$$\sigma(\lambda_i) + \sum_{j=1}^{n} h_j \lambda_i^{n-j} = \frac{1}{2}f(\lambda_i)\mu_i^2, \qquad i = 1, \ldots, n, \tag{2.5}$$

(systems of the above class are known in literature as *Benenti systems*) where moreover

$$f(\lambda) = \sum_{j=0}^{m} b_j \lambda^j, \qquad b_j \in \mathbf{R}, \quad m \in \{0, \ldots, n+1\}, \tag{2.6}$$

$$\sigma(\lambda) = \sum_{k \in I} \alpha_k \lambda^k, \qquad \alpha_k \in \mathbf{R}, \tag{2.7}$$

where $I \subset \mathbf{Z}$ is some finite index set (i.e., σ is a Laurent polynomial). Note that taking $k \in \{0, \ldots, n-1\}$ will only yield trivial terms in solutions (2.3) of (2.5), see the end of this section. Also, the parameters α_k will play a crucial roll in the sequel, when we discuss the Stäckel transform of the above systems. The metric tensor G attains in this case, due to (2.6), the form

$$G = \sum_{j=0}^{m} b_j G_j = \sum_{j=0}^{m} b_j L^j G_0, \tag{2.8}$$

where $L = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is a $(1,1)$-tensor (the so called special conformal Killing tensor, see for example [10]) on Q, while

$$G_j = \text{diag}\left(\frac{\lambda_1^j}{\Delta_1}, \ldots, \frac{\lambda_n^j}{\Delta_n}\right), \qquad j \in \mathbf{Z}, \qquad \Delta_i = \prod_{j \neq i}(\lambda_i - \lambda_j). \tag{2.9}$$

Remark 2.1. The metric (2.8) is flat for $m \leq n$ and of constant curvature for $m = n + 1$ (see for example [9, p. 788]). For higher m it would have a non-constant curvature.

Further, the Killing tensors K_i in (2.4) are in this case given by

$$K_i = \sum_{r=0}^{i-1} q_r L^{i-1-r} = -\operatorname{diag}\left(\frac{\partial q_i}{\partial \lambda_1}, \ldots, \frac{\partial q_i}{\partial \lambda_n}\right), \qquad i = 1, \ldots, n. \tag{2.10}$$

Here and below $q_i = q_i(\lambda)$ are Viète polynomials in the variables $\lambda_1, \ldots, \lambda_n$:

$$q_i(\lambda) = (-1)^i \sum_{1 \leq s_1 < s_2 < \cdots < s_i \leq n} \lambda_{s_1} \cdots \lambda_{s_i}, \qquad i = 1, \ldots, n, \tag{2.11}$$

that can also be considered as new coordinates on our Riemannian manifold Q (we will call them Viète coordinates on Q). Notice that q_i are coefficients of the characteristic polynomial of the tensor L. Notice also that the first form of K_i in (2.10) is of course valid in any coordinate system while the second form of K_i is valid in separation coordinates λ only.

Further, due to (2.7), the potentials $U_j(\lambda)$ in (2.3) are for the subclass (2.5) given by

$$U_j = \sum_{k \in I} \alpha_k V_j^{(k)}, \qquad j = 1, \ldots, n, \tag{2.12}$$

where the "basic" potentials V_i^k ($k \in \mathbf{Z}$) satisfy the linear system

$$\lambda_i^k + \sum_{j=1}^{n} V_j^{(k)} \lambda_i^{n-j} = 0, \qquad i = 1, \ldots, n, \qquad k \in \mathbf{Z},$$

and can be computed by the recursive formula [4, 8]

$$V^{(k)} = F^k V^{(0)}, \qquad k \in \mathbf{Z}, \tag{2.13}$$

where $V^{(k)} = \left(V_1^{(k)}, \ldots, V_n^{(k)}\right)^T$, $V^{(0)} = (0, 0, \ldots, 0, -1)^T$ and where F is an $n \times n$ matrix given by

$$F = \begin{pmatrix} -q_1(\lambda) & 1 & & \\ -q_2(\lambda) & & \ddots & \\ \vdots & & & 1 \\ -q_n(\lambda) & 0 & \cdots & 0 \end{pmatrix} \tag{2.14}$$

with $q_i(\lambda)$ given by (2.11). Note that the formulas (2.13), (2.14) are non tensorial in that they are the same in an arbitrary coordinate system, not only in the separation variables λ_i. As we mentioned above, the first potentials, i.e., $V^{(1)} = (0, 0, \ldots, 0, -1, 0)^T$ up to $V^{(n-1)} = (-1, 0, \ldots, 0)^T$ are constant, $V^{(n)} = (q_1, \ldots, q_n)$ is the first nonconstant positive potential while $V^{(-1)} = (1/q_n, q_1/q_n, \ldots, q_{n-1}/q_n)^T$. The potentials $V^{(k)}$ are for $k < 0$ rational functions of q that quickly become complicated with decreasing k.

To summarize, the Hamiltonians h_i generated by (2.5)–(2.7) can be explicitly written as

$$h_r(\lambda) = -\frac{1}{2} \sum_{i=0}^{n} \frac{\partial q_r}{\partial \lambda_i} \frac{f(\lambda_i)\mu_i^2 - \sigma(\lambda_i)}{\Delta_i} = -\frac{1}{2} \sum_{i=0}^{n} \frac{\partial q_r}{\partial \lambda_i} \frac{f(\lambda_i)\mu_i^2}{\Delta_i} + U_r(\lambda), \qquad r = 1, \ldots, n.$$

3   Maximally superintegrable flat and constant curvature Stäckel systems

Suppose that we have an integrable system, i.e., n functionally independent Hamiltonians on a $2n$-dimensional phase space M that pairwise commute: $\{h_i, h_j\} = 0$ for all $i, j = 1, \ldots, n$. If there exists an additional function P commuting to a constant with one of the Hamiltonians, say with h_1 (so that $\{h_1, P\} = c$) and if the $n - 1$ functions

$$h_{n+i} = \{h_{i+1}, P\}, \qquad i = 1, \ldots, n - 1$$

together with all h_i are functionally independent, then the system becomes maximally superintegrable (with respect to this particular Hamiltonian h_1) since then by the Jacobi identity

$$\{h_{n+i}, h_1\} = -\{\{P, h_1\}, h_{i+1}\} - \{\{h_1, h_{i+1}\}, P\} = 0, \qquad i = 1, \ldots, n - 1.$$

If moreover the first n integrals of motion h_i are quadratic in momenta and if P is linear in momenta, then the resulting $n - 1$ extra integrals of motion h_{n+i} are also quadratic in momenta. Thus, in order to distinguish those constant curvature Stäckel systems that are maximally superintegrable and have quadratic in momenta extra integrals of motion we have to find P that commutes with h_1 up to a constant and that is linear in momenta. To do it in a systematic way, we need the following well-known result.

Lemma 3.1. *Suppose that $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ are Darboux (canonical) coordinates on a $2n$-dimensional phase space $M = T^*Q$. Consider two functions on M:*

$$h = \frac{1}{2} \sum_{i,j=1}^{n} p_i A^{ij}(q) p_j + U(q) \qquad with \quad A = A^T \qquad and \qquad P = \sum_{i=1}^{n} y^i(q) p_i.$$

Then

$$\{h, P\} = \frac{1}{2} \sum_{i,j=1}^{n} p_i (L_Y A)^{ij} p_j + Y(U),$$

where Y is the vector field on Q given by $Y = \sum\limits_{i=1}^{n} y^i(q) \frac{\partial}{\partial q_i}$ and where L_Y is the Lie derivative (on Q) along Y.

One can thus say that h and P commute if the corresponding vector field Y is the Killing vector for the metric defined by the $(2,0)$-tensor A (i.e., if $L_Y A = 0$) and if moreover Y is symmetry of U (i.e., if $Y(U) = 0$).

Consider now the Stäckel system given by (2.5). The coordinates (λ, μ) are Darboux (so that the above lemma applies to this situation) but the components of the metric (2.8) expressed in λ-coordinates are rational functions making computations very complicated. We will therefore perform the search for the function P in the coordinates (q, p) on M such that q_i are Viète coordinates (2.11) and such that

$$p_i = -\sum_{k=1}^{n} \frac{(\lambda_k)^{n-i} \mu_k}{\Delta_k} \tag{3.1}$$

are the conjugated momenta. Since the transformation from (λ, μ) to (q, p) is a point transformation the coordinates (q, p) are also Darboux coordinates four our Poisson tensor. It can be shown [7] that in the (q, p)-coordinates

$$(L)_j^i = -\delta_j^1 q_i + \delta_j^{i+1}, \qquad (G_0)^{ij} = \sum_{k=0}^{n-1} q_k \delta_{n+k+1}^{i+j}, \tag{3.2}$$

and moreover

$$(G_r)^{ij} = \begin{cases} \sum_{k=0}^{n-r-1} q_k \delta_{n-r+k+1}^{i+j}, & i,j = 1,\ldots,n-r, \\ -\sum_{k=n-r+1}^{n} q_k \delta_{n-r+k+1}^{i+j}, & i,j = n-r+1,\ldots,n, \\ 0 & \text{otherwise}, \end{cases} \quad r = 1,\ldots,n, \tag{3.3}$$

$$(G_{n+1})^{ij} = q_i q_j - q_{i+j}, \qquad i,j = 1,\ldots,n,$$

where we set $q_0 \equiv 1$ and $q_r = 0$ for $r > n$. An advantage of these new coordinates is that the geodesic parts of h_i are polynomial in q.

Example 3.2. For $n = 3$ and in Viète coordinates (2.11) we have

$$L = \begin{pmatrix} -q_1 & 1 & 0 \\ -q_2 & 0 & 1 \\ -q_3 & 0 & 0 \end{pmatrix}, \qquad G_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q_1 \\ 1 & q_1 & q_2 \end{pmatrix}, \tag{3.4}$$

and hence the metric tensors G_j have the form

$$G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & q_1 & 0 \\ 0 & 0 & -q_3 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -q_2 & -q_3 \\ 0 & -q_3 & 0 \end{pmatrix}, \tag{3.5}$$

$$G_3 = \begin{pmatrix} -q_1 & -q_2 & -q_3 \\ -q_2 & -q_3 & 0 \\ -q_3 & 0 & 0 \end{pmatrix}, \qquad G_4 = \begin{pmatrix} q_1^2 - q_2 & q_1 q_2 - q_3 & q_1 q_3 \\ q_1 q_2 - q_3 & q_2^2 & q_2 q_3 \\ q_1 q_3 & q_2 q_3 & q_3^2 \end{pmatrix}. \tag{3.6}$$

In accordance with Remark 2.1, the metric tensors $G_0, \ldots, G_3$ are flat, while the metric G_4 is of constant curvature.

We are now in position to perform our search for P. We do this in the case when σ is the Laurent polynomial (2.7) and allow f to be polynomial as in (2.6); a particular case of $f = \lambda^m$ of the theorem below was formulated in [6].

Theorem 3.3. *The Stäckel system*

$$\sum_{k \in I} \alpha_k \lambda_i^k + \sum_{j=1}^{n} h_j \lambda_i^{n-j} = \frac{1}{2} f(\lambda_i) \mu_i^2, \qquad i = 1,\ldots,n,$$

(where $I \subset \mathbf{Z}$ is a finite index set) with $f(\lambda_i)$ given by

$$f(\lambda) = \sum_{j=0}^{m} b_j \lambda^j, \qquad b_j \in \mathbf{R}, \qquad m \in \{0,\ldots,n+1\} \tag{3.7}$$

is maximally superintegrable in the following cases:

(i) *case $m \in \{0,\ldots,n-1\}$: if $I \subset \{n,\ldots,2n-m-1\} \cup \{-1,\ldots,-r-1\}$, where r is such that $b_i = 0$ for $i = 0,\ldots,r \leq m-1$ (if all $b_i \neq 0$, then there is no such r and no second component in I);*

(ii) *case $m = n$ and $b_0 = b_1 = 0$: if $I \subset \{n,-1,\ldots,-r+1\}$, where r is such that $b_i = 0$ for $i = 2,\ldots,r \leq n-1$;*

(iii) *case $m = n+1$ (case of constant curvature) and $b_0 = b_1 = 0$: if $I \subset \{-1,\ldots,-r+1\}$, where r is such that $b_i = 0$ for $i = 2,\ldots,r \leq n$.*

The additional integrals h_{n+r} commuting with h_1 are given in (q,p)-coordinates by

$$h_{n+r} = \frac{1}{2} \sum_{i,j=1}^{n} p_i (L_Y A_{r+1})^{ij} p_j + Y(U_{r+1}), \qquad r = 1, \ldots, n-1, \tag{3.8}$$

where Y is a vector field on Q given by

(i) for $m \in \{0, \ldots, n-1\}$

$$Y = \sum_{i=0}^{m} b_{m-i} \frac{\partial}{\partial q_{n-m+i}}, \tag{3.9}$$

(ii) for $m = n$

$$Y = q_n \sum_{i=2}^{n} b_{n-i+2} \frac{\partial}{\partial q_i}, \tag{3.10}$$

(iii) for $m = n+1$

$$Y = q_n \sum_{i=1}^{n} b_{n-i+2} \frac{\partial}{\partial q_i}, \tag{3.11}$$

and where L_Y denotes the Lie derivative along Y.

Proof. We will search for a function P that commutes with h_1 and we will perform this search in the (q,p)-coordinates (2.11), (3.1). The Hamiltonian h_1 has in these coordinates the form

$$h_1 = \frac{1}{2} \sum_{i,j=1}^{n} G^{ij}(q) p_i p_j + V_1^{(k)}(q)$$

with G given by (2.8) and further by (3.2), (3.3) and with the potential $V_1^{(k)}(q)$ defined by (2.13) and (2.14).

(i) For $m = 0, \ldots, n-1$, the Killing equation $L_Y G = 0$ has a unique (up to a multiplicative constant) constant solution (3.9) which also satisfies $Y\big(V_1^{(k)}\big) = 0$ for $k = n, \ldots, 2n-m-2$ and $Y\big(V_1^{(2n-m-1)}\big) = c$. In consequence, due to Lemma 3.1, the function

$$P = b_m p_{n-m} + b_{m-1} p_{n-m+1} + \cdots + b_0 p_n$$

satisfies

$$\{h_1, P\} = 0 \tag{3.12}$$

for $k = n, \ldots, 2n-m-2$ and

$$\{h_1, P\} = c$$

for $k = 2n-m-1$. Moreover, if $b_i = 0$ for $i = 0, \ldots, r \leq m-1$, then (3.12) is satisfied also for $k = -1, \ldots, -r-1$.

(ii) For $m = n$, there is no constant solution of $L_Y G = 0$. This equation has a simple linear in q solution (3.10) provided that $b_0 = b_1 = 0$; Y is then also a symmetry for the single nontrivial potential $V_1^{(n)}$, i.e., $Y\big(V_1^{(n)}\big) = 0$. In consequence, the function

$$P = q_n (b_n p_2 + b_{n-1} p_3 + \cdots + b_2 p_n)$$

satisfies (3.12). Moreover, if $b_i = 0$ for $i = 2, \ldots, r \leq n - 1$ then (3.12) is satisfied also for $k = -1, \ldots, -r + 1$.

(iii) For $m = n + 1$ there is no constant solution of $L_Y G = 0$. This equation has a simple linear in q solution (3.11) provided that $b_0 = b_1 = 0$ but Y is not a symmetry for any nontrivial potential $V_1^{(k)}$. In consequence, the function

$$P = q_n(b_{n+1}p_1 + b_n p_2 + \cdots + b_2 p_n).$$

Poisson commutes only with the geodesic part E_1 of h_1: $\{E_1, P\} = 0$. However, if $b_i = 0$ for $i = 2, \ldots, r \leq n$ then (3.12) is satisfied for $k = -1, \ldots, -r + 1$.

Finally, the form of additional integrals h_{n+r} in (3.8) is obtained through $h_{n+r} = \{h_{r+1}, P\}$ by using Lemma 3.1. Due to their form, the functions $h_1, \ldots, h_{2n-1}$ are functionally independent. ∎

Remark 3.4. The above theorem provides us with a sufficient condition for maximal superintegrability of Stäckel systems of constant curvature (flat in particular) in case when $f(\lambda)$ is a polynomial of maximal order $n + 1$. In consequence, the case (i) of Theorem 3.3 yields an $(n + 1)$-parameter family of maximally superintegrable systems, parametrized by

$$\{b_r, \ldots, b_m, \alpha_{-r-1}, \ldots, \alpha_{-1}, \alpha_n, \ldots, \alpha_{2n-m-1}\}, \qquad r = 0, \ldots, m,$$

where b_j parametrize superintegrable metrics (2.6), (2.8) and α_j parametrize families of nontrivial superintegrable potentials U (2.12) (in case there is no r, i.e., all $b_i \neq 0$) then there is no α_{-j} in the above set. Similarly, in the cases (ii) and (iii) Theorem 3.3 yields appropriate n-parameter families of superintegrable systems. A particular case of that classification (for the monomial case $f(\lambda) = \lambda^m$) was presented in [21].

It is possible to calculate explicitly the structure of the geodesic parts E_{n+r} of the extra integrals h_{n+r} in the separation coordinates (λ, μ).

Proposition 3.5. *The geodesic parts*

$$E_{n+r} = \frac{1}{2} \sum_{i,j=1}^{n} \mu_i A_{n+r}^{ij}(\lambda) \mu_j, \qquad r = 1, \ldots, n - 1$$

of additional integrals of motion $h_{n+r} = \{h_{r+1}, P\}$ with $r = 1, \ldots, n - 1$ are given by

(i) *for $0 \leq m \leq n - 1$*

$$A_{n+r}^{ij} = -\frac{\partial^2 q_r}{\partial \lambda_i \partial \lambda_j} \frac{f(\lambda_i) f(\lambda_j)}{\Delta_i \Delta_j}, \qquad i \neq j,$$

$$A_{n+r}^{ii} = \frac{f(\lambda_i)}{\Delta_i} \sum_{j=1}^{n} \frac{\partial^2 q_r}{\partial \lambda_i \partial \lambda_j} \frac{f(\lambda_j)}{\Delta_j},$$

where $q_r = q_r(\lambda)$ are given by (2.11), $f(\lambda)$ are given by (2.6) while Δ_i by (2.9),

(ii) *for $m = n, n + 1$*

$$A_{n+r}^{ij} = -\frac{\partial^2 q_r}{\partial \lambda_i \partial \lambda_j} \frac{\partial q_n}{\partial \lambda_i} \frac{1}{\lambda_j} \frac{f(\lambda_i) f(\lambda_j)}{\Delta_i \Delta_j}, \qquad i \neq j,$$

$$A_{n+r}^{ii} = \frac{f(\lambda_i)}{\Delta_i} \sum_{j=1}^{n} \frac{\partial^2 q_r}{\partial \lambda_i \partial \lambda_j} \frac{f(\lambda_j)}{\Delta_j} \frac{\partial q_n}{\partial \lambda_j} \frac{1}{\lambda_j}.$$

Let us illustrate the above considerations by some examples.

Example 3.6. Consider the flat case $n = 3$, $m = 1$, $b_1 = 1$, (so that $f(\lambda) = b_0 + \lambda$) with $\sigma(\lambda) = \alpha\lambda^k$ and where $k = -1, 3$ or 4. The commuting Hamiltonians h_i are given by separation relations (2.5)

$$\alpha\lambda_i^k + h_1\lambda_i^2 + h_2\lambda_i + h_3 = \frac{1}{2}(b_0 + \lambda_i)\mu_i^2, \qquad i = 1, 2, 3.$$

Then, according to (2.10), (3.4), (3.6) and to (2.13), (2.14) the corresponding Stäckel Hamiltonians attain in the (q, p) coordinates (2.11), (3.1) the form

$$h_1 = p_1 p_2 + b_0 p_1 p_3 + b_0 q_1 p_2 p_3 + \frac{1}{2}(q_1 + b_0)p_2^2 + \frac{1}{2}(b_0 q_2 - q_3)p_3^2 + \alpha V_1^{(k)}(q),$$

$$h_2 = \frac{1}{2}p_1^2 + \frac{1}{2}(q_1^2 + 2b_0 q_1 - q_2)p_2^2 + \frac{1}{2}(b_0 q_1 q_2 - q_1 q_3 - b_0 q_3)p_3^2 + (q_1 + b_0)p_1 p_2$$
$$\quad + b_0 q_1 p_1 p_3 + (b_0 q_1^2 - q_3)p_2 p_3 + \alpha V_2^{(k)}(q),$$

$$h_3 = \frac{1}{2}b_0 p_1^2 + \frac{1}{2}(b_0 q_1^2 - q_3)p_2^2 + \frac{1}{2}(-b_0 q_1 q_3 + b_0 q_2^2 - q_2 q_3)p_3^2 + b_0 q_1 p_1 p_2$$
$$\quad + (b_0 q_2 - q_3)p_1 p_3 + (b_0 q_1 q_2 - q_1 q_3 - b_0 q_3)p_2 p_3 + \alpha V_3^{(k)}(q),$$

where

$$V_1^{(-1)} = \frac{1}{q_3}, \qquad V_2^{(-1)} = \frac{q_1}{q_3}, \qquad V_3^{(-1)} = \frac{q_2}{q_3},$$

$$V_1^{(3)} = q_1, \qquad V_2^{(3)} = q_2, \qquad V_3^{(3)} = q_3,$$

$$V_1^{(4)} = -q_1^2 + q_2, \qquad V_2^{(4)} = -q_1 q_2 + q_3, \qquad V_3^{(4)} = -q_1 q_3.$$

According to Theorem 3.3 $Y = \frac{\partial}{\partial q_2} + b_0 \frac{\partial}{\partial q_3}$ so that $P = p_2 + b_0 p_3$ and thus

$$\{h_1, P\} = \begin{cases} 0 & \text{for } k = -1 \text{ and } b_0 = 0, \\ 0 & \text{for } k = 3, \\ \alpha & \text{for } k = 4 \ \left(\text{then } Y\left(V_1^{(4)}\right) = 1\right). \end{cases}$$

Hence, the system is maximally superintegrable with additional constants of motion for h_1 given by:
for $k = -1$ and $b_0 = 0$

$$h_4 = \{h_2, P\} = -\frac{1}{2}p_2^2, \qquad h_5 = \{h_3, P\} = -\frac{1}{2}q_3 p_3^2 + \alpha\frac{q_3}{q_3^2},$$

for $k = 3$

$$h_4 = \{h_2, P\} = -\frac{1}{2}p_2^2 - \frac{1}{2}b_0^2 p_3^2 - b_0 p_2 p_3 + \alpha,$$

$$h_5 = \{h_3, P\} = -\frac{1}{2}b_0 p_2^2 + \left(\frac{1}{2}b_0 q_2 - \frac{1}{2}q_3 - \frac{1}{2}b_0^2 q_1\right)p_3^2 - b_0^2 p_2 p_3 + \alpha b_0,$$

and for $k = 4$

$$h_4 = \{h_2, P\} = -\frac{1}{2}p_2^2 - \frac{1}{2}b_0^2 p_3^2 - b_0 p_2 p_3 + \alpha(b_0 - q_1),$$

$$h_5 = \{h_3, P\} = -\frac{1}{2}b_0 p_2^2 + \frac{1}{2}(b_0 q_2 - q_3 - b_0^2 q_1)p_3^2 - b_0^2 p_2 p_3 - \alpha b_0 q_1.$$

Example 3.7. Consider the case $n = 3$, $m = 1$, with the monomial $f(\lambda) = \lambda$, given by the separation relations

$$\alpha_4\lambda_i^4 + \alpha_3\lambda_i^3 + h_1\lambda_i^2 + h_2\lambda_i + h_3 + \alpha_{-1}\lambda_i^{-1} = \frac{1}{2}\lambda_i\mu_i^2, \qquad i = 1, 2, 3, \tag{3.13}$$

so that $I = \{-1, 3, 4\}$ and satisfies the condition in part (i) of Theorem 3.3. The system is thus maximally superintegrable and has a three-parameter family of potentials (cf. Remark 3.4). Consider now the point transformation from (q, p)-coordinates (2.11), (3.1) to non-orthogonal coordinates (r, s) such that r_i are given by [7]

$$q_1 = r_1, \qquad q_2 = r_2 + \frac{1}{4}r_1^2, \qquad q_3 = -\frac{1}{4}r_3^2, \tag{3.14}$$

while

$$s_j = \sum_{i=1}^{3} \frac{\partial q_i}{\partial r_j} p_i, \qquad j = 1, 2, 3 \tag{3.15}$$

are new conjugated momenta. Then r_i are flat coordinates for the metric $G_1 = A_1$ in h_1. In these coordinates we get in this case

$$G = G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad L = \begin{pmatrix} -\frac{1}{2}r_1 & 1 & 0 \\ -r_2 & -\frac{1}{2}r_1 & -\frac{1}{2}r_3 \\ -\frac{1}{2}r_3 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

while the first three commuting Hamiltonians in (r, s)-variables become

$$h_1 = s_1 s_2 + \frac{1}{2}s_3^2 + \alpha_{-1}V_1^{(-1)}(r) + \alpha_3 V_1^{(3)}(r) + \alpha_4 V_1^{(4)}(r),$$

$$h_2 = \frac{1}{2}s_1^2 - \frac{1}{2}r_2 s_2^2 + \frac{1}{2}r_1 s_3^2 + \frac{1}{2}r_1 s_1 s_2 - \frac{1}{2}r_3 s_2 s_3 + \alpha_{-1}V_2^{(-1)}(r) + \alpha_3 V_2^{(3)}(r) + \alpha_4 V_2^{(4)}(r),$$

$$h_3 = \frac{1}{8}r_3^2 s_2^2 + \left(\frac{1}{2}r_2 + \frac{1}{8}r_1^2\right)s_3^2 - \frac{1}{2}r_3 s_1 s_3 - \frac{1}{4}r_1 r_3 s_2 s_3$$

$$\qquad + \alpha_{-1}V_3^{(-1)}(r) + \alpha_3 V_3^{(3)}(r) + \alpha_4 V_3^{(4)}(r), \tag{3.17}$$

with

$$V_1^{(-1)} = \frac{4}{r_3^2}, \qquad V_2^{(-1)} = \frac{4r_1}{r_3^2}, \qquad V_3^{(-1)} = \frac{r_1^2 + 4r_2}{r_3^2}, \tag{3.18}$$

$$V_1^{(3)} = r_1, \qquad V_2^{(3)} = \left(r_2 + \frac{1}{4}r_1^2\right), \qquad V_3^{(3)} = -\frac{1}{4}r_3^2, \tag{3.19}$$

$$V_1^{(4)} = r_2 - \frac{3}{4}r_1^2, \qquad V_2^{(4)} = -\left(r_1 r_2 + \frac{1}{4}r_1^3 + \frac{1}{4}r_3^2\right), \qquad V_3^{(4)} = \frac{1}{4}r_1 r_3^2. \tag{3.20}$$

In accordance with Theorem 3.3 and after the transformation to (r, s)-coordinates we have $P = s_2$, and $Y = \frac{\partial}{\partial r_2}$ so the additional constants of motion h_{n+i} of h_1 are

$$h_4 = \{h_2, P\} = -\frac{1}{2}s_2^2 + \alpha_3 - \alpha_4 r_1, \qquad h_5 = \{h_3, P\} = \frac{1}{2}s_3^2 + \frac{4\alpha_{-1}}{r_3^2}. \tag{3.21}$$

Example 3.8. Consider the constant curvature case $n = 3$, $m = 4$ and $I = \{-2, -1\}$. In order to apply part (iii) of Theorem 3.3 we have to put $b_0 = b_1 = 0$. Assume further that also

$b_2 = b_3 = 0$ and $b_4 = 1$ (so that $f(\lambda) = \lambda^4$ is again a monomial). The commuting Hamiltonians are then given by the separation relations

$$\alpha_{-2}\lambda_i^{-2} + \alpha_{-1}\lambda_i^{-1} + h_1\lambda_i^2 + h_2\lambda_i + h_3 = \frac{1}{2}\lambda_i^4\mu_i^2, \qquad i = 1, 2, 3.$$

Then again, according to (2.10), (3.4)–(3.6) and to (2.13), (2.14), the corresponding Stäckel Hamiltonians attain in the (q, p)-variables the form

$$
\begin{aligned}
h_1 &= \frac{1}{2}(q_1^2 - q_2)p_1^2 + \frac{1}{2}q_2^2p_2^2 + \frac{1}{2}q_3^2p_3^2 + (q_1q_2 - q_3)p_1p_2 + q_1q_3p_1p_3 + q_2q_3p_2p_3 \\
&\quad + \alpha_{-2}V_1^{(-2)} + \alpha_{-1}V_1^{(-1)}, \\
h_2 &= \frac{1}{2}(q_1q_2 - q_3)p_1^2 + q_2q_3p_2^2 + q_2^2p_1p_2 + q_2q_3p_1p_3 + q_3^2p_2p_3 + \alpha_{-2}V_2^{(-2)} + \alpha_{-1}V_2^{(-1)}, \\
h_3 &= \frac{1}{2}q_1q_3p_1^2 + \frac{1}{2}q_3^2p_2^2 + q_2q_3p_1p_2 + q_3^2p_1p_3 + \alpha_{-2}V_3^{(-2)} + \alpha_{-1}V_3^{(-1)}
\end{aligned}
$$

with

$$V_1^{(-1)} = \frac{1}{q_3}, \qquad V_2^{(-1)} = \frac{q_1}{q_3}, \qquad V_3^{(-1)} = \frac{q_2}{q_3},$$

$$V_1^{(-2)} = -\frac{q_2}{q_3^2}, \qquad V_2^{(-2)} = \frac{1}{q_3} - \frac{q_1q_2}{q_3^2}, \qquad V_3^{(-2)} = \frac{q_1}{q_3} - \frac{q_2^2}{q_3^2}.$$

Now, according to part (iii) of Theorem 3.3, $P = q_3p_1$, $Y = q_3\frac{\partial}{\partial q_1}$ and $\{h_1, P\} = 0$ so the additional constants of motion are

$$h_4 = \{h_2, P\} = -\frac{1}{2}q_2q_3p_1^2 - q_3^2p_1p_2 + \alpha_{-1} - \alpha_{-2}\frac{q_2}{q_3}, \qquad h_5 = \{h_3, P\} = -\frac{1}{2}q_3^2p_1^2 + \alpha_{-2}.$$

4 Stäckel transforms preserving maximal superintegrability

In this chapter we apply a 1-parameter Stäckel transform to our systems (2.5)–(2.7) to produce new maximally superintegrable Stäckel systems. As the transformation parameter α we will always use one of the α_i from (2.7).

Stäckel transform is a functional transform that maps a Liouville integrable systems into a new integrable system. It was first introduced in [13] (where it was called the coupling-constant metamorphosis) and later developed in [9]. When applied to a Stäckel separable system, this transformation yields a new Stäckel separable system, which explains its name. In the original paper [13] the authors used only one parameter (one coupling constant). In [22] the authors introduced a multiparameter generalization of this transform. This idea has been further developed in [8] and later in [4].

In this section we prove a theorem (Theorem 4.2) that yields sufficient conditions for Stäckel transform to preserve maximal superintegrability of a Stäckel system.

Let us first, following [4], remind the definition of the multiparameter Stäckel transform. Consider again a manifold M equipped with a Poisson tensor Π and the corresponding Poisson bracket $\{\cdot, \cdot\}$. Suppose we have r smooth functions $h_i \colon M \to \mathbf{R}$ on M, each depending on $k \leq r$ parameters $\alpha_1, \ldots, \alpha_k$ so that

$$h_i = h_i(x, \alpha_1, \ldots, \alpha_k), \qquad i = 1, \ldots, r, \tag{4.1}$$

where $x \in M$. Let us now from r functions in (4.1) choose k functions h_{s_i}, $i = 1, \ldots, k$, where $\{s_1, \ldots, s_k\} \subset \{1, \ldots, r\}$. Assume also that the system of equations

$$h_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \qquad i = 1, \ldots, k,$$

(where $\tilde{\alpha}_i$ is another set of k free parameters, or values of Hamiltonians h_{s_i}) involving the functions h_{s_i} can be solved for the parameters α_i yielding

$$\alpha_i = \tilde{h}_{s_i}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \qquad i = 1, \ldots, k, \tag{4.2}$$

where the right hand sides of these solutions define k new functions $\tilde{h}_{s_i}$ on M, each depending on k parameters $\tilde{\alpha}_i$. Finally, let us define $r - k$ functions $\tilde{h}_i$ for $i = 1, \ldots, r$, $i \notin \{s_1, \ldots, s_k\}$, by substituting $\tilde{h}_{s_i}$ from (4.2) instead of α_i in h_i:

$$\tilde{h}_i = h_i|_{\alpha_1 \to \tilde{h}_{s_1}, \ldots, \alpha_k \to \tilde{h}_{s_k}}, \qquad i = 1, \ldots, r, \qquad i \notin \{s_1, \ldots, s_k\}. \tag{4.3}$$

Definition 4.1. The functions $\tilde{h}_i = \tilde{h}_i(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$, $i = 1, \ldots, r$, defined through (4.2) and (4.3) are called the (generalized) Stäckel transform of the functions (4.1) with respect to the indices $\{s_1, \ldots, s_k\}$ (or with respect to the functions $h_{s_1}, \ldots, h_{s_k}$).

Unless we extend the manifold M this operation cannot be obtained by any coordinate change of variables. Moreover, if we perform again the Stäckel transform on the functions $\tilde{h}_i$ with respect to $\tilde{h}_{s_i}$ we will receive back the functions h_i in (4.1). In this sense the Stäckel transform is a reciprocal transform. Note also that neither r nor k are related to the dimension of the manifold M.

In [4] we proved that if $\dim M = 2n$, $k = r = n$ and if all h_i are functionally independent then also all $\tilde{h}_i$ will be functionally independent and if all h_i are pairwise in involution with respect to Π then also all $\tilde{h}_i$ will pairwise Poisson-commute. That means that if the functions h_i, $i = 1, \ldots, n$ constitute a Liouville integrable system then also $\tilde{h}_i$ will constitute a Liouville integrable system. In other words, Stäckel transform preserves Liouville integrability. But what about superintegrability?

Theorem 4.2. *Consider a maximally superintegrable system on a $2n$-dimensional Poisson manifold, i.e., a set of $2n - 1$ functionally independent Hamiltonians $h_1, \ldots, h_{2n-1}$ such that the first n Hamiltonians pairwise commute, and assume that all the Hamiltonians depend on $k \leq n$ parameters α_i:*

$$
\begin{aligned}
h_i &= h_i(x, \alpha_1, \ldots, \alpha_k), & i &= 1, \ldots, 2n - 1, \\
\{h_i, h_j\} &= 0, & i, j &= 1, \ldots, n, \quad \text{for all } \alpha_i, \\
\{h_1, h_{n+j}\} &= 0, & j &= 1, \ldots, n - 1, \quad \text{for all } \alpha_i.
\end{aligned}
\tag{4.4}
$$

Suppose that $\{s_1, \ldots, s_k\} \subset \{1, \ldots, 2n - 1\}$ are chosen so that $s_1 = 1$ and that $\{s_2, \ldots, s_k\} \subset \{2, \ldots, n\}$ and moreover that $h_1 = h_1(x, \alpha_1)$. Then the Stäckel transform $\tilde{h}_i$, $i = 1, \ldots, 2n-1$ given by (4.2), (4.3) also satisfy (4.4) and therefore constitute a maximally superintegrable system.

Note that the Hamiltonian h_1 is now distinguished as the one that commutes with all the remaining h_i and as it can only depend on one parameter. Note also that the first n functions h_i pairwise commute with each other and therefore constitute a Liouville integrable system. The same is true about the first n functions $\tilde{h}_i$.

Proof. Differentiating the identity

$$h_{s_i}\big(x, \tilde{h}_{s_1}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \ldots, \tilde{h}_{s_k}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)\big) = \tilde{\alpha}_i, \qquad i = 1, \ldots, k$$

with respect to x we get

$$dh_{s_i} = -\sum_{j=1}^{k} \frac{\partial h_{s_i}}{\partial \alpha_j} d\tilde{h}_{s_j}, \qquad i = 1, \ldots, k, \tag{4.5}$$

while differentiation of (4.3) yields

$$dh_i = d\tilde{h}_i - \sum_{j=1}^{k} \frac{\partial h_i}{\partial \alpha_j} d\tilde{h}_{s_j}, \qquad i = 1, \ldots, 2n-1, \qquad i \notin \{s_1, \ldots, s_k\}. \qquad (4.6)$$

The transformation (4.5), (4.6) can be written in a matrix form as

$$dh = A d\tilde{h},$$

where we denote $dh = (dh_1, \ldots, dh_{2n-1})^T$ and $d\tilde{h} = (d\tilde{h}_1, \ldots, d\tilde{h}_{2n-1})^T$ and where the $(2n-1) \times (2n-1)$ matrix A has the form

$$A_{ij} = \delta_{ij} \qquad \text{for} \quad j \notin \{s_1, \ldots, s_k\}, \qquad A_{is_j} = -\frac{\partial h_i}{\partial \alpha_j} \qquad \text{for} \quad j = 1, \ldots, k.$$

Since

$$\det A = \pm \det \left(\frac{\partial h_{s_i}}{\partial \alpha_j} \right)$$

is not zero and since h_i are by assumption functionally independent on M we conclude that also the functions $\tilde{h}_i$ are functionally independent on M. Further, since $s_k \leq n$ (the Stäckel transform is taken with respect to the Hamiltonians belonging to the Liouville integrable system $h_1, \ldots, h_n$) the columns with derivatives of h_i with respect to parameters α_j all lie in the left hand side of the matrix A. Moreover, the fact that $h_1 = h_1(x, \alpha_1)$ also means that the first row of A is zero except $A_{11} = -\frac{\partial h_i}{\partial \alpha_1}$. Let us now introduce the $(2n-1) \times (2n-1)$ matrices C and D through $C_{ij} = \{h_i, h_j\}$ and $D_{ij} = \{\tilde{h}_i, \tilde{h}_j\}$. A direct calculation yields

$$\{\tilde{h}_i, \tilde{h}_j\} = \sum_{l_1, l_2 = 1}^{2n-1} \left(A^{-1} \right)_{il_1} \left(A^{-1} \right)_{jl_2} \{h_{l_1}, h_{l_2}\}_{\Pi}$$

or in matrix form

$$D = A^{-1} C \left(A^{-1} \right)^T,$$

and due to the aforementioned structure of A we have $D_{ij} = 0$ for $i, j = 1, \ldots, n$ (meaning that $\tilde{h}_1, \ldots, \tilde{h}_n$ constitute a Liouville integrable system) and moreover that $D_{1i} = D_{i1} = 0$ for $i = 1, \ldots, 2n-1$, so that $\{\tilde{h}_1, \tilde{h}_i\} = 0$ for all i. That concludes the proof. ∎

Remark 4.3. A similar statement with an analogous proof is valid for any superintegrable system of the form (4.4), not only the maximally superintegrable one.

5 Stäckel transform of maximally superintegrable Stäckel systems

In this section we perform those Stäckel transforms of our systems (2.5)–(2.7) that preserve maximal superintegrability. According to Theorem 4.2, the Hamiltonian h_1 of the considered system can only depend on one parameter $h_1 = h_1(x, \alpha)$. It is then natural to choose one of the a_k in (2.7) as this parameter.

Consider thus a maximally superintegrable system $(h_1, \ldots, h_{2n-1})$ with the first n commuting Hamiltonians $h_1, \ldots, h_n$ defined by our separation relations

$$\sum_{s \in I} \alpha_s \lambda_i^s + h_1 \lambda_i^{n-1} + h_2 \lambda_i^{n-2} + \cdots + h_n = \frac{1}{2} f(\lambda_i) \mu_i^2, \qquad i = 1, \ldots, n,$$

where the index set I satisfies the assumptions of Theorem 3.3 and where the higher integrals h_{n+r} are constructed as usual through $h_{n+r} = \{h_{r+1}, P\}$ with P constructed as in Theorem 3.3. Let us now choose one of the parameters α_s, with $s \in I$, say α_k, (we will suppose that $k \geq n$ or $k < 0$ otherwise the corresponding potential is trivial, as explained earlier) and define the functions H_r, $r = 1, \ldots, 2n - 1$, through

$$h_r = H_r + \alpha_k V_r^{(k)}, \qquad r = 1, \ldots, 2n - 1. \tag{5.1}$$

Then $V_r^{(k)}$ for $r = 1, \ldots, n$ obviously coincide with $V_r^{(k)}$ defined through (2.5)–(2.7) or equivalently through (2.13), (2.14).

We now perform the Stäckel transform on this system $(h_1, \ldots, h_{2n-1})$ with respect to the chosen parameter α_k as described in Theorem 4.2. It means that we first solve the relation $h_1 = \tilde{\alpha}$, i.e., $H_1 + \alpha_k V_1^{(k)} = \tilde{\alpha}$ with respect to α_k which yields

$$\tilde{h}_1 = \alpha_k = -\frac{1}{V_1^{(k)}} H_1 + \tilde{\alpha} \frac{1}{V_1^{(k)}}, \tag{5.2}$$

and then replace α_k with $\tilde{h}_1$ in all the remaining Hamiltonians:

$$\tilde{h}_r = H_r - \frac{V_r^{(k)}}{V_1^{(k)}} H_1 + \tilde{\alpha} \frac{V_r^{(k)}}{V_1^{(k)}}, \qquad r = 2, \ldots, 2n - 1. \tag{5.3}$$

We obtain in this way a new superintegrable system $(\tilde{h}_1, \ldots, \tilde{h}_{2n-1})$ where the first n commuting Hamiltonians $\tilde{h}_r$ are defined by (see [4]) the following separation relations

$$\tilde{h}_1 \lambda_i^k + \sum_{s \in I, \, s \neq k} \alpha_s \lambda_i^s + \tilde{\alpha} \lambda_i^{n-1} + \tilde{h}_2 \lambda_i^{n-2} + \cdots + \tilde{h}_n = \frac{1}{2} f(\lambda_i) \mu_i^2, \qquad i = 1, \ldots, n, \tag{5.4}$$

as it is easy to see, since on the level of the separation relations our Stäckel transform replaces α_k with $\tilde{h}_1$ and h_1 with $\tilde{\alpha}$. For $k \geq n$ or $k < -1$ the system (5.4) is no longer in the class (2.5), while for $k = -1$ it can be easily transformed by a simple point transformation to the form (2.5).

Lemma 5.1. *The separable system*

$$\alpha_k \lambda_i^k + \sum_{s \in I, \, s \neq k} \alpha_s \lambda_i^s + h_1 \lambda_i^{n-1} + h_2 \lambda_i^{n-2} + \cdots + h_n = \frac{1}{2} \lambda_i^m \mu_i^2, \qquad i = 1, \ldots, n$$

attains after the Stäckel transform (5.2), (5.3) *and after the consecutive point transformation on M given by*

$$\lambda_i \to 1/\lambda_i, \qquad \mu_i \to -\lambda_i^2 \mu_i, \qquad i = 1, \ldots, n \tag{5.5}$$

the form

$$\tilde{\alpha} \lambda_i^{-1} + \sum_{s \in I, \, s \neq k} \alpha_s \lambda_i^{n-2-s} + \tilde{h}_1 \lambda_i^{n-k-2} + \tilde{h}_n \lambda_i^{n-2} + \cdots + \tilde{h}_2$$

$$= \frac{1}{2} \lambda_i^{n-m+2} \mu_i^2, \qquad i = 1, \ldots, n. \tag{5.6}$$

Note that the transformation (5.5) on M does not change the separation web of the system on Q. Denoting, as before

$$\tilde{h}_r = \widetilde{H}_r + \tilde{\alpha} \tilde{V}_r, \qquad r = 1, \ldots, 2n - 1, \tag{5.7}$$

where $\tilde{h}_r$ for $r = 1, \ldots, n$ are defined by (5.4) while $\tilde{h}_r$ for $r = n+1, \ldots, 2n-1$ are obtained as usual through $\tilde{h}_{n+r} = \{\tilde{h}_{r+1}, P\}$, we see from (5.3) that

$$\tilde{V}_r = V_r - \frac{V_r^{(k)}}{V_1^{(k)}} V_1, \qquad r = 2, \ldots, 2n-1,$$

and from (5.2) it also follows that the geodesic part $\tilde{E}_1$ of $\tilde{h}_1$ has the form

$$\tilde{E}_1 = \sum_{i,j=1}^{n} \tilde{G}^{ij} p_i p_j, \qquad \tilde{G} = -\frac{1}{V_1^{(k)}} G. \tag{5.8}$$

It means that the metric $\tilde{G}$ is a *conformal deformation* of either a flat or a constant curvature metric G. In the following theorem we list the cases when the metric $\tilde{G}$ is actually flat or of constant curvature as well. The theorem is formulated only for f in (2.6) being a monomial, $f = \lambda^m$ (in this case there is a maximum number of flat metrics $\tilde{G}$).

Theorem 5.2. *Consider the system* (5.4) *with* $f = \lambda^m$ *where* $m \in \{0, \ldots, n+1\}$.

(i) *For* $0 \le m \le n-1$ *the system* (5.4) *is maximally superintegrable for* $k \in \{-m, \ldots, -1, n, \ldots, 2n-m-1\}$. *The metric* $\tilde{G}$ *in* (5.8) *is flat for* $k \in \{-[m/2], \ldots, -1, n, \ldots, n-1+[(n-m)/2]\}$, *where* $[\cdot]$ *denotes the integer part. Moreover, for* $m = 1$ *and* $k = -1$ $\tilde{G}$ *is of constant curvature. Otherwise* $\tilde{G}$ *is conformally flat.*

(ii) *For* $m = n$ *the system* (5.4) *is maximally superintegrable for* $k \in \{-(n-2), \ldots, -1, n\}$. *The metric* $\tilde{G}$ *in* (5.8) *is flat for* $k \in \{-[n/2], \ldots, -1\}$. *Otherwise* $\tilde{G}$ *is conformally flat.*

(iii) *For* $m = n+1$ *the system (5.4) is maximally superintegrable for* $k \in \{-(n-1), \ldots, -1\}$. *The metric* $\tilde{G}$ *in* (5.8) *is flat for* $k \in \{-[(n+1)/2], \ldots, -1\}$. *Otherwise* $\tilde{G}$ *is conformally flat.*

If f is a polynomial then the admissible values of k must satisfy the above type of bonds for all powers of λ in f, not only for the highest power m so we choose not to present this more general theorem, only to maintain the simplicity of the picture. In order to prove Theorem 5.2 we need one more lemma.

Lemma 5.3 ([20]). *The Ricci scalars* R *and* $\tilde{R}$ *of the conformally related* (*covariant*) *metric tensors* g *and* $\tilde{g} = \sigma g$ *are related through*

$$\tilde{R} = \sigma^{-1} R - \frac{1}{2}(n-1)\sigma^{-1} s_{ij} G^{ij}, \tag{5.9}$$

where $G = g^{-1}$ *and where*

$$s_{ij} = s_{ji} = 2\nabla_i s_j - s_i s_j + \frac{1}{2} g_{ij} s_k s^k \qquad with \quad s_i = \sigma^{-1} \frac{\partial \sigma}{\partial x_i},$$

where x_i *are any coordinates on the manifold.*

Proof of Theorem 5.2. The values of k for which (5.1) is maximally superintegrable follows from the specification of Theorem 3.3 to the case $f = \lambda^m$. For (i) and (ii) the metric G in (5.8) is flat so that its Ricci scalar $R = 0$. Therefore, according to (5.9), $\tilde{R} = 0$ if and only if $s_{ij} G^{ij} = 0$. This condition can be effectively calculated in flat coordinates r_i of the metric G given by [7]

$$q_i = r_i + \frac{1}{4} \sum_{j=1}^{i-1} r_j r_{i-j}, \qquad i = 1, \ldots, n-m,$$

$$q_i = -\frac{1}{4} \sum_{j=i}^{n} r_j r_{n-j+i}, \qquad i = n-m+1, \ldots, m.$$

In these coordinates

$$(G_m)^{kl} = \delta^{k+l}_{n-m+1} + \delta^{k+l}_{2n-m+1}$$

and the condition $s_{ij}G^{ij} = 0$ yields both statements. The case in (i) when $\tilde{G}$ is of constant curvature $(m = 1, k = -1)$ can be however more effectively proven using Lemma 5.1 since in this case the system (5.4) attains after the transformation (5.5) the form

$$\tilde{h}_1\lambda_i^{n-1} + \sum_{s \in I,\, s \neq k} \alpha_s\lambda_i^{n-2-s} + \tilde{h}_n\lambda_i^{n-2} + \cdots + \tilde{h}_2 + \tilde{\alpha}\lambda_i^{-1} = \frac{1}{2}\lambda_i^{n+1}\mu_i^2, \qquad i = 1, \ldots, n.$$

Due to Remark 2.1 the metric $\tilde{G}$ of this system has constant curvature. Finally, in the case (iii) $(m = n + 1)$ we have only negative potentials so by using Lemma 5.1 we transform this system to

$$\tilde{h}_1\lambda_i^{n-k-2} + \sum_{s \in I,\, s \neq k} \alpha_s\lambda_i^{n-2-s} + \tilde{h}_n\lambda_i^{n-2} + \cdots + \tilde{h}_2 + \tilde{\alpha}\lambda_i^{-1} = \frac{1}{2}\lambda_i\mu_i^2, \qquad i = 1, \ldots, n,$$

where $k < 0$, and this is the system from case (i) with $m = 1$ and therefore $\tilde{G}$ is flat for $k \geq -[(n + 1)/2]$. For other values of k the metric $\tilde{G}$ is conformally flat. ∎

If $Y(V_1^{(k)}) = 0$ then $Y(1/V_1^{(k)}) = 0$ and due to (5.8) also $L_Y\tilde{G} = 0$ so that $\{\tilde{h}_1, P\} = 0$ as well and the same P as in the "non-tilde"-case (i.e., before the Stäckel transform) can be used as an alternative definition of extra Hamiltonians through $\bar{h}_{n+r} = \{\tilde{h}_{r+1}, P\}$, $r = 1, \ldots, n - 1$. This is however no longer true if $Y(V_1^{(k)}) = c \neq 0$ (according to Theorem 3.3, it happens only in the case when $m < n$ and $k = 2n - m - 1$). It turns out that it leads to the same extra integrals of motion, as the following theorem states

Theorem 5.4. *If $Y(V_1^{(k)}) = 0$ then both sets of extra integrals of motion:*

$$\bar{h}_{n+r} = \{\tilde{h}_{r+1}, P\}, \qquad r = 1, \ldots, n - 1$$

and

$$\tilde{h}_{n+r} = h_{n+r}\big|_{\alpha = \tilde{h}_1(\tilde{\alpha})}, \qquad r = 1, \ldots, n - 1$$

coincide.

Proof. On one hand, according to (5.3) and due to the fact that $\{\tilde{h}_1, P\} = 0$ we have

$$\bar{h}_{n+r} = \{\tilde{h}_{r+1}, P\} = \left\{H_{r+1} - \frac{V_{r+1}^{(k)}}{V_1^{(k)}}H_1 + \tilde{\alpha}\frac{V_{r+1}^{(k)}}{V_1^{(k)}}, P\right\}$$

$$= \{H_{r+1}, P\} - \frac{H_1}{V_1^{(k)}}\{V_{r+1}^{(k)}, P\} + \frac{\tilde{\alpha}}{V_1^{(k)}}\{V_{r+1}^{(k)}, P\} = \{H_{r+1}, P\} + \tilde{h}_1\{V_{r+1}^{(k)}, P\}.$$

On the other hand, due to

$$\tilde{h}_{n+r} = h_{n+r}\big|_{\alpha = \tilde{h}_1(\tilde{\alpha})} = \{h_{r+1}, P\}\big|_{\alpha = \tilde{h}_1(\tilde{\alpha})} = \{H_{r+1}, P\} + \alpha\{V_{r+1}^{(k)}, P\}\big|_{\alpha = \tilde{h}_1(\tilde{\alpha})},$$

which yields the same result. ∎

Thus, if $Y(V_1^{(k)}) = 0$, the diagram below commutes

$$
\begin{array}{ccc}
(h_1,\ldots,h_n) & \xrightarrow{\ P\ } & (h_1,\ldots,h_{2n-1}) \text{ with } h_{n+r} = \{h_{r+1}, P\} \\
\big| & & \big| \\
\text{Stäckel transform} & & \text{Stäckel transform} \\
\downarrow & & \downarrow \\
(\tilde{h}_1,\ldots,\tilde{h}_n) & \xrightarrow{\ P\ } & (\tilde{h}_1,\ldots,\tilde{h}_{2n-1}) \text{ with } \tilde{h}_{n+r} = \{\tilde{h}_{r+1}, P\}.
\end{array}
$$

Example 5.5. Let us apply the relations (5.2), (5.3) to perform the Stäckel transform on the system from Example 3.7. To keep the formulas simple, we assume that all the α_s in (2.7) are zero except the transformation parameter α_k. Thus, we consider again the system given by the separation relations

$$
\alpha_k \lambda_i^k + h_1 \lambda_i^2 + h_2 \lambda_i + h_3 = \frac{1}{2}\lambda_i \mu_i^2, \qquad i = 1,2,3
$$

with $k = -1$, 3 or 4, respectively. Applying Stäckel transform to the resulting Hamiltonians (3.17)–(3.21) we obtain a maximally superintegrable system with the separation relations of the form:

$$
\tilde{h}_1 \lambda_i^k + \tilde{\alpha}\lambda_i^2 + \tilde{h}_2 \lambda_i + \tilde{h}_3 = \frac{1}{2}\lambda_i \mu_i^2, \qquad i = 1,2,3. \tag{5.10}
$$

Again we perform our calculations in the (r,s)-variables (3.14), (3.15). Explicitly, we obtain for $k = -1$

$$
\begin{aligned}
\tilde{h}_1 &= \frac{1}{8}r_3^2 s_3^2 + \frac{1}{4}r_3^2 s_1 s_2 - \frac{1}{4}\tilde{\alpha}r_3^2, \\
\tilde{h}_2 &= \frac{1}{2}s_1^2 - \frac{1}{2}r_2 s_2^2 - \frac{1}{2}r_1 s_1 s_2 - \frac{1}{2}r_3 s_2 s_3 + \tilde{\alpha}r_1, \\
\tilde{h}_3 &= \frac{1}{8}r_3^2 s_2^2 - \left(\frac{1}{4}r_1^2 + r_2\right)s_1 s_2 - \frac{1}{2}r_3 s_1 s_3 - \frac{1}{4}r_1 r_3 s_2 s_3 + \frac{1}{4}\tilde{\alpha}\left(r_1^2 + 4r_2\right), \\
\tilde{h}_4 &= -\frac{1}{2}s_2^2, \qquad \tilde{h}_5 = -s_1 s_2 + \tilde{\alpha},
\end{aligned} \tag{5.11}
$$

for $k = 3$

$$
\begin{aligned}
\tilde{h}_1 &= -\frac{1}{r_1}s_1 s_2 - \frac{1}{2}\frac{1}{r_1}s_3^2 + \tilde{\alpha}\frac{1}{r_1}, \\
\tilde{h}_2 &= \frac{1}{2}s_1^2 + \frac{1}{4}\frac{r_1^2 - 4r_2}{r_1}s_1 s_2 - \frac{1}{2}r_2 s_2^2 - \frac{1}{2}r_3 s_2 s_3 + \frac{1}{8}\frac{3r_1^2 - 4r_2}{r_1}s_3^2 + \frac{1}{4}\tilde{\alpha}\frac{r_1^2 + 4r_2}{r_1}, \\
\tilde{h}_3 &= \frac{1}{4}\frac{r_3^2}{r_1}s_1 s_2 - \frac{1}{2}r_3 s_1 s_3 + \frac{1}{8}r_3^2 s_2^2 - \frac{1}{4}r_1 r_3 s_2 s_3 + \frac{1}{8}\frac{r_1^3 + 4r_1 r_2 + r_3^2}{r_1}s_3^2 - \frac{1}{4}\tilde{\alpha}\frac{r_3^2}{r_1}, \\
\tilde{h}_4 &= -\frac{1}{r_1}s_1 s_2 - \frac{1}{2}s_2^2 - \frac{1}{2}\frac{1}{r_1}s_3^2 + \tilde{\alpha}\frac{1}{r_1}, \qquad \tilde{h}_5 = \frac{1}{2}s_3^2,
\end{aligned}
$$

and for $k = 4$

$$
\begin{aligned}
\tilde{h}_1 &= -\frac{1}{r_2 - \frac{3}{4}r_1^2}s_1 s_2 - \frac{1}{2}\frac{1}{r_2 - \frac{3}{4}r_1^2}s_3^2 + \tilde{\alpha}\frac{1}{r_2 - \frac{3}{4}r_1^2}, \\
\tilde{h}_2 &= \frac{1}{2}s_1^2 - \frac{1}{2}r_2 s_2^2 - \frac{1}{8}\frac{2r_1^3 - 8r_1 r_2 - r_3^2}{r_2 - \frac{3}{4}r_1^2}s_3^2 - \frac{1}{8}\frac{r_1^3 - 12r_1 r_2 - 2r_3^2}{r_2 - \frac{3}{4}r_1^2}s_1 s_2 - \frac{1}{2}r_3 s_2 s_3 \\
&\quad - \tilde{\alpha}\frac{r_1 r_2 + \frac{1}{4}r_1^3 + \frac{1}{4}r_3^2}{r_2 - \frac{3}{4}r_1^2},
\end{aligned}
$$

$$\tilde{h}_3 = \frac{1}{8}r_3^2 s_2^2 - \frac{1}{32}\frac{3r_1^4 + 8r_1^2 r_2 + 4r_1 r_3^2 + 16r_2^2}{r_2 - \frac{3}{4}r_1^2}s_3^2 + \frac{1}{4}\frac{r_1 r_3^2}{r_2 - \frac{3}{4}r_1^2}s_1 s_2 - \frac{1}{2}r_3 s_1 s_3 - \frac{1}{4}r_1 r_3 s_2 s_3$$

$$+ \frac{1}{4}\tilde{\alpha}\frac{r_1 r_3^2}{r_2 - \frac{3}{4}r_1^2},$$

$$\tilde{h}_4 = -\frac{1}{2}s_2^2 + \frac{1}{2}\frac{r_1}{r_2 - \frac{3}{4}r_1^2}s_3^2 + \frac{r_1}{r_2 - \frac{3}{4}r_1^2}s_1 s_2 - \tilde{\alpha}\frac{r_1}{r_2 - \frac{3}{4}r_1^2}, \qquad \tilde{h}_5 = \frac{1}{2}s_3^2. \tag{5.12}$$

According to part (i) of Theorem 5.2 the metrics of $\tilde{h}_1$ are of constant curvature, flat and conformally flat, respectively.

6 Quantization of maximally superintegrable Stäckel systems

This section is devoted to separable quantizations of Stäckel systems that were considered in the classical setting in the previous sections. Let us consider, as in the classical case, an n-dimensional Riemannian space Q equipped with a matric tensor g and the quadratic in momenta Hamiltonian on the cotangent bundle T^*Q:

$$h = \frac{1}{2}\sum_{i,j=1}^{n} p_i A^{ij}(x) p_j + U(x).$$

By its *minimal quantization* [5] we mean the following self-adjoint operator

$$\widehat{h} = -\frac{1}{2}\hbar^2 \sum_{i,j=1}^{n} \nabla_i A^{ij}(x)\nabla_j + U(x) = -\frac{1}{2}\hbar^2 \sum_{i,j=1}^{n} \frac{1}{\sqrt{|g|}}\partial_i \sqrt{|g|}A^{ij}(x)\partial_j + U(x) \tag{6.1}$$

(both expressions on the right hand side of (6.1) are equivalent) acting in the Hilbert space

$$\mathcal{H} = L^2(Q, d\mu), \qquad d\mu = |g|^{1/2}dx, \qquad |g| = |\det g|,$$

where ∇ is the Levi-Civita connection of the metric g. Note that a priori there is no relation between the tensor A and the metric g. Let us now consider an arbitrary Stäckel system of the form (2.3) coming from the separation relations (2.2). Applying the procedure of minimal quantization to this system will in general yield a non-integrable and non-separable quantum system. In order to preserve integrability and separability we have to carefully choose the metric g. To do this, we will use the following theorem, proved in [5].

Theorem 6.1. *Suppose that h_j are Hamiltonian functions* (2.3), *defined by separation relations* (2.2). *Suppose also that θ is an arbitrary function of one variable. Applying to h_j the procedure of minimal quantization* (6.1) *with the metric tensor*

$$g = \varphi^{\frac{2}{n}}g_\theta, \tag{6.2}$$

where $g_\theta = G_\theta^{-1}$ with G_θ given by

$$G_\theta = \text{diag}\left(\frac{\theta(\lambda_1)}{\Delta_1}, \ldots, \frac{\theta(\lambda_n)}{\Delta_n}\right), \tag{6.3}$$

and with φ being a particular function of $\lambda_1, \ldots, \lambda_n$, uniquely defined by (2.2) *(see formula* (27) *in* [5] *for details), we obtain a quantum integrable and separable system. More precisely, we obtain n operators $\widehat{h}_i$ of the form* (6.1) *such that* (i) $[\widehat{h}_i, \widehat{h}_j] = 0$ *for all i, j and* (ii) *eigenvalue problems for all $\widehat{h}_i$*

$$\widehat{h}_i\Psi = \varepsilon_i\Psi, \qquad i = 1, \ldots, n$$

have for each choice of eigenvalues ε_i of $\widehat{h}_i$ the common multiplicatively separable eigenfunction $\Psi(\lambda_1, \ldots, \lambda_n) = \prod\limits_{i=1}^{n} \psi(\lambda_i)$ with ψ satisfying the following ODE (quantum separation relation)

$$\left(\varepsilon_1 \lambda^{\gamma_1} + \varepsilon_2 \lambda^{\gamma_2} + \cdots + \varepsilon_n\right)\psi(\lambda)$$
$$= -\frac{1}{2}\hbar^2 f(\lambda)\left[\frac{d^2\psi(\lambda)}{d\lambda^2} + \left(\frac{f'(\lambda)}{f(\lambda)} - \frac{1}{2}\frac{\theta'(\lambda)}{\theta(\lambda)}\right)\frac{d\psi(\lambda)}{d\lambda}\right] + \sigma(\lambda)\psi(\lambda). \qquad (6.4)$$

Remark 6.2. For Stäckel systems defined by (2.5), when $(\gamma_1, \ldots, \gamma_n) = (n-1, \ldots, 0)$ in (2.2), we have $\varphi = 1$ and the most natural choice in (6.3) is to put $\theta = f$ which yields the metric for quantization

$$G = G_f = A_1. \qquad (6.5)$$

On the other hand, for Stäckel systems defined by (5.4) we have $\varphi = -V_1^{(k)}$ (as it follows from (5.2) and the formula (27) in [5]) and again the simplest choice in (6.3) is to put $\theta = f$ which yields according to (6.2) and (5.8) the metric for quantization

$$G = \varphi^{-\frac{2}{n}} G_f = \varphi^{1-\frac{2}{n}} \tilde{A}_1. \qquad (6.6)$$

For the choice (6.5) and (6.6) the quantum separation equation (6.4) reduce to

$$\left(\varepsilon_1 \lambda^{\gamma_1} + \varepsilon_2 \lambda^{\gamma_2} + \cdots + \varepsilon_n\right)\psi(\lambda) = -\frac{1}{2}\hbar^2 f(\lambda)\left[\frac{d^2\psi(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{f'(\lambda)}{f(\lambda)}\frac{d\psi(\lambda)}{d\lambda}\right] + \sigma(\lambda)\psi(\lambda), \quad (6.7)$$

where $(\gamma_1, \ldots, \gamma_n) = (n-1, n-2, \ldots, 0)$ in the first case (6.5) and $(\gamma_1, \ldots, \gamma_n) = (k, n-2, n-3, \ldots, 0)$ in the second case (6.6).

Let us now pass to the issue of quantum superintegrability of considered Stäckel systems. We formulate now a quantum analogue of Lemma 3.1.

Lemma 6.3. *Suppose that $\widehat{h}$ is given by (6.1) and that $Y = \sum\limits_{i=1}^{n} y^i(x)\nabla_i$ is a vector field on the Riemannian manifold Q with a metric g. Then*

$$\left[\widehat{h}, Y\right] = \frac{1}{2}\hbar^2 \sum_{i,j=1}^{n} \nabla_i (L_Y A)^{ij} \nabla_j + \frac{1}{2}\hbar^2 \sum_{i,j,k=1}^{n} A^{ij}\left(\nabla_j \nabla_k y^k\right)\nabla_i - Y(U).$$

One proves this lemma by a direct computation. Thus, a sufficient condition for $[\widehat{h}, Y] = c$ is satisfied when Y is a Killing vector for both A and g and if moreover U is constant along Y, that is when

$$L_Y A = 0, \qquad L_Y g = 0, \qquad Y(U) = c \qquad (6.8)$$

(note that $L_Y g = 0$ implies $\sum\limits_{i=1}^{n} \nabla_k y^k = 0$).

Corollary 6.4. *Suppose we have a quantum integrable system on the configuration space Q, that is a set of n commuting and algebraically independent operators $\widehat{h}_1, \ldots, \widehat{h}_n$ of the form (6.1) acting in the Hilbert space $L^2(Q, |g|^{1/2}dx)$ where g is some metric on Q. Suppose also that a vector field Y satisfies (6.8) with A_1 and U_1 instead of A and U (so that $[\widehat{h}_1, Y] = c$). Then, analogously to the classical case, the operators*

$$\widehat{h}_{n+r} = \left[\widehat{h}_{r+1}, Y\right] = \frac{1}{2}\hbar^2 \sum_{i,j=1}^{n} \nabla_i (L_Y A_{r+1})^{ij} \nabla_j - Y(U_{r+1}), \qquad r = 1, \ldots, n-1 \qquad (6.9)$$

satisfy $[\widehat{h}_{n+r}, \widehat{h}_1] = 0$ and the system $\widehat{h}_1, \ldots, \widehat{h}_{2n-1}$ is algebraically independent; that is we obtain a quantum separable and quantum superintegrable system.

We can now apply this corollary to construct quantum superintegrable counterparts of classical systems considered in previous sections. According to Remark 6.2, for the systems generated by the separation relations (2.5) the most natural choice of the metric g is to take $G = A_1$ as in (6.5). Then, by construction, $[\widehat{h}_i, \widehat{h}_j] = 0$ for $i, j = 1, \ldots, n$ while the remaining operators $\widehat{h}_{n+r}$ are constructed by the formula (6.9) and are – up to a sign – identical with minimal quantization (in the metric G) of the extra integrals h_{n+r} obtained in (3.8).

Example 6.5. Consider again separation relations (3.13) from Example 3.7, so that $f(\lambda) = \lambda$ and $\sigma = \alpha_{-1}\lambda^{-1} + \alpha_3\lambda^3 + \alpha_4\lambda^4$. Performing the minimal quantization of the Hamiltonians (3.17) in the metric $G = A_1$, i.e., given by (3.16), we obtain, in the flat r-coordinates (3.14)

$$\widehat{h}_1 = -\frac{1}{2}\hbar^2\left(\partial_1\partial_2 + \frac{1}{2}\partial_3^2\right) + \alpha_{-1}V_1^{(-1)}(r) + \alpha_3 V_1^{(3)}(r) + \alpha_4 V_1^{(4)}(r),$$

$$\widehat{h}_2 = -\frac{1}{4}\hbar^2\left(\partial_1^2 - \partial_2 r_2\partial_2 + r_1\partial_3\partial_3 + \frac{1}{2}\partial_1 r_1\partial_2 + \frac{1}{2}r_1\partial_2\partial_1 - r_3\frac{1}{2}\partial_2\partial_3 - \frac{1}{2}\partial_3 r_3\partial_2\right)$$
$$+ \alpha_{-1}V_2^{(-1)}(r) + \alpha_3 V_2^{(3)}(r) + \alpha_4 V_2^{(4)}(r),$$

$$\widehat{h}_3 = -\frac{1}{8}\hbar^2\left(\frac{1}{2}r_3^2\partial_2^2 + \left(2r_2 + \frac{1}{2}r_1^2\right)\partial_3^2 - r_3\partial_1\partial_3 - \partial_3 r_3\partial_1 - \frac{1}{2}r_1 r_3\partial_2\partial_3 - \frac{1}{2}r_1\partial_3 r_3\partial_2\right)$$
$$+ \alpha_{-1}V_3^{(-1)}(r) + \alpha_3 V_3^{(3)}(r) + \alpha_4 V_3^{(4)}(r),$$

where $\partial_i = \partial/\partial r_i$ and $V_i^{(k)}$ are given by (3.18)–(3.20). The respective separation equation, according to (3.13) and (6.7), is of the form

$$\left(\alpha_{-1}\lambda^{-1} + \alpha_3\lambda^3 + \alpha_4\lambda^4 + \varepsilon_1\lambda^2 + \varepsilon_2\lambda + \varepsilon_3\right)\psi(\lambda) = -\frac{1}{2}\hbar^2\left[\lambda\frac{d^2\psi(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{d\psi(\lambda)}{d\lambda}\right].$$

Now $Y = \partial_2$ satisfies the conditions (6.8) and the extra operators $\widehat{h}_4$, $\widehat{h}_5$ can be obtained either by using the formula (6.9) or directly by minimal quantization of functions h_4, h_5 in (3.21). The result is (up to a sign)

$$\widehat{h}_4 = \frac{1}{4}\hbar^2\partial_2^2 - \alpha_3 + \alpha_4 r_1, \qquad \widehat{h}_5 = -\frac{1}{4}\hbar^2\partial_3^2 + \frac{4\alpha_{-1}}{r_3^2}.$$

If we want to perform the separable quantization of superintegrable systems obtained by the Stäckel transform, as in Section 5, we have two cases: either the system – after the Stäckel transform – belongs again to the same class (2.5) or belongs to the other class, given by the separation relations (5.4) that are different from (2.5) as soon as $k \neq -1$. Again by Remark 6.2, in the first case the natural choice of the metric in which we perform the minimal quantization is to take $\tilde{G} = \widetilde{A}_1$, i.e., $\tilde{G}$ as given by (5.8). In the second case we have to use the metric given by (6.2) which in our case is given by (6.6), i.e., by $G = \varphi^{1-\frac{2}{n}}\widetilde{A}_1$ with $\varphi = -V_1^{(k)}$.

Example 6.6. Let us now minimally quantize the Stäckel Hamiltonians $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ given in (5.11), obtained through a Stäckel transform in Example 5.5, generated by the separation relations (5.10) with $k = -1$, that is by

$$\tilde{h}_1\lambda_i^{-1} + \tilde{\alpha}\lambda_i^2 + \tilde{h}_2\lambda_i + \tilde{h}_3 = \frac{1}{2}\lambda_i\mu_i^2, \qquad i = 1, 2, 3.$$

The metric associated with $\tilde{h}_1$

$$\tilde{G} = \frac{1}{4}r_3^2\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.10}$$

is of constant curvature as – by Lemma 5.1 – after applying transformation (5.5), in the new separation coordinates the separation relations (5.10) turns to

$$\tilde{\alpha}\lambda_i^{-1} + \tilde{h}_1\lambda_i^2 + \tilde{h}_3\lambda_i + \tilde{h}_2 = \frac{1}{2}\lambda_i^4\mu_i^2, \qquad i = 1, 2, 3$$

and belong again to the class (2.5). Thus, by Remark 6.2, we have to perform the minimal quantization of this system with respect to the original metric $\tilde{A}_1$ of the system which is just (6.10). Observing that $\sqrt{|\tilde{g}|} = 8/r_3^3$, we obtain the following quantum superintegrable system (we use the second expression in (6.1)):

$$\widehat{\tilde{h}}_1 = -\frac{1}{4}\hbar^2 r_3^2\left(\frac{1}{2}r_3\partial_3\frac{1}{r_3}\partial_3 + \partial_1\partial_2\right) - \frac{1}{4}\tilde{\alpha}r_3^2,$$

$$\widehat{\tilde{h}}_2 = \frac{1}{4}\hbar^2\left(-2\partial_1^2 + 2\partial_2 r_2\partial_2 + \partial_1 r_1\partial_2 + r_1\partial_2\partial_1 + r_3\partial_2\partial_3 + r_3^3\partial_3\frac{1}{r_3^2}\partial_2\right) + \tilde{\alpha}r_1,$$

$$\widehat{\tilde{h}}_3 = \frac{1}{8}\hbar^2\left[-r_3^2\partial_2^2 + (r_1^2 + 4r_2)\partial_1\partial_2 + \partial_1 r_1^2\partial_2 + 4\partial_2 r_2\partial_1 + 2r_3\partial_1\partial_3 + 2r_3^3\partial_3\frac{1}{r_3^2}\partial_1\right.$$
$$\left. + r_1 r_3\partial_2\partial_3 + r_1 r_3^3\partial_3\frac{1}{r_3^2}\partial_1\right] + \frac{1}{4}\tilde{\alpha}\left(r_1^2 + 4r_2\right),$$

$$\widehat{\tilde{h}}_4 = \frac{1}{2}\hbar^2\partial_2^2, \qquad \widehat{\tilde{h}}_5 = \hbar^2\partial_1\partial_2 + \tilde{\alpha}.$$

Example 6.7. Let us finally minimally quantize the Stäckel Hamiltonians $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ given in (5.12), obtained through a Stäckel transform in Example 5.5) and generated by separation relations (5.10) with $k = 4$

$$\tilde{h}_1\lambda_i^4 + \tilde{\alpha}\lambda_i^2 + \tilde{h}_2\lambda_i + \tilde{h}_3 = \frac{1}{2}\lambda_i\mu_i^2, \qquad i = 1, 2, 3.$$

The metric associated with $\tilde{h}_1$

$$\tilde{G} = \frac{1}{\frac{3}{4}r_1^2 - r_2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is conformally flat. By Remark 6.2, we have to perform minimal quantization of this system with respect to the metric (6.6) given by

$$G = \left(-V_1^{(4)}\right)^{1-\frac{2}{3}}\tilde{G} = \left(r_2 - \frac{3}{4}r_1^2\right)^{-\frac{2}{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observing that $\sqrt{|g|} = V_1^{(4)} = r_2 - \frac{3}{4}r_1^2$, we obtain the following quantum operators (we use again the second expression in (6.1)):

$$\widehat{\tilde{h}}_1 = \frac{1}{2}\hbar^2\left(r_2 - \frac{3}{4}r_1^2\right)^{-1}\left(2\partial_1\partial_2 + \partial_3^2\right) + \frac{\tilde{\alpha}}{r_2 - \frac{3}{4}r_1^2},$$

$$\widehat{\tilde{h}}_2 = -\frac{1}{2}\hbar^2\left(r_2 - \frac{3}{4}r_1^2\right)^{-1}\sum_{i,j}\partial_i B_2^{ij}\partial_j - \tilde{\alpha}\frac{r_1 r_2 + \frac{1}{4}r_1^3 + \frac{1}{4}r_3^2}{r_2 - \frac{3}{4}r_1^2},$$

$$\widehat{\tilde{h}}_3 = -\frac{1}{2}\hbar^2\left(r_2 - \frac{3}{4}r_1^2\right)^{-1}\sum_{i,j}\partial_i B_3^{ij}\partial_j + \frac{1}{4}\tilde{\alpha}\frac{r_1 r_3^2}{r_2 - \frac{3}{4}r_1^2},$$

$$\widehat{\tilde{h}}_4 = -\frac{1}{2}\hbar^2\left(r_2 - \frac{3}{4}r_1^2\right)^{-1}\left[\partial_1 r_1 \partial_2 + r_1 \partial_2 \partial_1 - \partial_2\left(r_2 - \frac{3}{4}r_1^2\right)\partial_2 + r_1\partial_3^2\right] - \tilde{\alpha}\frac{r_1}{r_2 - \frac{3}{4}r_1^2},$$

$$\widehat{\tilde{h}}_5 = -\frac{1}{2}\hbar^2\partial_3^2,$$

where

$$B_2 = \begin{pmatrix} r_2 - \frac{3}{4}r_1^2 & \frac{3}{2}r_1 r_2 - \frac{1}{8}r_1^3 + \frac{1}{4}r_3^2 & 0 \\ \frac{3}{2}r_1 r_2 - \frac{1}{8}r_1^3 + \frac{1}{4}r_3^2 & -r_2\left(r_2 - \frac{3}{4}r_1^2\right) & -\frac{1}{2}r_3\left(r_2 - \frac{3}{4}r_1^2\right) \\ 0 & -\frac{1}{2}r_3\left(r_2 - \frac{3}{4}r_1^2\right) & 2r_1 r_2 - \frac{1}{2}r_1^3 + \frac{1}{4}r_3^2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & -\frac{1}{4}r_1 r_3^2 & -\frac{1}{2}r_3\left(r_2 - \frac{3}{4}r_1^2\right) \\ -\frac{1}{4}r_1 r_3^2 & \frac{1}{4}r_3^2\left(r_2 - \frac{3}{4}r_1^2\right) & -\frac{1}{4}r_1 r_3\left(r_2 - \frac{3}{4}r_1^2\right) \\ -\frac{1}{2}r_3\left(r_2 - \frac{3}{4}r_1^2\right) & -\frac{1}{4}r_1 r_3\left(r_2 - \frac{3}{4}r_1^2\right) & -\frac{1}{2}r_1^2 r_2 + r_2^2 - \frac{1}{4}r_1 r_3^2 - \frac{3}{16}r_1^4 \end{pmatrix}$$

with $B = \sqrt{|g|}A$ in (6.1). It can be checked that it is again a quantum superintegrable system.

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