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Bruhat Order in the Full Symmetric $\mathfrak{sl}_n$ Toda Lattice on Partial Flag Space

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Abstract. In our previous paper [*Comm. Math. Phys.* **330** (2014), 367–399] we described the asymptotic behaviour of trajectories of the full symmetric $\mathfrak{sl}_n$ Toda lattice in the case of distinct eigenvalues of the Lax matrix. It turned out that it is completely determined by the Bruhat order on the permutation group. In the present paper we extend this result to the case when some eigenvalues of the Lax matrix coincide. In that case the trajectories are described in terms of the projection to a partial flag space where the induced dynamical system verifies the same properties as before: we show that when $t \to \pm\infty$ the trajectories of the induced dynamical system converge to a finite set of points in the partial flag space indexed by the Schubert cells so that any two points of this set are connected by a trajectory if and only if the corresponding cells are adjacent. This relation can be explained in terms of the Bruhat order on multiset permutations.

Key words: full symmetric Toda lattice; Bruhat order; integrals and semi-invariants; partial flag space; Morse function; multiset permutation

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1 Introduction

1.1 Toda system

The present paper is devoted to the study of the full symmetric $\mathfrak{sl}_n$ Toda lattice which can be considered as a straightforward generalization of the non-periodic Toda lattice. Let us briefly remind that the non-periodic Toda lattice (Toda chain) is the dynamical system of n particles on a straight line with interactions between neighbours. This system was first considered in [22, 23]; in paper [16] n functionally independent integrals of the motion were found. The involution of the integrals was proved in [11, 12].

The non-periodic Toda lattice has the Lax representation and the Lax operator matrix has the following form in Flaschka's variables:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & a_{n-2} & b_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & a_{n-1} & b_n \end{pmatrix}$$

One can show that the Hamilton equations are equivalent to the following matrix equation

$$\dot{L} = [B, L], \tag{1.1}$$

where B is

$$B = \begin{pmatrix} 0 & -a_1 & 0 & \cdots & 0 \\ a_1 & 0 & -a_2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & a_{n-2} & 0 & -a_{n-1} \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix}.$$

Equation (1.1) is the compatibility condition of the system

$$L\Psi = \Psi\Lambda, \qquad \frac{\partial}{\partial t}\Psi = B\Psi,$$

where $\Psi \in \mathrm{SO}(n, \mathbb{R})$ and Λ is the eigenvalue matrix of the Lax operator.

One can show that the non-periodic Toda lattice can be treated as a dynamical system on the orbits of the coadjoint action of the Borel subgroup B_n^+ of $\mathrm{SL}(n, \mathbb{R})$ (equal to the group of upper triangular matrices with the determinant equal to 1) see [1, 2, 19, 21]. It is also possible to give an alternative description of the phase space of this dynamical system if we identify the algebra $\mathfrak{sl}_n$ with its dual using the Killing form on $\mathfrak{sl}_n$. In this way one obtains generalizations of the classical Toda lattice (tri-diagonal Toda chain). Namely, use the following identifications:

$$\begin{aligned} &\mathfrak{sl}_n = \mathfrak{so}_n \oplus \mathfrak{b}_n^+, \qquad \mathfrak{sl}_n^* = (\mathfrak{b}_n^+)^* \oplus (\mathfrak{so}_n)^* \cong \mathrm{Symm}_n \oplus \mathfrak{n}_n^+, \\ &(\mathfrak{b}_n^+)^* \cong (\mathfrak{so}_n)^\perp = \mathrm{Symm}_n, \qquad (\mathfrak{so}_n)^* \cong (\mathfrak{b}_n^+)^\perp = \mathfrak{n}_n^+, \end{aligned}$$

where $\mathfrak{b}_n^+$ is the algebra of the upper triangular matrices and $\mathfrak{n}_n^+$ is the algebra of the strictly upper triangular matrices. As one can see this identification maps the space of symmetric matrices into the dual space of the Lie algebra of the Borel subgroup: $\mathrm{Symm}_n \cong (\mathfrak{b}_n^+)^*$ and hence we can introduce Kirillov–Kostant symplectic structure on Symm_n pulling it back from $(\mathfrak{b}_n^+)^*$; it is by restriction of this pullback that one obtains the symplectic structure on the tridiagonal symmetric matrices used in the Toda system.

Based on this approach one can get further generalizations of the non-periodic Toda lattice just by plugging in other Cartan pairs in this construction. In particular in this way one obtains the full symmetric $\mathfrak{sl}_n$ Toda lattice (elsewhere FS Toda lattice). In contrast to the Lax matrix of the non-periodic Toda lattice the Lax matrix of the FS Toda lattice is not a tri-diagonal symmetric matrix but an arbitrary symmetric matrix:

$$L = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{pmatrix}.$$

As one knows every symmetric matrix can be diagonalized in an orthogonal basis, i.e., the Lax matrix L can be decomposed as follows

$$L = \Psi\Lambda\Psi^{-1},$$

where Ψ is an orthogonal matrix. It turns out, that the FS Toda lattice is also integrable (we refer the reader to [2, 6, 8, 10, 13] for details). Moser in [20] showed that as $t \to -\infty$ the Lax operator of the usual tri-diagonal Toda lattice converges to the diagonal matrix with eigenvalues put in increasing order, and when $t \to +\infty$ it converges to the diagonal matrix with decreasing eigenvalues. This property was further studied in [5, 9, 13, 17, 18].

1.2 Outline of the paper

The present paper deals with the FS Toda lattice in an arbitrary dimension n. It is a natural continuation of the previous one [5] in which we study the behavior of the Lax operator L as $t \to \pm\infty$; this question is again in the center of our attention here. The difference is that in [5] we treated the Lax matrices L with n distinct eigenvalues and now we let the eigenvalues of L coincide. In the previous case the Bruhat order on the symmetric group S_n played a crucial rôle. On the other hand, as one knows there exists a Bruhat order on the permutations of multisets induced in a natural way from the Bruhat order on the symmetric group (one can regard this phenomenon as a combinatoric manifestation of the fact that the Schubert cells in the full flag space project into the Schubert cells in the partial flag manifolds). So it is natural to ask if this (induced) order has something to do with the asymptotic behavior of the Lax matrix with coinciding eigenvalues.

We show that one can regard the restriction of the FS Toda lattice on the space of symmetric matrices with multiple eigenvalues as a gradient system on a partial flag space so that the set of singular points of the gradient vector field is naturally identified with the permutations of multisets (see Section 3, Proposition 3.3) and we show that the gradient system at hand verifies the same properties as the system with distinct eigenvalues. That is we prove that for $t \to \pm\infty$ the trajectories of the FS Toda lattice converge to a finite set of singular points in the partial flag space indexed by the Schubert cells so that any two points of this set are connected by a trajectory if and only if the corresponding cells are adjacent (see Theorem 4.2).

The paper is organized as follows: in Section 2 we give a list of facts from the geometry of partial flag spaces, describe the Bruhat order and introduce Schubert cells; all this is used in the rest of the paper. In Section 3 we consider the FS Toda lattice and describe how it induces a gradient flow on partial flag manifolds so that some elementary facts from the Morse theory are applicable. Finally, in Section 4.1 we give two simple explicit examples that illustrate our theorem and prove the main results of this paper (see Section 4.3).

1.3 Notation and assumptions

In what follows, unless otherwise stated, all manifolds will be assumed smooth and compact (without boundary), all vector spaces are assumed to be real and finite-dimensional.

We shall also consider the full symmetric Toda system in a generic dimension n, so we let L denote the real symmetric $n \times n$ Lax matrix of the system and Λ the diagonal matrix of eigenvalues of L. As L is symmetric, its eigenvalues are real; so we assume that they are ordered naturally in Λ.

We shall use the notation $\mathrm{O}(n,\mathbb{R})$ (respectively $\mathrm{SO}(n,\mathbb{R})$) for the group of n-dimensional orthogonal matrices (respectively the group of orthogonal matrices with positive determinant), and $\mathfrak{so}(n)$ will denote its Lie algebra that is the space of real antisymmetric $n \times n$-matrices. Similarly $\mathrm{SL}(n,\mathbb{R})$ will denote the n-dimensional special linear group over real numbers and

$B_n^+ \subset \mathrm{SL}(n,\mathbb{R})$ (respectively B_n^-) the Borel group of upper (respectively lower) triangular matrices with unit determinant.

2 Bruhat order and Schubert cells

The notion of the Bruhat order, Schubert cells and their generalizations have long been crucial instruments in the research of geometry of Lie groups and homogenous spaces. In what follows we give a brief introduction to the subject. In most part of this section we draw on the exposition from classical books, see [14] and references therein.

2.1 Flag spaces, Grassmanians and their generalizations

First we recall some definitions:

Definition 2.1. Let $I = (i_1, i_2, \dots, i_k)$ be a set of positive integers, $i_1 + i_2 + \cdots + i_k = n$. Then the real partial flag space $\mathrm{Fl}_{i_1,i_2,\dots,i_k}(\mathbb{R})$ is the set of all sequences of vector subspaces

$$\left\{0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_k = \mathbb{R}^n\right\}$$

such that $\dim V_l = i_1 + i_2 + \cdots + i_l$.

Let us make a few remarks before we proceed. First of all, similar definitions work literally for vector spaces over arbitrary field $\mathbb{k}$ giving us the flag spaces $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{k})$. These spaces are easy to define on the level of sets; however, their topology depends a lot on $\mathbb{k}$. Below we shall work only with $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$ with a usual topology. Second, an important particular case of this construction is $k = n$, $i_1 = i_2 = \cdots = i_n = 1$, which is called the *full flag manifold* and is denoted by $\mathrm{Fl}_n(\mathbb{R})$ (or in a general case $\mathrm{Fl}_n(\mathbb{k})$). Another important particular case of the flag space is the case $k = 1$, i.e., when there is only one subspace chosen inside $\mathbb{R}^n$. In this case the flag manifold is called *the Grassman space* or just *Grassmannian* and is denoted by $\mathrm{Gr}_{d,n}(\mathbb{R})$, where $d = i_1$ is the dimension of the subspaces we choose.

An important property of flag spaces is that they are homogeneous spaces, the quotient spaces of the groups of linear transformations of $\mathbb{R}^n$ by their subgroups. The isomorphism is induced by the choice of a compatible basis in the subspaces V_i. In particular,

$$\mathrm{Fl}_n(\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/B_n^+, \qquad \mathrm{Gr}_{d,n}(\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/P,$$

for a *maximal parabolic subgroup* $P \subset \mathrm{SL}(n,\mathbb{R})$. Similarly, the general partial flag spaces are isomorphic to the quotient spaces of $\mathrm{SL}(n,\mathbb{R})$ by a suitable parabolic subgroup. On the other hand, if we choose an orthogonal basis of the corresponding subspaces so that the orientation on $\mathbb{R}^n$ would match with the given one, we obtain homeomorphisms with the quotients of a special orthogonal group. For instance

$$\begin{aligned}
\mathrm{Fl}_n(\mathbb{R}) &= \mathrm{O}(n;\mathbb{R})/T_n = \mathrm{SO}(n,\mathbb{R})/T_n^+,\\
\mathrm{Gr}_{d,n}(\mathbb{R}) &= \mathrm{O}(n;\mathbb{R})/\mathrm{O}(d;\mathbb{R}) \times \mathrm{O}(n-d;\mathbb{R})\\
&= \mathrm{SO}(n;\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \bigcap \left(\mathrm{O}(d;\mathbb{R}) \times \mathrm{O}(n-d;\mathbb{R})\right).
\end{aligned}$$

Here we have denoted by T_n the group of diagonal matrices with eigenavalues equal to ± 1, and T_n^+ the intersection $\mathrm{SO}(n,\mathbb{R}) \cap T_n$. In both cases we see that the groups $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(n,\mathbb{R})$ act transitively on the flag spaces.

The important rôle of the flag spaces in our investigation follows from the next observation:

Proposition 2.2. *The space of real symmetric $n \times n$ matrices with a fixed set of eigenvalues $\lambda_1, \dots, \lambda_k$ with multiplicities $i_1, \dots, i_k$ can be identified with the partial flag manifold* $\mathrm{Fl}_{i_1,\dots,i_k}$.

Before we *prove* this in full generality, let us consider the simplest case: $n = 3$. Let $\alpha < \beta$ be real numbers. Then the set of all symmetric 3×3 matrices with eigenvalues α, α, β is equal to the orbit of the diagonal matrix $\mathrm{diag}(\alpha, \alpha, \beta)$ under conjugations by the elements of $\mathrm{SO}(3, \mathbb{R})$. This action is not free, the stabilizer of $\mathrm{diag}(\alpha, \alpha, \beta)$ being equal to the subgroup $\widetilde{\mathrm{SO}}(2, \mathbb{R}) \subset \mathrm{SO}(3, \mathbb{R})$ of matrices Ψ that have one of the following forms:

$$\Psi = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \Psi = \begin{pmatrix} \cos t & -\sin t & 0 \\ -\sin t & -\cos t & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

So the orbit is equal to the quotient space of the sphere $\mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(2, \mathbb{R}) = S^2$ by the antipodal action of $\mathbb{Z}/2\mathbb{Z}$; to see this observe that the action is induced from the conjugation by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In fact, the subgroup $\widetilde{\mathrm{SO}}(2, \mathbb{R}) \subset \mathrm{SO}(3, \mathbb{R})$ is generated by A and the subgroup of rotations around the Oz axis. Since the homeomorphism $\mathrm{SO}(3, \mathbb{R})/\mathrm{SO}(2, \mathbb{R}) = S^2$ is given by the image of the point $(0, 0, 1)$ under the action of $\mathrm{SO}(3, \mathbb{R})$, we see that the action of A on S^2 is given by the antipodal map. So the quotient space that we need is $S^2/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}P^2$. As one can see this construction amounts to sending each matrix L from this set to the straight line spanned by the eigenvectors corresponding to β. Alternatively, one can consider $\mathbb{R}P^2$ in the dual sense as the space of all 2-dimensional subspaces in $\mathbb{R}^3$, and identify L with the eigenspace corresponding to α.

More generally, assume that the diagonal matrix Λ has $k < n$ coinciding eigenvalues, and all the other eigenvalues of Λ are distinct. Without loss of generality we can think that $\lambda_1 = \lambda_2 = \cdots = \lambda_k$. Then reasoning just as before we can identify the set of symmetric matrices with such eigenvalues with the quotient space of $\mathrm{SO}(n, \mathbb{R})$ modulo the subgroup $\widetilde{\mathrm{SO}}(k, \mathbb{R})$ generated by $\mathrm{SO}(k, \mathbb{R})$ (orthogonal transformations of the subspace $\mathbb{R}^k \subset \mathbb{R}^n$ spanned by the first k-axes) and the subgroup of diagonal matrices in $\mathrm{SO}(n, \mathbb{R})$. This subgroup is equal to the intersection of $\mathrm{SO}(n, \mathbb{R})$ and the cartesian product

$$\mathrm{O}(k) \times \underbrace{\mathrm{O}(1) \times \cdots \times \mathrm{O}(1)}_{n-k \text{ times}},$$

and the quotient space $\mathrm{SO}(n, \mathbb{R})/\widetilde{\mathrm{SO}}(k, \mathbb{R})$ is equal to the partial flag space $\mathrm{Fl}_{k,1,\ldots,1}$ (with $n - k$ units):

$$\mathrm{Fl}_{k,1,\ldots,1} = \bigl\{0 \subset W \subset V_1 \subset V_2 \subset \cdots \subset V_{n-k} = \mathbb{R}^n\bigr\},$$

where $\dim W = k$, $\dim V_i = k + i$.

Finally, consider the most general case. Assume that the eigenvalues of Λ are divided into several "clusters": $\lambda_1 = \cdots = \lambda_{i_1}, \lambda_{i_1+1} = \cdots = \lambda_{i_1+i_2}, \ldots, \lambda_{i_1+\cdots+i_{k-1}} = \cdots = \lambda_{i_1+\cdots+i_k}$, where $i_1 + i_2 + \cdots + i_k = n$. Then the orbit $\Psi\Lambda\Psi^{-1}$ will be equal to the partial flag space $\mathrm{Fl}_{i_1,\ldots,i_k}$:

$$\mathrm{Fl}_{i_1,i_2,\ldots,i_k} = \bigl\{0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_k = \mathbb{R}^n\bigr\},$$

where $\dim V_l = i_1 + i_2 + \cdots + i_l$. This can be proved either by the considerations of the symmetry (i.e., by finding the subgroup of matrices commuting with Λ) as before or one can use the following observation: every symmetric matrix $L \in \mathrm{Symm}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_k$ of multiplicities $i_1, \ldots, i_k$ is uniquely determined by the collection of eigenspaces:

$$L_j = \bigl\{0 \neq v \in \mathbb{R}^n \mid Lv = \lambda_j v\bigr\}, \qquad j = 1, \ldots, k.$$

Clearly $\dim L_j = i_j$. So one naturally identifies this matrix with a point in $\mathrm{Fl}_{i_1,\dots,i_k}$ by putting $V_l = L_1 \oplus L_2 \oplus \cdots \oplus L_l$.

An important property of the partial flag spaces is the existence of surjective projections

$$\pi\colon\ \mathrm{Fl}_n(\mathbb{R}) \to \mathrm{Fl}_{i_1,i_2,\dots,i_n}(\mathbb{R}) \tag{2.1}$$

given by "forgetting" the unnecessary subspaces. Alternatively, these projections can be regarded as the additional factorization

$$\mathrm{Fl}_n(\mathbb{R}) = \mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \bigcap (\underbrace{\mathrm{O}(1)\times\cdots\times\mathrm{O}(1)}_{n\ \text{times}}),$$
$$\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R}) = \mathrm{SO}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \bigcap (\mathrm{O}_{i_1}(\mathbb{R})\times\cdots\times\mathrm{O}_{i_k}(\mathbb{R})).$$

It follows from this description that π is a locally trivial fibre bundle with the fibre equal to

$$X = \mathrm{SO}(n,\mathbb{R}) \bigcap (\mathrm{O}_{i_1}(\mathbb{R})\times\cdots\times\mathrm{O}_{i_k}(\mathbb{R}))/\mathrm{SO}(n,\mathbb{R}) \bigcap (\underbrace{\mathrm{O}(1)\times\cdots\times\mathrm{O}(1)}_{n\ \text{times}}).$$

Or using the notation we introduced earlier

$$X = \mathrm{SO}(n,\mathbb{R}) \bigcap (\mathrm{O}_{i_1}(\mathbb{R})\times\cdots\times\mathrm{O}_{i_k}(\mathbb{R}))/T_n^+.$$

2.2 Bruhat order in S_n and full flag spaces

In this paragraph we will closely follow the exposition in [3, 14], with only a few notations changed. Let ω be a permutation,

$$\omega \in S_n, \qquad \omega\colon\ \{1,2,\dots,n\} \to \{1,2,\dots,n\}.$$

This permutation can be abbreviated to $(\omega(1),\dots,\omega(n))$. One defines the *length* of the permutation ω as the total number of involutions in the sequence $(\omega(1),\dots,\omega(n))$; that is,

$$l_\omega = \#\{j_1 < j_2 \,|\, \omega(j_2) < \omega(j_1)\}.$$

Let the numbers $r_\omega[p,q]$ be equal to the "number of involutions with respect to p, q":

$$r_\omega[p,q] = \#\{j \le p \,|\, \omega(j) \ge q\}, \qquad 1 \le p,q \le n. \tag{2.2}$$

There are many equivalent definitions of the Bruhat order on permutations; we give the following (see [3]):

Definition 2.3. The *Bruhat order* on S_n is the partial order determined by the following relation: for any two permutations u and v in S_n, one says that u precedes v ($u \prec v$) if and only if

$$r_u[p,q] \le r_v[p,q] \qquad \text{for all } p,\ q.$$

A simple way to compute $r_\omega[p,q]$ is to consider the matrix A_ω representing the permutation ω:

$$(A_\omega)_{ij} = \begin{cases} 1, & \omega(j) = n-i+1, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that for all p, q the number $r_\omega[p,q]$ is equal to the rank of the submatrix $A_\omega^{pq} = ((A_\omega)_{ij})_{i\le q,\, j\le p,}$ of A_ω (informally one can say that A_ω^{pq} is the submatrix in the upper left corner of A_ω determined by the element a_{pq}). Since there is only one nonzero entry in every row and column of A_ω, it is enough to count the number of such nonzero elements in A_ω^{pq}. E.g., for $\omega = (4213)$ we obtain $r_\omega[3,2] = 2$ from the following table (we replace 1 by $\bullet$):

4	$\bullet$			
3				$\bullet$
2		$\bullet$	$\leftarrow\uparrow$	
1			$\bullet$	
	1	2	3	4

The following lemma is proved in [14]:

Lemma 2.4. *Let $u \prec v$, $u \neq v$. Let j, $1 \leq j \leq n$ be the smallest integer, for which $u(j) \neq v(j)$ (and hence $u(j) < v(j)$). Let $n \geq k > j$ be the least integer for which $u(j) \leq v(k) < v(j)$ and let $v' = v \cdot (j,k)$ denote the composition of v with the swap of j and k. Then $u \prec v' \prec v$.*

The relation between this order and the geometry of the full flag space is determined by the structure of the Schubert cell decomposition of this space. There are many ways to define the Schubert cells, we shall use the following one (more definitions can be found in the literature, see [3, 4, 14] and cf. Section 2.3).

Definition 2.5. Embed the group S_n into $\mathrm{SL}(n,\mathbb{R})$ as it is explained above (one should only replace 1 in the definition of A_ω to ± 1 so as to make sure that the determinant is equal to 1 and not -1). Using the projection

$$p\colon\ \mathrm{SL}(n,\mathbb{R}) \to \mathrm{Fl}_n(\mathbb{R}) = \mathrm{SL}(n,\mathbb{R})/B_n^+,$$

we obtain the points $p(A_\omega) \in \mathrm{Fl}_n(\mathbb{R})$, which we shall denote by $[A_\omega]$. The subgroup B_n^+ of $\mathrm{SL}(n,\mathbb{R})$ acts on $\mathrm{Fl}_n(\mathbb{R})$ and *the Schubert cell $X_\omega \subset \mathrm{Fl}_n(\mathbb{R})$ for $\omega \in S_n$* is the orbit of $[A_\omega]$ with respect to this action:

$$X_\omega = B_n^+ \cdot [A_\omega] \subseteq \mathrm{Fl}_n(\mathbb{R}).$$

One can show (cf. [14] and Section 2.3) that X_ω is indeed a cell. In effect,

$$X_\omega \cong \mathbb{R}^{l(\omega)}.$$

The closure $\overline{X}_\omega$ of X_ω in the flag space is a singular algebraic variety. It is called *the Schubert variety*. The following statement explains the relation of the Schubert cells and the Bruhat order (see [14, Proposition 7, p. 175]):

Proposition 2.6. *An element $w \in S_n$ precedes $v \in S_n$ with respect to the Bruhat order, $w \prec v$, if and only if $\overline{X}_w \subseteq \overline{X}_v$.*

This proposition is sometimes used to give an alternative definition of the Bruhat order: *the Bruhat order is the partial order induced by the contiguity of the cells in the Schubert cell decomposition of the flag manifold.*

There is another important observation relating the Schubert cells and Bruhat order. Namely, consider the B_n^--orbits of the same points $[A_\omega]$:

$$X_\omega^\vee = B_n^- \cdot [A_\omega].$$

They are called *the dual Bruhat cells*. These sets are as well homeomorphic to the Euclidean spaces, their closures are called *the dual Schubert varieties*. Then the following is true (see again [14]):

Proposition 2.7. *Let $v, w \in S_n$ be two elements. Then $X_v \bigcap X_w^\vee \neq \varnothing$ if and only if $v \prec w$ in Bruhat order. In the latter case the intersection of cells is transversal.*

This property of the Schubert cells was crucial in our description of the asymptotic behavior of the FS Toda lattice, see [5]. Below we shall make use of analogous properties of the partial flag spaces.

2.3 Bruhat order on multiset permutations

In this section we give a description of the Schubert cell decomposition of the partial flag space and the Bruhat order associated with it. Since we were not able to find a combinatoric description of this order in literature, we do not confine ourselves here to mere definitions and statement of results, and give proofs of a few facts (in particular, see Proposition 2.10). Besides this one can refer to [4] and book [3] for more details.

Let $n > k$ be two natural numbers and let $I = (i_1, i_2, \dots, i_k)$ be a partition of n into k parts, i.e., for each $j = 1, \dots, k$ we let i_j be a positive integer so that

$$i_1 + i_2 + \cdots + i_k = n.$$

Then we give the following definition:

Definition 2.8. An *I-permutation of multiset* (or permutation with multiplicities, or permutation with repetitions) is any surjective map

$$\tau\colon \ \{1, 2, \dots, n\} \to \{1, 2, \dots, k\},$$

such that $|\tau^{-1}(j)| = i_j$ for all $j = 1, \dots, k$. We shall denote the set of all such permutations by S_n^I.

One can regard an element $\tau \in S_n^I$ as a string of elements of the form

$$\tau = (\tau(1), \tau(2), \dots, \tau(n)),$$

where $1 \leq \tau(j) \leq k$ and every element j, $1 \leq j \leq k$ appears exactly i_j times (which explains the name of these objects that we use).

Below we shall make an extensive use of the following map $\tau_1^*\colon S_n \to S_n^I$, where $\tau_1 \in S_n^I$ is the following multiset permutation which we regard as a map

$$\tau_1\colon \ \{1, 2, \dots, n\} \to \{1, 2, \dots, k\},$$

$$\tau_1(l) = \begin{cases} 1, & 1 \leq l \leq i_1, \\ 2, & i_1 + 1 \leq l \leq i_1 + i_2, \\ & \dots\dots\dots\dots\dots\dots \\ j, & i_1 + \cdots + i_{j-1} + 1 \leq l \leq i_1 + \cdots + i_{j-1} + i_j, \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ k, & i_1 + \cdots + i_{k-1} + 1 \leq l \leq n. \end{cases}$$

Then for every permutation $w \in S_n$ we define the multiset permutation $\tau_1^*(w) = \tau_w$ by the formula

$$\tau_w = \tau_1 \circ w,$$

or in the "linear form":

$$\tau_w = (\tau_1(w(1)), \tau_1(w(2)), \dots, \tau_1(w(n))).$$

The map τ_1^* is evidently an epimorphism and

$$(\tau_1^*)^{-1}(\tau_1) = S_{i_1} \times S_{i_2} \times \cdots \times S_{i_k} \subseteq S_n,$$

where we embed S_{i_j} into S_n by letting it permute the elements l, $i_1+\cdots+i_{j-1} < l \leq i_1+\cdots+i_j$. For all other elements $\tau \in S_n^I$ we see that $(\tau_1^*)^{-1}(\tau)$ is a right coset of S_n by the subgroup $S_{i_1}\times S_{i_2}\times\cdots\times S_{i_k}$, i.e., the set of all permutations of the form $x_\tau \cdot w$ for any $w \in S_{i_1}\times S_{i_2}\times\cdots\times S_{i_k}$ and some $x_\tau \in S_n$ such that $\tau_1^*(x_\tau) = \tau$. We shall sometimes call these cosets *clusters in* S_n, *corresponding to* $\tau \in S_n^I$; clearly every such cluster contains $N = i_1!i_2!\cdots i_k!$ elements. One can describe elements in S_n^I in terms of clusters; for instance, the string $(1,2,2,1) \in S_4^{(2,2)}$ corresponds to the cluster $\{(1,3,4,2), (2,3,4,1), (1,4,3,2), (2,4,3,1)\}$.

Now one uses these constructions to introduce the Bruhat order on S_n^I. Loosely speaking it is obtained by "pulling back" from S_n. However, this procedure is not always well defined, so we give some details.

First of all, just as in Section 2.2 we introduce the notion of *length* of a permutation with repeating elements: let $\tau\colon \{1,2,\ldots,n\} \to \{1,2,\ldots,k\}$ be an element in S_n^I, then

$$l_\tau = \#\{j_1 < j_2 \,|\, \tau(j_2) > \tau(j_1)\},$$

i.e., l_τ is the number of involutions in the sequence $(\tau(1),\ldots,\tau(n))$. Observe that we do not count the pairs $j_1 < j_2$ for which $\tau(j_1) = \tau(j_2)$.

Second, similarly to the Definition 2.3 we define the Bruhat order on S_n^I with the help of ranks of some matrices: consider the $n \times k$ matrices A_τ corresponding to the elements of S_n^I. Just as before we define the numbers $r_\tau[p,q]$ (see formula (2.2)) for all $1 \leq p \leq n$ and $1 \leq q \leq k$. Then:

Definition 2.9. We say that $\tau \prec \upsilon$ (where both $\tau, \upsilon \in S_n^I$) if and only if $r_\tau[p,q] \leq r_\upsilon[p,q]$ for all p, q. Observe that we use $\leq$ rather than the strict inequality $<$ here. Of course at least one inequality must be strict if $\tau \neq \upsilon$

The following example illustrates this definition very well: let $I = (2,2)$ as before and $\tau = (2,1,1,2)$. Then A_τ has this shape:

2	$\bullet$			$\bullet$
1		$\bullet$	$\bullet$	
	1	2	3	4

Similarly, the lengths of the elements of $S_4^{(2,2)}$ are equal to

$$\begin{array}{llllll}
(1,1,2,2), & l_{(1,1,2,2)} = 0, & (1,2,1,2), & l_{(1,2,1,2)} = 1, & (1,2,2,1), & l_{(1,2,2,1)} = 2,\\
(2,1,1,2), & l_{(2,1,1,2)} = 2, & (2,1,2,1), & l_{(2,1,2,1)} = 3, & (2,2,1,1), & l_{(2,2,1,1)} = 4.
\end{array}$$

These data are sufficient to draw the Hasse diagram of the Bruhat order in this case (cf. Fig. 2)[1].

The following propositions (Propositions 2.10 and 2.12, and Lemma 2.11) explain, in what sense one can "pull back" the order from S_n to S_n^I.

Proposition 2.10. *Let $u, v \in S_n$ be two permutations such that $u \prec v$ in the Bruhat order on S_n and $\tau_1^*(u) \neq \tau_1^*(v)$. Then $\tau_1^*(u) \prec \tau_1^*(v)$ in the Bruhat order in S_n^I.*

[1] Recall that the Hasse diagram of a partially ordered set is the oriented graph whose vertices are given by the elements of the set and edges connect elements a and b if and only if $a \prec b$ and there is no c such that $a \prec c \prec b$.

Proof. Let us denote $\tau_1^*(u) = U$ and $\tau_1^*(v) = V$; we will show that $U \prec V$. First of all we observe that it is enough to show this in a particular case

$$v = u \cdot (i,j), \qquad l(v) = l(u) + 1$$

(here (i,j) denotes the transposition of i and j for $i < j$). Indeed, it follows directly from Lemma 2.4 that every pair of Bruhat-comparable elements in S_n can be connected by a "path" of intermediate elements verifying this condition at every stage.

So let u and v be as above. Since $u \prec v$ we have $r_u[p,q] \le r_v[p,q]$. We shall show that $r_U[p,q] \le r_V[p,q]$ for all $1 \le p \le n$, $1 \le q \le k$. It is impossible that all the numbers $r_U[p,q]$ and $r_V[p,q]$ are equal (we assumed that $U \ne V$) and so we have that $U \prec V$. Recall now that we assumed that $v = u(i,j)$, $i < j$ and $l(v) = l(u) + 1$. This means, in particular, that $u(i) < u(j)$ (otherwise the number of inversions would decrease from multiplication by (i,j)). Let $u(i) = a$, $u(j) = b$ then $v(i) = b$, $v(j) = a$; let $i < l < j$ and for $c = u(l)$ there are three possibilities: $c < a < b$, $a < b < c$ or $a < c < b$. Counting the number of inversions in all three cases, we see that only the first two options are possible (in the third case the number of inversions will change at least by 3 when we swap i and j).

In terms of the rectangular matrices associated with the elements of S_n we can say that the rectangle P_{abij} cut from A_u by the a-th and b-th rows and i-th and j-th columns contains no nonzero elements, see (2.3).

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
n & & & & & & & & & & & \\ \hline
\dots & & & & & & & & & & & \\ \hline
b & & & & & & & & & 1 & & \\ \hline
 & & & & & & & & & & & \\ \hline
 & & & & & & & & & & & \\ \hline
 & & & & & & & & & & & \\ \hline
a & & & & 1 & & & & & & & \\ \hline
\dots & & & & & & & & & & & \\ \hline
2 & & & & & & & & & & & \\ \hline
1 & & & & & & & & & & & \\ \hline
 & 1 & 2 & \dots & i & & & & & j & \dots & n
\end{array} \tag{2.3}$$

Clearly, no nonzero elements will appear in this rectangle when we swap i and j.

When we pass from u and v to U and V, this matrix "shrinks" vertically and the interior of the corresponding rectangle $P_{\tau_1(a)\tau_1(b)ij}$ in the matrices A_U and A_V does not contain nonzero elements: new nonzero elements can appear only in the top and bottom rows of the rectangle (i.e., in the rows number $\tau_1(a)$ and $\tau_1(b)$) and this can happen only simultaneously for U and V. Hence the matrices A_U and A_V differ only by positions of 1's in the corners of $P_{k_a k_b ij}$: in A_U they stand at the entries (k_a, i) and (k_b, j), while in A_V they stand at (k_a, j) and (k_b, i). Now a simple inspection shows that $r_U[p,q] < r_V[p,q]$ for all p and q, see (2.4).

$$A_U = \begin{array}{c|c|c|c|c|c|c|c|}
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
k_b & & & & \dots & & 1 & \\ \hline
 & & & & \dots & & & \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
 & & & & \dots & & & \\ \hline
k_a & & 1 & & \dots & & & \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
 & \dots & i & & \dots & & j & \dots
\end{array} \tag{2.4}$$

$$
A_V = \begin{array}{c|c|c|c|c|c|c|c|}
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\hline
k_b & & 1 & & \dots & & & \\
\hline
 & & & & \dots & & & \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\hline
 & & & & \dots & & & \\
\hline
k_a & & & & \dots & & 1 & \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\hline
 & \dots & i & & \dots & & j & \dots
\end{array}
$$
■

The following statement is similar to Lemma 2.4; it gives a procedure by which we can move between comparable elements of S_n^I.

Lemma 2.11. *Let $U, V \in S_n^I$, $U \prec V$, $U \neq V$. Let j_0 be the least j, $1 \leq j \leq n$, for which $U(j_0) \neq V(j_0)$; then $U(j_0) < V(j_0)$. Let $l \geq j_0$ be the maximal number for which $V(j_0) = V(j_0+1) = \cdots = V(l)$ so that $U(j_0) < V(l)$. Let m denote the least integer $l \leq m \leq n$ verifying the inequality $U(j) \leq V(m) < V(l)$. Put $V' = V(l,m)$, i.e., in terms of the corresponding strings V' is V in which the values of $V(l)$ and $V(m)$ are swapped. Then $U \prec V' \prec V$.*

The proof is by direct inspection of the definitions and we omit it. Finally we use this procedure to move "in the opposite direction": from S_n^I to S_n.

Proposition 2.12. *Every coset $(\tau_1^*)^{-1}(v) \subset S_n$, $v \in S_n^I$ contains the unique least element $u \in S_n$ (with respect to the Bruhat order in S_n) so that if $\xi \prec \zeta$ in S_n^I, then a similar inequality holds for the corresponding least elements in S_n: $x \prec z$.*

Recall that the *least element* in a partial order is the element, which is less than all others, in particular, it is comparable with all others. It is clear that it is unique if it exists.

Proof. Observe that every coset $(\tau_1^*)^{-1}(v)$ contains a unique element s_v such that s_v^{-1} is a I-*shuffle*, i.e., a permutation which preserves the order of the elements inside the "blocks" of the partition I. We shall call such s_v the I-*deshuffle*; clearly, this deshuffle is the least element that we need. This follows easily from the fact that for any permutation $u \in (\tau_1^*)^{-1}(v)$ and any $i < k < j$ such that $v(i) = v(j) \neq v(k)$ we shall have either $u(k) < u(i)$, $u(k) < u(j)$, or $u(i) < u(k)$, $u(j) < u(k)$ (i.e., $u(k)$ cannot lie between $u(i)$ and $u(j)$). Hence, the order in $(\tau_1^*)^{-1}(v)$ (induced from S_n) depends only on the permutation of the elements inside separate blocks of the partition, and not on v. In particular, the shuffle corresponds to the "unit" with respect to the permutations inside the blocks; hence, it is least.

Consider now arbitrary $U, V \in S_n^I$ and let s_U, s_V be the corresponding deshuffles. By Lemma 2.11 we can assume that V is obtained from U by swapping two elements chosen as it is explained in its condition: $V = U(l,m)$. Then, as one can see $s_V = s_U(l,m)$: clearly $s_U(l,m) \in (\tau_1^*)^{-1}(V)$ and the fact that it coincides with s_V follows from the choice of l and m, see Lemma 2.11. Similarly, from the same choice it follows that all the elements in the string corresponding to s_U that lie between l and m, are either greater than both $s_U(l)$ and $s_U(m)$, or less than both. Hence,

$$l_{s_V} = l_{s_U(l,m)} = l_{s_U} + 1.$$
■

Observe that we have in fact established that the Bruhat order on S_n, when restricted to *clusters* $(\tau_1^*)^{-1}(v)$, $v \in S_n^I$, coincides with the *lexicographic order* on the direct product $S_{i_1} \times S_{i_2} \times \cdots \times S_{i_k}$ where each factor is equipped with the usual Bruhat order on S_l. In particular, there is a unique *greatest* element in it too.

2.4 Schubert cells in partial flags and Bruhat order

The definition of the Schubert cells given in Section 2.2 can be extended to a more general situation of the partial flag spaces, in particular to Grassmannians, so that the canonical projection (2.1) maps the Schubert cells in $\mathrm{Fl}_n(\mathbb{R})$ to the cells in the partial flags. We shall use this observation and the properties of the Bruhat order to describe some of the structures of the Schubert cells, their duals and their intersections in the partial flag spaces. The main references for this section are [4, 14, 15].

We begin with the general definition of the Schubert cells in an important particular case: $k = 1$, i.e., the flag space is equal to a Grassmannian $\mathrm{Gr}_{d,n}(\mathbb{R})$, the space of d-planes in $\mathbb{R}^n$.

Let $\{e_1, \dots, e_n\}$ be a basis of $\mathbb{R}^n$. Any d-dimensional hyperplane inside $\mathbb{R}^n$ is determined by a collection of d linearly-independent vectors in $\mathbb{R}^n$, which is the same (when the basis of $\mathbb{R}^n$ is fixed) as a $d \times n$, $d < n$ matrix with a maximal rank. It is clear that two matrices Λ, Λ' determine the same d-plane if and only if there exists $g \in \mathrm{GL}(d, \mathbb{R})$ such that:

$$\Lambda' = g \cdot \Lambda.$$

We shall use the same symbol Λ to denote the d-plane corresponding to the matrix Λ. Observe that the natural $\mathrm{GL}(n, \mathbb{R})$-action on the Grassmannian translates into the right multiplication of Λ by the matrices $B \in \mathrm{GL}(n, \mathbb{R})$. Also observe that in our notation the left and right actions exchange their roles: in Section 2.3 we define $\mathrm{Gr}_{d,n}$ as the quotient of $\mathrm{SL}(n, \mathbb{R})$ by the right action of a parabolic subgroup so that $\mathrm{GL}(n, \mathbb{R})$ acts on it from the left, and when we draw rectangular matrices, the action and factorization change directions. This can be amended by the use of the $n \times d$ matrices instead of the $d \times n$ matrices.

Let for any multi-index $A = (a_1, \dots, a_d)$, $1 \le a_1 < a_2 < \dots < a_d \le n$ symbol V_A denote the subspace in $\mathbb{R}^n$ spanned by $\{e_{a_1}, e_{a_2}, \dots, e_{a_d}\}$. Then

Definition 2.13. The Schubert cell X_A is the B_n^+-orbit of the point V_A in Grassmannian $\mathrm{Gr}_{d,n}(\mathbb{R})$.

One can give a more geometric definition of the Schubert cells. To this end consider the subspaces V_i, $i = 1, \dots, n$, $V_i \subseteq \mathbb{R}^n$ spanned for each i by the vectors $\{e_1, \dots, e_i\}$ and consider for an arbitrary d-plane $\Lambda \in \mathrm{Gr}_{d,n}(\mathbb{R})$ the intersections:

$$0 \subset \Lambda \cap V_1 \subset \Lambda \cap V_2 \subset \dots \subset \Lambda \cap V_{n-1} \subset \Lambda \cap V_n = \Lambda.$$

Then for the same multi-index $A = (a_1, \dots, a_d)$ we have

Definition 2.14. The Schubert cell X_A is the set of such planes $\Lambda \in \mathrm{Gr}_{d,n}(\mathbb{R})$ that

$$\dim(\Lambda \cap V_j) = \#\{k \,|\, 1 \le k \le d,\ a_k < j\}, \qquad \text{for all } j = 1, \dots, n.$$

The rectangular matrices Λ corresponding to the planes in X_A can be chosen in a special form:

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & 0 & * & 0 & \cdots & \cdots & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & * & 0 & \cdots & \cdots & * \\ \cdots & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & \cdots & * \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & 0 & 1 & * & \cdots & * \end{pmatrix}.$$

Here 1's stand in the intersections of the i-th rows and $(n - a_{d-i+1} + 1)$-th columns and are the only nonzero elements of the column; asterisks are used to denote arbitrary real numbers. This flip of indices is caused by the fact that we have substituted the left action of B_n^+ for the right.

As one can see the sets X_A are indeed cells: it follows from the shape of the matrices we use here that

$$X_A \cong R^{dn-\sum(n+i-a_i)}.$$

Here dn is the dimension of the space of the $d\times n$ matrices and we subtract the number of fixed elements in a matrix. One can also introduce the "length" of the sequence A by putting

$$l(A) = \sum_{k=1}^{d} a_k - k.$$

Then it is easy to see that $l(A) = dn - \sum(n+i-a_i)$ and so $X_A \cong \mathbb{R}^{l(A)}$. It turns out that $\bigcup_A X_A$ is a cell decomposition of $\mathrm{Gr}_{d,n}(\mathbb{R})$. Moreover, one can show (see [4]) that a cell X_B is adjacent to X_A if and only if $b_k \le a_k$ for all k. In this case we shall write $B \le A$; this is a partial order on the set of all sequences.

Finally, we associate a sequence A with the multiset permutation $\omega \in S_n^{(d,n-d)}$ given by

$$\omega(a_1) = \cdots = \omega(a_d) = 1, \qquad \omega(j) = 2, \qquad j \notin \{a_1,\ldots,a_d\}.$$

It is easy to see that the Bruhat order on $S_n^{(d,n-d)}$ corresponds to the partial order given by the inequalities $B \le A$.

Example 2.15. Consider the Grassmannian $\mathrm{Gr}_{2,4}(\mathbb{R})$, the space of all real 2-planes in $\mathbb{R}^4$. In this case there are 6 cells corresponding to the sequences $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$ and $(3,4)$. The corresponding Schubert cells are spanned by the matrices of the following shapes (we use the equality signs to denote this):

$$C_{(1,2)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C_{(1,3)} = \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C_{(1,4)} = \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$C_{(2,3)} = \begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \qquad C_{(2,4)} = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}, \qquad C_{(3,4)} = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}.$$

In the general case of an arbitrary partial flag space $\mathrm{Fl}_{i_1,\ldots,i_k}(\mathbb{R})$ (which we shall often abbreviate to just $\mathrm{Fl}_I(\mathbb{R})$) we can define the Schubert cells decomposition similarly by choosing a basis $\{e_1,\ldots,e_n\}$ of $\mathbb{R}^n$; then for any partial flag

$$E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_k)$$

of vector subspaces in $\mathbb{R}^n$ and any multiset permutation $\omega \in S_n^I$ (where I is a k-partition of n) we say that $E_\bullet$ is in the cell X_ω if and only if

$$\dim(E_p \cap E_q) = \#\{i \le p \,|\, \omega(i) \le q\} \tag{2.5}$$

for all $1 \le p \le n$, $1 \le q \le k$. It is easy to see that these sets are indeed cells, that the cells defined for different $\omega \in S_n^I$ do not intersect and that $\mathrm{Fl}_I(\mathbb{R})$ is equal to their union. The closures of these cells will be denoted by $\overline{X}_\omega$; they are called *the Schubert varieties* in $\mathrm{Fl}_I(\mathbb{R})$. One can describe them by replacing the equalities in (2.5) by the non-strict inequality $\ge$; they are indeed singular algebraic subvarieties in $\mathrm{Fl}_I(\mathbb{R})$.

As before, the same sets can be described as orbits of the group of upper-triangular matrices B_n^+; if we view the partial flag space as the quotient space of $\mathrm{SL}(n,\mathbb{R})$ (or $\mathrm{SO}(n,\mathbb{R})$) by some parabolic subgroup P, we can define the Schubert cells as the orbits of certain elements within $\mathrm{Fl}_I(\mathbb{R})$ with respect to the action of B_n^+ (or some block-diagonal matrices, if we speak about $\mathrm{SO}(n,\mathbb{R})$): take the matrix A_w corresponding to the minimal representative w in S_n of an element $\omega \in S_n^I$ (see Section 2.3), then

$$X_\omega = B_n^+(A_w)P/P \subset \mathrm{SL}(n;\mathbb{R})/P.$$

It clearly follows from this definition that *the Schubert cells in the partial flag spaces are equal to the projections of the Schubert cells in the full flag space under the projection* π, *see* (2.1). Moreover, one can show (see [4]) that they are in effect homeomorphic to the cell of the least element w in $\mathrm{Fl}_n(\mathbb{R})$.

Now by comparing this description with the results of the previous section (see Propositions 2.10 and 2.12 and Lemma 2.11) we obtain the following statement:

Proposition 2.16. *For any two permutations with repetitions $\psi, \omega \in S_n^I$ we have $\psi \prec \omega$ in the Bruhat order if and only if the corresponding cells X_ψ and X_ω are adjacent; on the level of Schubert varieties this implies the inclusion $\overline{X}_\psi \subset \overline{X}_\omega$.*

Finally, similarly to the case of the full flag space $\mathrm{Fl}_n(\mathbb{R})$ in $\mathrm{Fl}_I(\mathbb{R})$ one can also define the dual Schubert cells $X_\omega^\wedge$, $\omega \in S_n^I$ (and the corresponding Schubert varieties). This can be done either by a modification of the rank (in)equalities (2.5), or as orbits of the corresponding elements in $\mathrm{Fl}_I(\mathbb{R})$ with respect to the action of B_n^-, or finally simply as projections of the dual cells in $\mathrm{Fl}_n(\mathbb{R})$. As before a quick comparison of the definitions proves the following proposition:

Proposition 2.17. *A Schubert cell $X_\psi \subset \mathrm{Fl}_I(\mathbb{R})$ intersects with the dual Schubert cell $X_\omega^\wedge \subset \mathrm{Fl}_I(\mathbb{R})$ if and only if $\psi \prec \omega$ in the Bruhat order in S_n^I and in the latter case the intersection is transversal.*

The proof follows from the observation that $X_\omega^\wedge$ corresponds to the cell determined by the greatest element w' in the cosets $(\tau_1^*)^{-1}(\omega)$ just as X_ω corresponds to the least one: i.e., $X_\omega^\wedge$ is a homeomorphic image of the dual cell corresponding to w' in $\mathrm{Fl}_n(\mathbb{R})$ under the projection π. Since the statement of the Proposition 2.17 holds in $\mathrm{Fl}_n(\mathbb{R})$, it must hold in $\mathrm{Fl}_I(\mathbb{R})$ as well (we must also use the fact that π is submersion).

3 The FS Toda lattice on partial flag spaces

3.1 Fibre bundles and Bott–Morse functions

In our study of the Toda system on symmetric matrices with coinciding eigenvalues we shall need a few statements from the differential geometry of homogeneous spaces of Lie groups. We prove them here for the sake of completeness.

Let M be a smooth closed manifold with a smooth right transitive action of a compact Lie group $\tilde{G}$, $h \in \tilde{G}$, $x \in M$, $x \mapsto x^h$. Let $G \subset \tilde{G}$ be a compact subgroup of $\tilde{G}$. We assume that $\pi\colon M \to M/G$ is a locally trivial bundle (in which the fibre containing x is homeomorphic to the stabilizer of x, $G^x \subset G$); in particular, we assume that $X = M/G$ is a smooth manifold. Let g be a G-invariant Riemannian metric on M and f be a G-invariant smooth function. In this case the gradient vector field $\xi = \mathrm{grad}_g f$ is G-invariant as well. Since g and f are G-invariant they induce a Riemannian structure g_X and a smooth function f_X on X; similarly, the vector field ξ induces a field ξ_X on X; we just put $\xi_X(\pi(m))$ to be equal to the projection $d\pi_m(\xi(m))$ for an arbitrary point m in M.

We shall need the following statement:

Proposition 3.1. *The field ξ_X is equal to the gradient of the function f_X with respect to the metric g_X:*

$$\xi_X = \mathrm{grad}_{g_X} f_X.$$

Proof. The statement follows almost immediately from the definitions of a gradient vector field and induced metric g_X on M/G: recall that the field $\mathrm{grad}_{g_X} f_X$ is characterized by the equality

$$g_X(\eta_X, \mathrm{grad}_{g_X} f_X) = \eta_X(f_X);$$

and that $g_X(\eta_X,\, \zeta_X)$ for two vectors $\eta_X, \zeta_X \in T_x X$ (here $X = M/G$) is defined as

$$g_X(\eta_X, \zeta_X) = g(\eta, \zeta),$$

for any $\eta, \zeta \in T_m M$ ($m \in \pi^{-1}(x)$ is an arbitrary point), such that $d\pi_m(\eta) = \eta_X$, $d\pi_m(\zeta) = \zeta_X$ and they are both perpendicular to the "vertical" subspace $T_m(mG)$ of $T_m M$.

Now since the function f is constant along the fibres of π, the field $\mathrm{grad}_g f$ is orthogonal to the vertical subspaces; so

$$g_X(\eta_X, d\pi(\mathrm{grad}_g f)) = g(\eta, \mathrm{grad}_g f) = \eta(f) = \eta_X(f_X),$$

where the second equality follows from the definition of $\mathrm{grad}_g f$, and the third one from the definition of f_X (it is enough to consider the local structure of a cartesian product in M). ■

Below we shall need to know what conditions should be imposed on f to provide that f_X is a Morse function. Observe that in the general case f cannot be a Morse function itself unless the group G is discrete (and finite): because of the G-invariance of f every critical point of f_X in $X = M/G$ will correspond to a G-orbit consisting of critical points of f in M. So let us give a simple criterion for f_X to be Morse:

Proposition 3.2. *Let f be a G-invariant function and g be a G-invariant metric on M. The function $f_X \in C^\infty(M/G)$, induced from f, is a Morse function if and only if at every critical point m of f the Hessian $H(f)$ is nondegenerate when restricted to the orthogonal complement to the space of "vertical" vectors (i.e., vectors tangent to the orbit).*

Proof. First, recall that the gradient of a function is always perpendicular to its level sets. So in the case we consider it to be perpendicular to the G-orbits. It follows that it is not in the kernel of the projection $d\pi_m \colon T_m M \to T_{\pi(m)} M/G$ (as we observed earlier, the gradient of f_X is equal to the projection of the gradient of f). Hence the critical points of f_X correspond to the "critical orbits" of f in M, i.e., to the G-orbits consisting of the critical points of f.

Second, Hessian of f_X at a critical point $x_0 \in X$ is the symmetric quadratic form on tangent space given by the formula

$$H(f_X)_{x_0}(\xi(x_0), \eta(x_0)) = \xi(\eta(f_X))(x_0),$$

where ξ, η are two vector fields in the neighborhood of x_0: it is easy to show that the right-hand side of this formula is indeed symmetric 2-form on vectors in $T_{x_0} X$ if x_0 is singular (in particular, it depends only on the values of ξ and η at the point x_0).

Now the proposition follows from the fact that every vector field ξ on M/G can be locally lifted to a G-invariant vector field $\tilde{\xi}$ in the neighborhood of $\pi^{-1}(x_0) \subset M$ orthogonal to the fibre $\pi^{-1}(x_0)$: to see this it is enough to use the local trivialization $\pi^{-1}(U) = U \times G^{x_0}$ for some open neighborhood U of x_0. So we have the following formula:

$$H(f_X)_{x_0}(\xi(x_0), \eta(x_0)) = \tilde{\xi}(\tilde{\eta}(f))(\tilde{x}_0) = H(f)_{\tilde{x}_0}(\tilde{\xi}(\tilde{x}_0), \tilde{\eta}(\tilde{x}_0)),$$

where $\tilde{x}_0$ is a point in the fibre $\pi^{-1}(x_0)$; the right-hand side of this formula is G-equivariant, so the statement of the proposition follows. ■

As a matter of fact the condition of Proposition 3.2 is a variation of the Morse–Bott condition. Recall that a function $f \in C^\infty(X)$ (X is a smooth manifold) is called *the Morse–Bott function* if its critical set is a smooth (not connected) submanifold in X and the Hessian is nondegenerate in the normal direction to this submanifold. In our case the set of critical points is equal to the G^x-orbits through critical points x of f; hence, it is smooth (by assumption on the nature of the action) and normal direction is the direction orthogonal to the vertical vectors.

3.2 The FS Toda lattice on partial flags

As we have explained in the introduction, the FS Toda lattice in dimension n induces a gradient flow on the orthogonal group $\mathrm{SO}(n,\mathbb{R})$, which we here shall call *the Toda flow* or just *Toda system* on $\mathrm{SO}(n,\mathbb{R})$. This flow is determined by the vector field

$$M(\Psi) = \big(\big(\Psi\Lambda\Psi^{-1}\big)_+ - \big(\Psi\Lambda\Psi^{-1}\big)_-\big)\Psi, \tag{3.1}$$

where Λ is the diagonal matrix of eigenvalues of the symmetric Lax matrix L; in addition we identify the tangent space $T_\Psi\mathrm{SO}(n,\mathbb{R})$ with the right translation of the Lie algebra $\mathfrak{so}_n(\mathbb{R})$ by Ψ. In fact, this vector field is equal to the gradient of a function with respect to an invariant metric on $\mathrm{SO}(n,\mathbb{R})$: for the fixed eigenvalues matrix Λ one can take

$$F(\Psi) = \mathrm{Tr}\,\big(\Psi\Lambda\Psi^T N\big), \qquad \text{where} \quad N = \mathrm{diag}(0,1,\dots,n-1). \tag{3.2}$$

The $\mathrm{SO}(n,\mathbb{R})$-invariant Riemannian structure is determined by its values on $\mathfrak{so}_n$, where it is given by the formula

$$\langle A,B\rangle_J = -\,\mathrm{Tr}\,\big(AJ^{-1}(B)\big), \tag{3.3}$$

for any antisymmetric matrices A and B and a linear isomorphism $J\colon \mathfrak{so}_n \to \mathfrak{so}_n$. Then

$$M(\Psi) = \mathrm{grad}_{\langle\,,\,\rangle_J}\, F,$$

see [5, 7] for details. This property does not depend on the eigenvalues of L, in particular it holds for both distinct and coinciding eigenvalues.

Since all the objects here are T_n^+-invariant (see Section 2.1) the same formulas (3.1), (3.2) and (3.3) induce a vector field, a function and a Riemannian structure on the full flag space

$$\mathrm{Fl}_n(\mathbb{R}) = \mathrm{SO}(n;\mathbb{R})/T_n^+.$$

It follows from the discussion of Section 3.1 that the field M on $\mathrm{Fl}_n(\mathbb{R})$ is equal to the gradient of the corresponding function. It can be shown (see [5, 7]) that the function F is a Morse function on both $\mathrm{SO}(n,\mathbb{R})$ and the flag space $\mathrm{Fl}_n(\mathbb{R})$.

In a similar way one can use the propositions of Section 3.1 to construct gradient flows on the partial flag spaces. Suppose there are coinciding eigenvalues of Λ; for definiteness we can assume that they are

$$\begin{aligned}&\lambda_1 = \lambda_2 = \dots = \lambda_{i_1} < \lambda_{i_1+1} = \lambda_{i_1+2} = \dots = \lambda_{i_1+i_2} < \cdots\\ &\quad < \lambda_{i_1+\dots+i_{k-1}+1} = \lambda_{i_1+\dots+i_{k-1}+2} = \dots = \lambda_{i_1+\dots+i_{k-1}+i_k},\end{aligned} \tag{3.4}$$

where $i_1+i_2+\dots+i_k = n$; that is we assume there are $k<n$ distinct eigenvalues of Λ and the multiplicity of the j-th eigenvalue is i_j. In this case the vector field $M(\Psi)$ on $\mathrm{SO}(n;\mathbb{R})$ is invariant with respect to $\mathrm{O}(i_1,\mathbb{R})\times\dots\times\mathrm{O}(i_k,\mathbb{R})$, i.e., $M(\Psi g) = M(\Psi)g$ for all $g \in \mathrm{SO}(n,\mathbb{R})\bigcap(\mathrm{O}(i_1,\mathbb{R})\times \dots\times \mathrm{O}(i_k,\mathbb{R}))$. So we obtain a vector field $\widetilde{M}$ on the partial flag space. In effect, all the conditions of Proposition 3.1 hold (i.e., the function F and the Riemannian structure $\langle\,,\,\rangle_J$ are

also invariant with respect to the group action); hence, the field $\widetilde{M}$ is equal to the gradient of a function $\widetilde{F}$ with respect to the induced Riemannian structure.

Now we are going to show that the function $\widetilde{F}$ on $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R})$ is Morse. To this end (see Proposition 3.2) we should show that the restriction of the Hessian of F on the directions orthogonal to the "vertical" vectors is nondegenerate. This Hessian of F is given by formula (39) in [5]: let $s_w \in \mathrm{SO}(n;\mathbb{R})$ be a positive permutation matrix corresponding to $w \in S_n$ (i.e., the matrix in $\mathrm{SO}(n,\mathbb{R})$ which permutes the vectors $\pm e_i$, $i = 1,\dots,n$ so that the permutation $w \in S_n$ emerges when the signs are dropped); then s_w is a singular point in $\mathrm{SO}(n;\mathbb{R})$ and the Hessian of F at s_w is

$$d^2_{s_w}F = \sum_{i<j} \theta^2_{ij}(\lambda_{w(i)} - \lambda_{w(j)})(j-i). \tag{3.5}$$

Here θ_{ij} denote the local coordinates on $\mathrm{SO}(n,\mathbb{R})$ at the point $s_w \in \mathrm{SO}(n,\mathbb{R})$ obtained from the standard coordinates on $\mathfrak{so}_n$ by right translation. Put

$$G = \mathrm{SO}(n,\mathbb{R}) \bigcap \big(\mathrm{O}(i_1,\mathbb{R}) \times \cdots \times \mathrm{O}(i_k,\mathbb{R})\big).$$

Recall that we assume the eigenvalues of Λ to be partitioned into k "blocks", see (3.4), so that everything is G-invariant. In this case the "vertical" directions at s_w are equal to the tangent space of the corresponding orbit

$$T^v_{s_w}\mathrm{SO}(n;\mathbb{R}) = T_{s_w}(s_w \cdot G).$$

The Lie algebra $\mathfrak{g}$ of G is equal to the space of all antisymmetric matrices, spanned by the following set of elementary antisymmetric matrices:

$$e_{ij} - e_{ji}, \qquad \text{such that} \quad \exists\, p, \quad 1 \le p \le k, \quad i_0 + \cdots + i_{p-1} < i < j \le i_0 + \cdots + i_p,$$

where we have put $i_0 = 0$ and use symbols e_{ij} to denote the matrix units. Then the vertical directions at s_w are equal to the linear span of

$$(e_{w(i)w(j)} - e_{w(j)w(i)})s_w \in T_{s_w}\mathrm{SO}(n;\mathbb{R})$$

for the same set of indices i, j, as above. This follows either from the direct computations, or from the fact that the conjugate action Ad_{s_w} of s_w on $\mathfrak{so}_n$ amounts to the permutation of indices. Now the non-degeneracy of $d^2_{s_w}$ on the linear complement of $T^v_{s_w}\mathrm{SO}(n,\mathbb{R}) \subset T_{s_w}\mathrm{SO}(n,\mathbb{R})$ can be seen from the comparison of formula (3.5) with this description of the vertical subspace. Due to the G-invariance of the constructions, the same is true for an arbitrary point in the orbit through s_w.

We sum up our observations in the following proposition:

Proposition 3.3. *The full symmetric Toda system on the matrices with non-distinct eigenvalues induces a dynamical system on the partial flag manifold* $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R})$ *(where* $i_1,\dots,i_k$ *are the multiplicities of eigenvalues). This dynamical system is equal to the gradient flow of a Morse function on* $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R})$.

We conclude this section by an important observation: consider the natural projection (see Section 2.1)

$$\pi\colon\ \mathrm{Fl}_n(\mathbb{R}) \to \mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R}).$$

As we have shown above, when the eigenvalues of Λ are given by (3.4), the spaces on both sides of this diagram can be equipped with a Morse–Bott function and a Riemannian metric, which give rise to the gradient vector fields $\widetilde{M}$ on them. In both cases these structures are pulled to the flag spaces (full or partial) from the group $\mathrm{SO}(n,\mathbb{R})$. On the other hand we can use this projection π to pull the structures we need from $\mathrm{Fl}_n(\mathbb{R})$ to the partial flag space. It is clear that in either way we shall obtain the same result. In other words, when the eigenvalues of Λ are not distinct, the function F on $\mathrm{SO}(n,\mathbb{R})$ is a Morse–Bott function, which induces a Morse function on the base $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R})$.

4 The asymptotic behavior

In this section we prove the main theorem of the paper: the asymptotic behavior of the trajectories of the vector field induced on $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R})$ by the FS Toda lattice on the set of symmetric matrices with multiple eigenvalues (with multiplicities $i_1,\dots,i_k$) is completely determined by the Bruhat order on S_n^I, see Theorem 4.2. We begin with two particular cases in which the structure of the trajectories is easy to perceive: the case of real projective spaces and the case of Grassmannian $\mathrm{Gr}_{2,4}(\mathbb{R})$. The general case is treated at the end.

4.1 Example 1: projective spaces

Let L be the Lax matrix of the FS Toda lattice; Λ is the diagonal matrix of its eigenvalues so that

$$L = \Psi\Lambda\Psi^{-1}, \qquad \Psi \in \mathrm{SO}(n,\mathbb{R}).$$

As we have explained this relation allows one to introduce an analogue of the FS Toda lattice on the partial flag space $\mathrm{Fl}_{i_1,\dots,i_k}(\mathbb{R})$, where $i_1, i_2, \dots, i_k$ are the multiplicities of the eigenvalues of L. The geometry of the flag space (and hence the structure of Toda trajectories) depends to a great measure on the partition $I = (i_1, i_2, \dots, i_k)$. In this section we consider the simplest possible case; namely, we assume that $I = (1, n-1)$, i.e., we assume that there are only two different eigenvalues $\lambda < \mu$ with multiplicities 1 and $n-1$, respectively (we can also assume that $\lambda + (n-1)\mu = 0$). The flag spaces here are equal to the projective spaces $\mathbb{R}P^{n-1}$.

We begin with the smallest possible dimension $n = 3$ and eigenvalues $\lambda_1 = \lambda < \lambda_2 = \lambda_3 = \mu$. In this case we consider the system on the projective plane $\mathbb{R}P^2$. One can see that there are exactly 3 singular points of the vector field induced by the Toda system on $\mathbb{R}P^2$ corresponding to the permutations $0 = (\lambda,\mu,\mu)$, $1 = (\mu,\lambda,\mu)$ and $2 = (\mu,\mu,\lambda)$; direct calculations in the local coordinates at these points (it is enough to transfer the coordinates from the unit of $\mathrm{SO}(3,\mathbb{R})$ and take the directions complementary to the vertical) show that the Morse function takes distinct values at these points so that their Morse indices are 2, 1 and 0, respectively (see formula (3.5)). Recall that the index of a singular point of a Morse function is equal to the dimension of the submanifold spanned by the trajectories exiting this point. Taking this into consideration we see that there is a unique way of connecting 0, 1 and 2 by trajectories:

$$\begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left] & 1 \arrow[r] & 2. \end{tikzcd}$$

It is easy to describe the structure of the vector field here: its pullback from $\mathbb{R}P^2$ to S^2 has 6 singular points, which can be identified with the intersections of the sphere with the coordinate axes; the indices of these points are $0, 1$ and 2 so that the opposite points have the same index. Besides this the vector field is invariant with respect to the natural involution of the sphere (exchange of the opposite points). We shall call this vector field *the Toda field* on $\mathbb{R}P^2$, and use the same name for similar fields on arbitrary projective spaces.

In the general case $n > 3$ we shall have an analogous picture: just remark that the diagonal embedding of $\mathrm{SO}(n,\mathbb{R})$ into $\mathrm{SO}(n+1,\mathbb{R})$, given by

$$\Psi \mapsto \begin{pmatrix} \Psi & 0 \\ 0 & 1 \end{pmatrix},$$

corresponds on the one hand, to the hyperplane embedding of projective spaces $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$, and on the other hand, it is induced by the evident embedding of symmetric $n \times n$ matrices into the space of the symmetric $(n+1)\times(n+1)$ matrices:

$$L \to \begin{pmatrix} L & 0 \\ 0 & \mu \end{pmatrix}.$$

This embedding sends the symmetric matrices with the spectrum

$$\lambda < \underbrace{\mu = \mu = \cdots = \mu}_{n-1 \text{ times}}$$

to the symmetric matrices with the spectrum

$$\lambda < \underbrace{\mu = \mu = \cdots = \mu}_{n \text{ times}}.$$

Also, one can see that this inclusion intertwines Toda systems on bigger matrices and smaller matrices. Thus, the Toda vector field on $\mathbb{R}P^{n-1}$ coincides with the restriction of the Toda field from $\mathbb{R}P^n$, and the latter field has one more singular point in the complement of $\mathbb{R}P^{n-1}$ in $\mathbb{R}P^n$. This new point has the maximum possible index. So we see that in the case of a generic projective space $\mathbb{R}P^n$ the phase diagram of singular points and trajectories between them is as follows

$$\begin{tikzcd} 0 \arrow[r] \arrow[rr, bend left] \arrow[rrrr, bend left=50] & 1 \arrow[r] \arrow[rrr, bend left=40] & 2 \arrow[r] \arrow[rr, bend left] & \dots \arrow[r] & n\,. \end{tikzcd}$$

It is also possible to rephrase the results of this section in terms of the asymptotic behavior of the symmetric Lax matrix L: as one can see, when $t \to \pm\infty$ the matrix L tends to a diagonal matrix with eigenvalues λ and μ of multiplicities 1 and $n-1$, respectively, in which λ stands on an arbitrary position (depending on the corresponding multiset permutation).

4.2 Example 2: $\mathbf{Gr_{2,4}}(\mathbb{R})$

The next simplest case is when there are only two distinct eigenvalues of L, but the dimensions of the corresponding eigenspaces are greater than 1. In this case the phase space of the system can be identified (see Section 2.1) with the Grassmann space of all dimension $d > 1$ hyperplanes in an $n > d+1$ dimensional Euclidian space $\mathbb{R}^n$. This case is already quite complicated, so we restrict our discussion to the least-dimensional case: $n = 4$, $d = 2$. We assume that the eigenvalues of the 4×4 symmetric Lax matrix L are $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda_4 = \mu$ and consider the induced gradient system on $\mathrm{Gr}_{2,4}(\mathbb{R})$.

We begin with a detailed description of the Grassmannian. This is the manifold parameterizing all 2-dimensional subspaces in $\mathbb{R}^4$. As we mentioned in Section 2.1 one has a homeomorphism of spaces

$$\mathrm{Gr}_{2,4}(\mathbb{R}) = \mathrm{SO}(4,\mathbb{R})/\mathrm{SO}(4,\mathbb{R}) \cap (\mathrm{O}(2,\mathbb{R}) \times \mathrm{O}(2,\mathbb{R})).$$

The identification is given by choosing an orthogonal basis in $\mathbb{R}^4$:

$$e_1 = \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}.$$

We can regard the 2-dimensional plane spanned by e_1, e_2 as the "origin" x of $\mathrm{Gr}_{2,4}(\mathbb{R})$ so that all the other planes in $\mathbb{R}^4$ are equal to translations of x by appropriate elements of $\mathrm{SO}(4,\mathbb{R})$. The stabilizer of x in $\mathrm{SO}(4,\mathbb{R})$ is the subgroup B_x of $\mathrm{SO}(4,\mathbb{R})$ comprised of the elements of the form

$$\begin{pmatrix} \cos(t_a) & \sin(t_a) & 0 & 0 \\ -\sin(t_a) & \cos(t_a) & 0 & 0 \\ 0 & 0 & \cos(t_b) & \sin(t_b) \\ 0 & 0 & -\sin(t_b) & \cos(t_b) \end{pmatrix}, \qquad \begin{pmatrix} -\cos(t_a) & \sin(t_a) & 0 & 0 \\ \sin(t_a) & \cos(t_a) & 0 & 0 \\ 0 & 0 & -\cos(t_b) & \sin(t_b) \\ 0 & 0 & \sin(t_b) & \cos(t_b) \end{pmatrix}.$$

As one can easily see, the plane x is preserved by the action of these elements:

$$e_1 B_x = e_1 \cos(t_a) + e_2(-\sin(t_a)), \qquad e_2 B_x = e_1 \sin(t_a) + e_2 \cos(t_a).$$

Now consider the generic point $y \in \mathrm{Gr}_{2,4}(\mathbb{R})$: as we know $y = g^y x$ for some $g^y \in \mathrm{SO}(4,\mathbb{R})$. Then the stabilizer of y is equal to the conjugation of B_x by g^y:

$$B_y = g^y B_x (g^y)^{-1}.$$

This simple observation allows one to find a suitable description of the tangent space of $\mathrm{Gr}_{2,4}(\mathbb{R})$ at y:

$$T_y \mathrm{Gr}_{2,4}(\mathbb{R}) \cong \mathfrak{so}(4)/\operatorname{Ad}_{g^y}(\mathfrak{g}),$$

where $\mathfrak{g}$ is the Lie algebra of the group B_x and Ad_g, $g \in \mathrm{SO}(4,\mathbb{R})$ denotes the adjoint action of the group $\mathrm{SO}(4,\mathbb{R})$ on its Lie algebra $\mathfrak{so}(4)$ (by conjugation of matrices).

We choose the standard basis in $\mathfrak{so}(4)$ (the space of all 4×4 anti-symmetric matrices) so that the generic element X of $\mathfrak{so}(4)$ takes the form

$$X = \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ -\theta_1 & 0 & \theta_4 & \theta_5 \\ -\theta_2 & -\theta_4 & 0 & \theta_6 \\ -\theta_3 & -\theta_5 & -\theta_6 & 0 \end{pmatrix}.$$

The functions $\theta_1,\dots,\theta_6$ are coordinates in $\mathfrak{so}(4)$. The exponential map allows one to pull these coordinates to an open neighbourhood of the unit matrix in $\mathrm{SO}(4,\mathbb{R})$ and the right translations by the group elements then give coordinate systems in open neighbourhoods of the points in $\mathrm{SO}(4,\mathbb{R})$ and on tangent spaces at these points. Below we shall use these coordinates extensively without explanation, preserving (by a slight abuse of the language) their original names.

This notation is well-suited for the description of the tangent spaces of $\mathrm{Gr}_{2,4}(\mathbb{R})$ at the singular points of the FS Toda lattice: recall that these singular points are equal to the projections to $\mathrm{Gr}_{2,4}(\mathbb{R})$ of the singular fibres of the FS Toda lattice in $\mathrm{SO}(4,\mathbb{R})$ (or in $\mathrm{Fl}_4(\mathbb{R})$). These singular fibres are determined by the singular points of the FS Toda lattice on the matrices with distinct eigenvalues, which they contain. As one knows, the singular points in $\mathrm{SO}(4,\mathbb{R})$ of the FS Toda lattice corresponding to the Lax matrices with distinct eigenvalues are given by the matrices, which permute the basis vectors and their opposites (see [5]). The conjugation by such matrices induces permutations of the coordinates $\theta_1,\dots,\theta_6$.

With the help of these observations one can draw the following table, describing the tangent spaces at the critical points, values of the Morse function $F_{2,4}$, positive and negative directions of Hessians at these critical points, i.e., of the singular points of the Toda vector field on $\mathrm{Gr}_{2,4}(\mathbb{R})$. We index the singular points on $\mathrm{Gr}_{2,4}(\mathbb{R})$ by the corresponding permutations of the eigenvalues' set $(\lambda,\lambda,\mu,\mu)$, $\lambda<\mu$.

Point	Morse index	Value of $F_{2,4}$	$+$	$-$	Minors
$(\lambda,\lambda,\mu,\mu)$	0	$\lambda+5\mu$	θ_2, θ_3, θ_4, θ_5	0	ψ_{13}, ψ_{14}, ψ_{41}, ψ_{42}
$(\lambda,\mu,\lambda,\mu)$	1	$2\lambda+4\mu$	θ_1, θ_4, θ_6	θ_2	ψ_{13}, ψ_{14}, ψ_{41}, ψ_{42}
$(\lambda,\mu,\mu,\lambda)$	2	$3\lambda+3\mu$	θ_5, θ_6	θ_1, θ_3	ψ_{13}, ψ_{14}, ψ_{43}, ψ_{44}
$(\mu,\lambda,\lambda,\mu)$	2	$3\lambda+3\mu$	θ_1, θ_3	θ_5, θ_6	ψ_{11}, ψ_{12}, ψ_{41}, ψ_{42}
$(\mu,\lambda,\mu,\lambda)$	3	$4\lambda+2\mu$	θ_2	θ_1, θ_4, θ_6	ψ_{11}, ψ_{12}, ψ_{43}, ψ_{44}
$(\mu,\mu,\lambda,\lambda)$	4	$5\lambda+\mu$	0	θ_2, θ_3, θ_4, θ_5	ψ_{11}, ψ_{12}, ψ_{43}, ψ_{44}

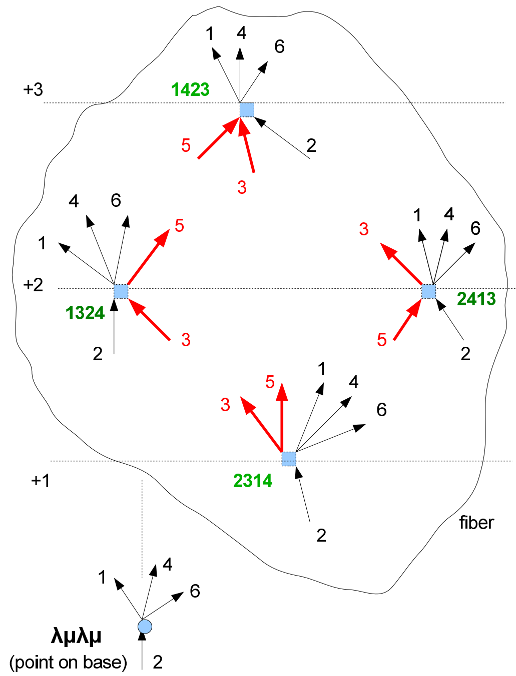

Figure 1. The bundle over $(\lambda, \mu, \lambda, \mu) \in \mathrm{Gr}_{2,4}(\mathbb{R})$.

In the fourth and the fifth columns we use the coordinates $\theta_1, \ldots, \theta_6$ on $\mathrm{SO}(4, \mathbb{R})$ introduced earlier and project them to $\mathrm{Gr}_{2,4}(\mathbb{R})$: on the level of tangent spaces the differential $d\pi$ of this projection is a linear epimorphism and we identify $T_{\pi(x)}\mathrm{Gr}_{2,4}(\mathbb{R})$ with a linear subspace, which maps isomorphically onto it. The last column gives the list of certain invariant surfaces of the Toda field in $\mathrm{SO}(4, \mathbb{R})$, which contain the corresponding "singular fibre" of the Toda field (i.e., the fibre of the projection $\pi\colon \mathrm{SO}(4, \mathbb{R}) \to \mathrm{Gr}_{2,4}(\mathbb{R})$, on which the Toda field vanishes identically). As one knows (see [5]) such surfaces can be given by the equations $\psi_{ij} = 0$; there are more general surfaces of this sort, in which the equations are given by the condition that a certain minor of the matrix $\Psi = (\psi_{ij})$ vanishes (hence the name "minor surfaces" that we use in this and previous paper) but we restrict our attention to the simplest set of invariants listed in this table.

We illustrate these constructions by Fig. 1, in which the case of the singular point in $\mathrm{Gr}_2(4, \mathbb{R})$, corresponding to the permutation $(\lambda, \mu, \lambda, \mu)$, is considered (below we shall often identify such points with the corresponding permutations of eigenvalues).

In this figure the little circle at the bottom represents the point in $\mathrm{Gr}_{2,4}(\mathbb{R})$ that corresponds to the permutation $(\lambda, \mu, \lambda, \mu)$ of the eigenvalues. The small squares above correspond to the permutation matrices in $\mathrm{SO}(4, \mathbb{R})$ that project into the chosen point. The arrows represent the coordinate directions in the tangent space of the group at the chosen points. The red (bold) arrows correspond to the directions inside the fibre, and the black (thin) arrows to all the rest. As one knows (see [5]) the coordinates θ_i are (up to infinitesimal correction terms) the canonical coordinates, in which the Morse function takes the form of the sums of the squares. So the arrows are directed towards or from the points, depending on whether the corresponding tangent directions are in positive or negative subspaces of the Hessian (the red arrows in fact correspond to the directions which lie in the kernel of the Hessian but we preserve them for the sake of simplicity; their directions are determined under the assumption that all the eigenvalues are distinct and are ordered in a natural way, otherwise one can take arbitrary vectors in the fibre direction).

Finally, we consider the minor surfaces that pass through the fibre: it is clear that if a surface is invariant under the Toda flow on $\mathrm{SO}(4, \mathbb{R})$, its projection to the Grassmann space is an invariant set of the generalized flows. Using this simple observation we come up with the

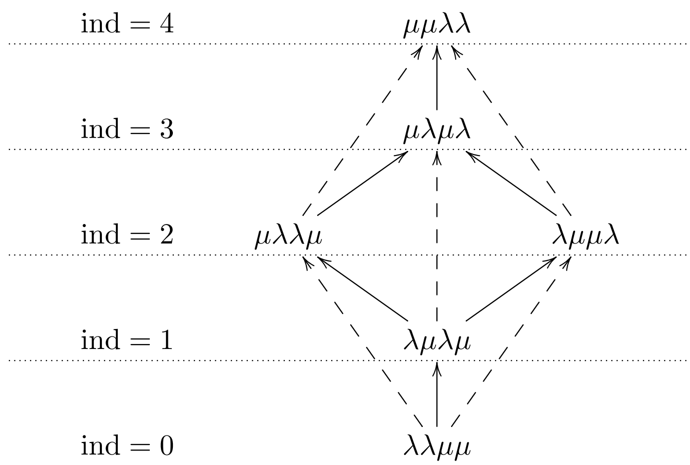

Figure 2. The generalized Toda flow on $\mathrm{Gr}_2(4,\mathbb{R})$.

following diagram, see Fig. 2, representing the flows in $\mathrm{Gr}_{2,4}(\mathbb{R})$; the dotted lines represent the 1-parameter families of trajectories that connect points whose Morse indices differ by 2. There are also 2-parameter families between the points with the Morse indices that differ by 3 and one 3-parametric family between the lowest and the highest points, which we have omitted to make the diagram more readable.

As one can see, this diagram coincides with the Hasse diagram of the Bruhat order on multiset permutations. Below (see Theorem 4.2), we shall show that it is always the case.

Let us give a brief explanation of how the diagram has been obtained: consider two singular points in $\mathrm{Gr}_{2,4}(\mathbb{R})$, which correspond to the permutations $(\lambda,\lambda,\mu,\mu)$ and $(\lambda,\mu,\lambda,\mu)$ of the eigenvalues. Let us show that there is a finite number of trajectories of the Toda flow between these points.

Observe that the surface

$$\Sigma = (\psi_{13}=0)\cap(\psi_{14}=0)\cap(\psi_{41}=0)\cap(\psi_{42}=0), \qquad \Sigma\subset \mathrm{SO}(4,\mathbb{R})$$

is invariant with respect to the $\mathrm{O}(2,\mathbb{R})\times\mathrm{O}(2,\mathbb{R})$-action and also is preserved by the Toda flow. Thus, it projects to an invariant surface $\widetilde{\Sigma}$ in $\mathrm{Gr}_{2,4}(\mathbb{R})$, which contains only the above mentioned singular points on the base. Of course, Σ and $\widetilde{\Sigma}$ are singular varieties in $\mathrm{SO}(4,\mathbb{R})$ and $\mathrm{Gr}_{2,4}(\mathbb{R})$, so for our purposes it is enough to compute their dimensions at the generic point. The generic matrix Ψ from Σ has the following form

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & 0 & 0\\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24}\\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34}\\ 0 & 0 & \psi_{43} & \psi_{44}\end{pmatrix}.$$

Moreover, Ψ being an orthogonal matrix the dimension of the surface Σ at the generic point is equal to 3: it is enough to compute the dimension of the tangent space to it at the unit matrix; the tangent space of $\mathrm{SO}(4,\mathbb{R})$ at the unit consists of antisymmetric matrices, so summing the conditions on the matrix elements we come to the above mentioned conclusion about the dimension of Σ.

Thus, the dimension $\widetilde{\Sigma}$ (at the generic point) is equal to 1. On the other hand $\widetilde{\Sigma}$ is invariant with respect to the Toda field on $\mathrm{Gr}_{2,4}(\mathbb{R})$; since it is 1-dimensional, it should consist of a finite number of trajectories connecting singular points in it. Thus, the points $(\lambda,\lambda,\mu,\mu)$ and $(\lambda,\mu,\lambda,\mu)$ must be connected by a discrete set of trajectories.

4.3 The general case

Finally, let us consider the most general distribution of the eigenvalues of L; let us fix the multi-index $I = (i_1, i_2, \dots, i_k)$ such that $0 < i_j$, $i_1 + \cdots + i_k = n$. As we have observed earlier, the set of all real symmetric matrices L with the given set Λ of eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, such that λ_j has multiplicity i_j, is homeomorphic to the flag space $\mathrm{Fl}_I(\mathbb{R})$ (see Section 2.1). We use the same symbol Λ for a diagonal matrix with the set of eigenvalues equal to Λ.

As we have shown earlier (see Section 3.2) the FS Toda lattice on the space of symmetric matrices conjugate with Λ is induced by a gradient vector field ξ_I on the partial flag space $\mathrm{Fl}_I(\mathbb{R})$: the image of the usual Toda vector field ξ on the full flag variety $\mathrm{Fl}_n(\mathbb{R})$. Moreover, the potential function generating this field is a Morse function; its singular points correspond to those fibres of the natural projection $\mathrm{Fl}_n(\mathbb{R}) \to \mathrm{Fl}_I(\mathbb{R})$ on which the Toda vector field vanishes identically (singular fibres with respect to the Toda potential function on $\mathrm{Fl}_n(\mathbb{R})$). As one knows, in this case the trajectories connect the singular points of the field when $t \to \pm\infty$. Our goal is to describe the order in which these points are connected: we shall show that this order is the same as the (strong) Bruhat order.

To this end recall (see Section 3.2 and [5]) that the singular points of the Toda vector field on the full flag space $\mathrm{Fl}_n(\mathbb{R})$ with the Lax matrix, whose eigenvalues are all distinct, correspond to the permutation matrices in $\mathrm{SO}(n, \mathbb{R})$, i.e., matrices which permute the basis vectors in $\mathbb{R}^n$ and if necessary, multiply them by -1 (to make sure the determinant is positive). In order to understand the structure of the singularities in $\mathrm{Fl}_I(\mathbb{R})$, it is convenient to look at the bundle

$$\mathrm{SO}(n, \mathbb{R}) \to \mathrm{Fl}_I(\mathbb{R})$$

with the fibres isomorphic to $G = \mathrm{SO}(n, \mathbb{R}) \bigcap (\mathrm{O}(i_1, \mathbb{R}) \times \cdots \times \mathrm{O}(i_k, \mathbb{R}))$. The vector field ξ can be raised to a vector field on the orthogonal group with singular points at the matrices A_w. The fibre through a permutation matrix A_w, which is equal to the conjugation of G by A_w, will contain all permutations of the form wuw^{-1}, where $u \in S_{i_1} \times \cdots \times S_{i_k}$. Comparing this description with Section 2.3 we conclude that the following proposition holds:

Proposition 4.1. *The singular points of the Morse field induced on $\mathrm{Fl}_I(\mathbb{R})$ by the FS Toda lattice with eigenvalues Λ (of multiplicities I) are indexed by the multiset permutations S_n^I.*

Now it is our purpose to describe the trajectories connecting different singular points in $\mathrm{Fl}_I(\mathbb{R})$. We shall prove the following statement:

Theorem 4.2. *Let $\psi, \omega \in S_n^I$ be two multiset permutations. Then the corresponding singular points of the FS Toda lattice in $\mathrm{Fl}_I(\mathbb{R})$ will be connected by a trajectory, if and only if $\psi \prec \omega$ in the Bruhat order on S_n^I (see Section 2.3). Moreover, the dimension of the subset swept by the trajectories connecting these points is equal to the length of the path in the Hasse diagram of S_n^I.*

Proof. First of all observe that the projection $\mathrm{Fl}_n(\mathbb{R}) \to \mathrm{Fl}_I(\mathbb{R})$ maps the Toda field ξ to the field ξ_I; hence it sends invariant subvarieties in the full flag space to invariant subsets in $\mathrm{Fl}_I(\mathbb{R})$. In particular, it means that the Schubert cells in $\mathrm{Fl}_I(\mathbb{R})$ are preserved by the generalized Toda flow since the Schubert cells in the full flags are. Moreover, since the Schubert cells in $\mathrm{Fl}_n(\mathbb{R})$ coincide with the *unstable* subspace of ξ (i.e., the possibly singular subspace in M swept by the trajectories tending to the given singular point of the gradient system) we conclude that their images in the partial flags coincide with the stable subspaces of ξ_I: this follows for example from the fact that the cells in $\mathrm{Fl}_I(\mathbb{R})$ are the homeomorphic images of *the minimal* cells in $\mathrm{Fl}_n(\mathbb{R})$, see the end of Section 2.4 (also compare formula (3.5)).

Similarly, the *stable* submanifolds of ξ_I (i.e., the subsets spanned by the outgoing trajectories of ξ_I) in $\mathrm{Fl}_I(\mathbb{R})$ coincide with the dual Schubert cells in this space since this is true for the dual cells in $\mathrm{Fl}_n(\mathbb{R})$ (this was proved in [5]). However, we know (see Section 2.4 again) that the

Schubert cell and dual Schubert cell in the flag space intersect if and only if the corresponding multiset permutations are comparable in the Bruhat order. The statement about the dimensions follows from the transversality of these intersections. ■

We conclude by the simple observation concerning the picture that emerges on the level of the Lax matrices:

Corollary 4.3. *The Toda flow $t \to L(t)$ on symmetric matrices converges to a diagonal matrix when $t \to \pm\infty$ (the set of eigenvalues of these matrices is fixed). Two such matrices are connected by a trajectory if and only if the corresponding permutations of the eigenvalues are comparable with respect to the Bruhat order on permutations (or permutations with repetitions if there are multiple eigenvalues).*

5 Further observations

As we have shown above (see Section 3), the FS Toda lattice on symmetric matrices with non-distinct eigenvalues can be regarded as a dynamical system on partial flag manifolds. It is interesting that although the system on these spaces can be described as the image of the FS Toda lattice, the usual invariants of the FS Toda system do not descend easily to the flag spaces: direct computations show that they can become constants or functionally-dependent. So the question is whether one can still find new invariants to prove Liuoville integrability of such systems (of course, one should first make up explicit definitions of the Poisson structures used there).

One can begin with studying a few low-dimensional cases, first of all those which correspond to the projective spaces (see Section 4.1). Already in the simplest case $n = 3$ (and when two eigenvalues coincide), that is for the system on $\mathbb{R}P^2$, we obtain a new integral of motion:

$$I_{(RP^2,\lambda_1=\lambda_2)} = \frac{1}{(\mu-\lambda)}\frac{\psi_{23}^2}{\psi_{13}^2\psi_{33}^2}.$$

Here $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \mu$, $\lambda < \mu$ are the eigenvalues. In terms of the matrix entries a_{ij} of the Lax matrix L this function can be rewritten in the following form:

$$I_{(RP^2,\lambda_1=\lambda_2)}(a_{ij}) = \frac{a_{12}a_{23}}{a_{13}^3}.$$

On the other hand, one can show that in this case the chopping procedure (see [8] for example) does not give any integrals which are different from the traces of the powers of the Lax matrix. Thus, this invariant is a new phenomenon, which makes the whole picture quite intriguing. Similar integrals can be found in the case of projective spaces in higher dimensions. These questions will be the subject of our further papers.

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