# An Analytic Formula for the $A_2$ Jack Polynomials*

*Vladimir V. MANGAZEEV*

*Department of Theoretical Physics, Research School of Physical Sciences and Engineering, The Australian National University, Canberra, Australia*
E-mail: *vladimir@maths.anu.edu.au*

Received November 01, 2006, in final form January 05, 2007; Published online January 24, 2007
Original article is available at http://www.emis.de/journals/SIGMA/2007/014/

**Abstract.** In this letter I shall review my joint results with Vadim Kuznetsov and Evgeny Sklyanin [*Indag. Math.* **14** (2003), 451–482] on separation of variables (SoV) for the $A_n$ Jack polynomials. This approach originated from the work [*RIMS Kokyuroku* **919** (1995), 27–34] where the integral representations for the $A_2$ Jack polynomials was derived. Using special polynomial bases I shall obtain a more explicit expression for the $A_2$ Jack polynomials in terms of generalised hypergeometric functions.

*Key words:* Jack polynomials; integral operators; hypergeometric functions

*2000 Mathematics Subject Classification:* 05E05; 33C20; 82B23

*Dedicated to the memory of Vadim Kuznetsov*

## 1 Introduction

In middle 1990's Vadim Kuznetsov and Evgeny Sklyanin started to work on applications of the Separation of Variables (SoV) method to symmetric polynomials. When I read their papers on SoV for Jack and Macdonald polynomials, I was quite charmed by a beauty of this approach. It connected together few different areas of mathematics and mathematical physics including symmetric polynomials, special functions and integrable models.

Later on we started to work together on development of SoV for symmetric functions. During my visits to Leeds around 2001–2003 we realised that the separating operator for the $A_n$ Jack polynomials can be constructed using Baxter's $Q$-operator following a general approach [8] developed by Vadim and Evgeny Sklyanin.

Vadim was a very enthusiastic and active person. His creative energy always encouraged me to work harder on joint projects. His area of expertise ranged from algebraic geometry to condensed matter physics. He was an excellent mathematician and a very strong physicist.

During my visits to Leeds University I was charmed by the English countryside. I will never forget long walks in the forest with Vadim's family and quiet dinners at his home.

The sad news about his death was a real shock for me. His untimely death is a big loss for the scientific community. However, he will be long remembered by many people for his outstanding scientific achievements.

## 2 Quantum Calogero–Sutherland model

The quantum Calogero–Sutherland model [1, 21] describes a system of $n$ particle on a circle with coordinates $0 \le q_i \le \pi$, $i = 1, \dots, n$. The Hamiltonian and momentum are

$$H = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial q_i^2} + \sum_{i<j}\frac{g(g-1)}{\sin^2(q_i - q_j)}, \qquad P = -i\sum_{j=1}^{n}\frac{\partial}{\partial q_j}. \tag{2.1}$$

*This paper is a contribution to the Vadim Kuznetsov Memorial Issue "Integrable Systems and Related Topics". The full collection is available at http://www.emis.de/journals/SIGMA/kuznetsov.html

The ground state of the model is given by

$$\Omega(\boldsymbol{q}) = \left[\prod_{i<j} \sin(q_i - q_j)\right]^g, \qquad \boldsymbol{q} \equiv (q_1, \ldots, q_n) \tag{2.2}$$

with the ground state energy $E_0 = \frac{1}{6}g^2(n^3 - n)$. Further we shall assume that $g > 0$ that simplifies description of the eigenvectors.

The Calogero–Sutherland model is completely integrable [14] and there is a commutative ring of differential operators $H_i$, $i = 1, \ldots, n$ [16], which contains the Hamiltonian and momentum (2.1).

The eigenfunctions are labelled by partitions $\lambda \equiv (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \geq \lambda_{i+1} \geq 0$, $i = 1, \ldots, n-1$ and can be written as

$$\Psi_\lambda(\boldsymbol{q}) = \Omega(\boldsymbol{q}) P_\lambda^{(1/g)}(\boldsymbol{x}), \tag{2.3}$$

where $P_\lambda^{(1/g)}(\boldsymbol{x})$, $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ are symmetric Jack polynomials in $n$ variables [3, 13, 20]

$$x_i = e^{2iq_i}, \qquad i = 1, \ldots, n. \tag{2.4}$$

Further for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we define its length $l(\lambda) = n$, its weight $|\lambda| = \lambda_1 + \cdots + \lambda_n$ (see [13]) and use a short notation $\lambda_{ij} = \lambda_i - \lambda_j$.

Conjugating the Hamiltonian $H$ with vacuum (2.2) we obtain

$$\Omega^{-1}(\boldsymbol{q}) H \Omega(\boldsymbol{q}) = \frac{1}{2} H_g + E_0, \tag{2.5}$$

$$H_g = \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i}\right)^2 + g \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j}\right). \tag{2.6}$$

Then we obtain that Jack polynomials $P_\lambda^{(1/g)}(\boldsymbol{x})$ are eigenfunctions of the operator $H_g$

$$H_g P_\lambda^{(1/g)}(\boldsymbol{x}) = E_g P_\lambda^{(1/g)}(\boldsymbol{x}) \tag{2.7}$$

and the eigenvalues are given by

$$E_g = \sum_{i=1}^n \lambda_i[\lambda_i + g(n + 1 - 2i)]. \tag{2.8}$$

It follows that symmetric Jack polynomials are orthogonal with respect to the scalar product

$$\begin{aligned}(P_\lambda, P_\mu) = \frac{1}{(2\pi i)^n} \oint_{|x_1|=1} \frac{dx_1}{x_1} \cdots \oint_{|x_n|=1} \frac{dx_n}{x_n} \left\{\prod_{i\neq j} (1 - x_i/x_j)\right\}^g \\ \times \overline{P_\lambda^{(1/g)}(x)} P_\mu^{(1/g)}(x) = 0 \qquad \text{if} \quad \lambda \neq \mu,\end{aligned} \tag{2.9}$$

where $\overline{x}$ stands for the complex conjugate of $x$ and due to conditions $g > 0$ and (2.4), where $q_i$ are real, we have $\overline{P_\lambda^{(1/g)}(x)} = P_\lambda^{(1/g)}(x^{-1})$.

Jack polynomials are homogeneous of the degree $|\lambda|$

$$P_\lambda^{(1/g)}(x_1, \ldots, x_n) = x_n^{|\lambda|} P_\lambda^{(1/g)}(x_1/x_n, \ldots, x_{n-1}/x_n, 1). \tag{2.10}$$

In fact, (2.10) corresponds to a centre-of-mass separation in the Calogero–Sutherland model.

Under simultaneous shift of all parts of the partition $\lambda$ by an integer the Jack polynomials undergo a simple multiplicative transformation, which can be written as

$$P_{\lambda_1,\ldots,\lambda_n}^{(1/g)} = (x_1 \ldots x_n)^{\lambda_n} P_{\lambda_1-\lambda_n,\ldots,\lambda_{n-1}-\lambda_n,0}^{(1/g)}(x_1, \ldots, x_n). \tag{2.11}$$

We shall use properties (2.10), (2.11) in our construction of the $A_2$ Jack polynomials.

## 3 Separation of variables for Jack polynomials

In this section we shall briefly review basic facts about separation of variables [17, 18] for Jack polynomials [7, 6]. Our main reference is [6].

We shall start with definition of separated polynomials. For each partition $\lambda$ defines a function $f_\lambda(x)$ [7, 9] as

$$f_\lambda(y) = y^{\lambda_n}(1-y)^{1-ng}\,{}_nF_{n-1}\left(\begin{matrix} a_1,\dots,a_n; \\ b_1,\dots,b_{n-1}\end{matrix}\;\middle|\; y\right), \tag{3.1}$$

where the hypergeometric function ${}_nF_{n-1}$ [2, 19] is defined as

$${}_nF_{n-1}\left(\begin{matrix} a_1,\dots,a_n; \\ b_1,\dots,b_{n-1}\end{matrix}\;\middle|\; y\right) = \sum_{m=0}^{\infty}\frac{(a_1)_m\cdots(a_n)_m}{(b_1)_m\cdots(b_{n-1})_m}\frac{y^m}{m!}, \qquad |y|<1, \tag{3.2}$$

the Pocchammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ and parameters $a_i$, $b_i$ are related to the partition $\lambda = \{\lambda_1,\dots,\lambda_n\}$ as follows

$$a_i = \lambda_n - \lambda_i + 1 - (n-i+1)g, \qquad b_j = a_j + g. \tag{3.3}$$

The important property of this function is described by the following

**Theorem 1.** *$f_\lambda(y)$ is a polynomial in $y$ of the cumulative degree $\lambda_1$ and also has another representation*

$$f_\lambda(y) = b_\lambda\, y^{\lambda_n}\sum_{k_1=0}^{\lambda_{1,2}}\cdots\sum_{k_{n-1}=0}^{\lambda_{n-1,n}}\prod_{i=1}^{n-1}(1-y)^{k_i}\frac{(-\lambda_{i,i+1})_{k_i}}{k_i!}\frac{(ig)_{k_1+\cdots+k_i}}{((i+1)g)_{k_1+\cdots+k_i}}, \tag{3.4}$$

*where*

$$b_\lambda = \prod_{i=1}^{n-1}\frac{((n-i+1)g)_{\lambda_{i,n}}}{((n-i)g)_{\lambda_{i,n}}}. \tag{3.5}$$

This theorem proved in [6] allows to evaluate $f_\lambda(1) = b_\lambda$ and we shall use this fact to calculate a correct normalisation of the $A_2$ Jack polynomials.

In fact, we can directly expand (3.1) into the product of two series in $y$ and multiply them to produce an expansion of the form

$$f_\lambda(y) = \sum_{k=\lambda_n}^{\lambda_1} y^k \xi_k(\lambda; g) \tag{3.6}$$

with coefficients $\xi_k(\lambda; g)$ being expressed in terms of ${}_{n+1}F_n$ hypergeometric functions (see [9] for the case of Macdonald polynomials). However, the coefficients of $y^k$ will be, in general, infinite series depending on $\lambda$ and $g$. The Theorem 1 shows that, in fact, all those coefficients are rational functions and the expansion really truncates at $y^{\lambda_1}$. It is not clear how to produce an explicit form of those coefficients in terms of hypergeometric functions of one variable with finite number of terms. It can be easily done for the case $n = 2$ and later we will show how to do that for the case $n = 3$.

Now we shall formulate two main theorems of [6].

**Theorem 2.** *There exists an integral operator $\mathcal{S}_n$, which maps symmetric polynomials in $n$ variables into symmetric polynomials,*

$$\mathcal{S}_n \mathbf{1} = \mathbf{1} \tag{3.7}$$

*and this operator factorises Jack polynomials*

$$\mathcal{S}_n[P_\lambda^{(1/g)}](x_1,\ldots,x_n) = c_\lambda b_\lambda^{-n} \prod_{i=1}^n f_\lambda(x_i), \tag{3.8}$$

*where*

$$c_\lambda = P_\lambda^{(1/g)}(1,\ldots,1) = \prod_{1\le i<j\le n} \frac{(g(j-i+1))_{\lambda_{i,j}}}{(g(j-i))_{\lambda_{i,j}}}. \tag{3.9}$$

Action of this operator $\mathcal{S}_n$ on elementary symmetric polynomials $e_k$ was described in [6]. This symmetric separation of variables corresponds to the so called dynamical normalization of the Baker–Akhiezer function (see, for example, [10]).

However, using the property (2.10) we can trivially separate one variable $x_n$. After that we need to separate the $n-1$ remaining variables. It results in the so called standard separation

**Theorem 3.** *There exist an integral operator $\mathcal{S}_n^s$ which maps symmetric polynomials in $n$ variables into polynomials symmetric only in $n-1$ variables,*

$$\mathcal{S}_n^s \mathbf{1} = \mathbf{1} \tag{3.10}$$

*and*

$$\mathcal{S}_n^s[P_\lambda^{(1/g)}](x_1,\ldots,x_n) = c_\lambda b_\lambda^{1-n} |x_n|^\lambda \prod_{i=1}^{n-1} f_\lambda(x_i). \tag{3.11}$$

In two next sections we shall describe the action of $\mathcal{S}_2$ and $\mathcal{S}_3^s$ explicitly.

## 4 The $A_1$ case

In this section we shall consider the simplest $n=2$ case in standard and dynamical normalisations.

In the standard normalisation the separating operator $\mathcal{S}_2^s$ is trivial, $c_\lambda = b_\lambda$, $\lambda = (\lambda_1,\lambda_2)$ and we simply obtain

$$P_\lambda^{(1/g)}(x_1,x_2) = x_2^{\lambda_1+\lambda_2} f_\lambda(x_1/x_2), \tag{4.1}$$

where the separating polynomial

$$\begin{aligned} f_\lambda(x) &= x^{\lambda_2}(1-x)^{1-2g}\, {}_2F_1(-\lambda_{12}+1-2g, 1-g; -\lambda_{12}+1-g; x) \\ &= x^{\lambda_2}\, {}_2F_1(-\lambda_{12}, g; -\lambda_{12}+1-g; x) \end{aligned} \tag{4.2}$$

and we used a well known formula for ${}_2F_1$ [2]. The last formula in (4.2) gives an explicit expansion of $f_\lambda(x)$ in series of $x$ with rational coefficients.

Although formulas (4.1), (4.2) give explicit representation for the $A_1$ Jack polynomials, a symmetry w.r.t. $x_1$ and $x_2$ is not obvious. To obtain such symmetric representation it is instructive to construct a separation which corresponds to the dynamical normalisation of the Baker–Akhiezer function.

Introduce the following basis in the space of symmetric polynomials $S(x_1, x_2)$ in two variables $x_1$, $x_2$

$$p_{mn} \equiv p_{mn}(x_1, x_2) = (x_1x_2)^m[(1 - x_1)(1 - x_2)]^n, \qquad m, n \geq 0 \tag{4.3}$$

and define the operator $\mathcal{S}_2$ acting on $S(x_1, x_2)$ as

$$\mathcal{S}_2[p_{mn}] = \frac{(g)_n}{(2g)_n} p_{mn}. \tag{4.4}$$

Such operator $\mathcal{S}_2$ can be easily represented as an integral operator with a simple kernel (see [9] for Macdonald polynomials) but we do not need here its explicit form. What is more important is that it maps the $A_1$ Jack polynomials into the product of separated polynomials

$$\mathcal{S}_2[P_\lambda^{(1/g)}](x_1, x_2) = \frac{(g)_{\lambda_{12}}}{(2g)_{\lambda_{12}}} f_\lambda(x_1) f_\lambda(x_2). \tag{4.5}$$

The action of the inverse operator $\mathcal{S}_2^{-1}$ on the basis $p_{mn}$ is easily constructed from (4.4) and to obtain a symmetric representation of $P_\lambda^{(1/g)}(x_1, x_2)$ we have to expand the rhs of (4.5) in $p_{mn}$.

To do that we use Watson's formula [22]

$$\begin{aligned} &{}_2F_1(-n, b; c; x_1)\, {}_2F_1(-n, b; c; x_2) \\ &\quad = \frac{(c - b)_n}{(c)_n} F_4(-n, b; c, 1 - n + b - c; x_1x_2, (1 - x_1)(1 - x_2)), \qquad n \in \mathbb{Z}_+ \end{aligned} \tag{4.6}$$

and the Appell's function $F_4$ is defined as

$$F_4(a, b; c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m y^n}{m!n!}. \tag{4.7}$$

After little algebra we obtain from (4.2)–(4.7)

$$P_\lambda^{(1/g)}(x_1, x_2) = (x_1x_2)^{\lambda_2} \sum_{0 \leq m+n \leq \lambda_{12}} \frac{(-\lambda_{12})_{m+n}(g)_{m+n}}{(-\lambda_{12} + 1 - g)_m(g)_n} \frac{p_{mn}(x_1, x_2)}{m!n!}. \tag{4.8}$$

Finally we can rewrite (4.8) in the basis of elementary symmetric polynomials $e_1 = x_1 + x_2$, $e_2 = x_1x_2$. Expanding $p_{mn}$ in $e_1$, $e_2$ and evaluating remaining sums we come to

$$P_\lambda^{(1/g)}(x_1, x_2) = (e_2)^{\lambda_2}(-1)^{\lambda_{12}} \frac{(\lambda_{12})!}{(g)_{\lambda_{12}}} \sum_{i+2j=\lambda_{12}} (g)_{i+j} \frac{e_1^i e_2^j}{i!j!}(-1)^{i+j}. \tag{4.9}$$

Another symmetric representation for $P_\lambda^{(1/g)}(x_1, x_2)$ which first appeared in [5] is given in terms of Gegenbauer polynomials $C_n^g(x)$. Expanding $e_1$ in series of $x_1$ and $x_2$ we obtain

$$P_\lambda^{(1/g)}(x_1, x_2) = (x_1x_2)^{\frac{1}{2}|\lambda|} \frac{(\lambda_{12})!}{(g)_{\lambda_{12}}} C_{\lambda_{12}}^g \left( \frac{1}{2} \left[ (x_1/x_2)^{1/2} + (x_2/x_1)^{1/2} \right] \right). \tag{4.10}$$

In fact, this formula can be directly obtained from representation (4.1), (4.2), but we preferred to use the symmetric separation (4.5). As was shown in [10], the separating operator $\mathcal{S}_{n-1}$ for the dynamical normalisation has the same structure as the separating operator $\mathcal{S}_n^s$ in the standard normalisation. In the next section we shall use this fact to construct a representation for the $A_2$ Jack polynomials.

## 5 The $A_2$ case and the expansion of products of separated polynomials

Now we shall consider $n = 3$ case in the standard normalisation. Using (2.10) we obtain

$$P_\lambda^{(1/g)}(x_1, x_2, x_3) = x_3^{|\lambda|} P_\lambda^{(1/g)}(x_1/x_3, x_2/x_3, 1) \equiv x_3^{|\lambda|} p_\lambda(x_1/x_3, x_2/x_3), \tag{5.1}$$

where we introduced a short notation $p_\lambda(x_1, x_2)$ for the $A_2$ Jack polynomials with $x_3 = 1$.

Define a reduced separating operator $\hat{\mathcal{S}}_3$ which acts only on the basis $p_{mn}$ (see (4.3)) in the space of symmetric polynomials in two variables $S(x_1, x_2)$ as

$$\hat{\mathcal{S}}_3[p_{mn}] = \frac{(2g)_n}{(3g)_n} p_{mn}. \tag{5.2}$$

The explicit construction of $\hat{\mathcal{S}}_3$ as an integral operator was given in [7, 6]. The operator $\hat{\mathcal{S}}_3$ separates variables in $p_\lambda(x_1, x_2)$

$$\hat{\mathcal{S}}_3[p_\lambda](x_1, x_2) = c_\lambda b_\lambda^{-2} f_\lambda(x_1) f_\lambda(x_2), \tag{5.3}$$

where $f_\lambda(x)$ is given by (3.1) with $n = 3$.

To construct $p_\lambda(x_1, x_2)$ we have to invert the action of $\hat{\mathcal{S}}_3$. It follows from (5.2) that in order to apply to (5.3) $\hat{\mathcal{S}}_3^{-1}$ we have to expand the rhs of (5.3) in $p_{mn}$

$$f_\lambda(x_1) f_\lambda(x_2) = (x_1 x_2)^{\lambda_3} \sum_{0 \le m+n \le \lambda_{1,3}} c_{m,n}(\lambda; g)\, p_{mn}(x_1, x_2) \tag{5.4}$$

and it is easy to see that coefficients $c_{m,n}(\lambda; g)$ depend on the partition $\lambda$ only via $\lambda_{13}$ and $\lambda_{23}$.

In fact, formula (5.4) is the main nontrivial result of this paper. Setting $x_2 = 1$ in (5.4) we obtain as a particular case the expansion for $f_\lambda(x_1)$ of the form (3.6) with coefficients

$$\xi_k(\lambda, g) = c_{m,0}(\lambda, g)/b_\lambda. \tag{5.5}$$

Now we shall calculate coefficients $c_{m,n}(\lambda, g)$. For convenience we shall omit dependence of $c_{m,n}(\lambda; g)$ on $\lambda$ and $g$.

Let us remind that the hypergeometric function ${}_3F_2$ satisfies 3-th order differential equation of the form [2, 19]

$$\left[\delta \prod_{i=1}^{2} (\delta + b_i - 1) - x \prod_{i=1}^{3} (\delta + a_i)\right] {}_3F_2(a_1, a_2, a_3; b_1, b_2; x) = 0, \qquad \delta \equiv x \frac{d}{dx}. \tag{5.6}$$

Using (3.1) we can easily obtain a third order differential equation for the functions $f_\lambda(x_i)$, $i = 1, 2$. Now we can rewrite these equations in terms of new variables

$$u = x_1 x_2, \qquad v = (1 - x_1)(1 - x_2), \qquad p_{m,n} = u^m v^n. \tag{5.7}$$

Therefore, for the rhs of (5.4) we obtain two differential equations in $u$ and $v$ which can be rewritten as recurrence relations for the coefficients $c_{m,n}$. All calculations are straightforward and we shall omit the details. The first recurrence relation for $c_{mn}$ looks amazingly simple and basically allows to find $c_{m,n}$ in a closed form

$$\begin{aligned} &c_{m+1,n}(m + 1)(1 - 2g + m - r_1)(1 - g + m - r_2) \\ &\quad + c_{m,n+1}(n + 1)(3g + n - 1)(3g + n) \\ &\quad + c_{mn}(2g + m + n)(r_1 - m - n)(g + m + n - r_2) = 0, \end{aligned} \tag{5.8}$$

where we denoted $r_1 = \lambda_{1,3}$, $r_2 = \lambda_{2,3}$.

The relation (5.8) immediately suggests to make a substitution

$$c_{mn} = \frac{(2g)_{m+n}(-r_1)_{m+n}(g-r_2)_{m+n}}{m!n!(3g-1)_n(3g)_n(1-2g-r_1)_m(1-g-r_2)_m}\, a_{mn}. \tag{5.9}$$

Then (5.8) reduces to

$$a_{m+1,n} + a_{m,n+1} = a_{mn} \tag{5.10}$$

which can be solved in terms of $a_{m,0}$

$$a_{m,n} = \sum_{l=0}^{n} (-1)^l \binom{n}{l} a_{m+l,0}. \tag{5.11}$$

The second recurrence relation for $c_{m,n}$ leads to the following relation for $a_{m,n}$

$$\begin{aligned} & m(2g+r_1-m)(g+r_2-m)a_{m-1,n} - n(3g+n-2)(3g+n-1)a_{m,n-1} \\ & \quad + (m+n-r_1)(2g+m+n)(g-r_2+m+n)a_{m+1,n} \\ & \quad + [n(3g+n-1)(3g+n-2) - 3m(m+1)n - n(r_1-1)(r_2-1) \\ & \quad + 2(m-r_2)[g(1+r_1)+(r_1-m)m] + 2(3g+r_1+r_2)mn \\ & \quad - g(g-1)(r_1+3r_2-5m) - g(2g-3+r_1+2r_2)n]\, a_{mn} = 0. \end{aligned} \tag{5.12}$$

However, due to (5.11) we need the latter relation only for the calculation of $a_{m,0}$ which corresponds to the case $n = 0$ in (5.12). In fact, it is much easier to obtain the 3-term recurrence relation for $a_{m,0}$ not from (5.12), but from the 3-term recursion for the coefficients $c_{m,0}$ itself. This latter recursion follows from (3.6), (5.5) and the third order differential equation for $f_\lambda(x)$.

Now let us introduce a generating function

$$h(x) = x^{1-\rho} \sum_{m\ge 0} a_{m,0}\, x^m \tag{5.13}$$

and rewrite the recurrence relation for $a_{m,0}$ as a third order differential equation for $h(x)$. Then this equation can be easily reduced to the standard hypergeometric equation (5.6) for the function ${}_3F_2$. This equation has three solutions with exponents

$$\rho = 2g, g-r_2, -r_1 \tag{5.14}$$

at $x = 0$ which can be written as

$$h(x) = x^{1-\rho}(1-x)^{3g-2}\, {}_4F_3\left( \begin{matrix} g-r_1-\rho, 3g-\rho, 2g-r_2-\rho, 1; \\ 2g+1-\rho, 1-\rho-r_1, 1-\rho+g-r_2 \end{matrix} \,\Big|\, x \right) \tag{5.15}$$

and for any choice of $\rho$ a hypergeometric function ${}_4F_3$ reduces to ${}_3F_2$.

Now we note that for $\rho = 2g$ and $\rho = g - r_2$ the hypergeometric function in the rhs of (5.15) truncates and we obtain from (5.13), (5.15) the following equivalent expressions for the coefficients $a_{m,0}$

$$a_{m,0} = \alpha_1 \frac{(1-g)_m}{(2g)_m}\, {}_4F_3\left( \begin{matrix} -r_2,\ g,\ -g-r_1, 1-2g-m; \\ 1-g-r_2, 1-2g-r_1, g-m \end{matrix} \,\Big|\, 1 \right) \tag{5.16}$$

and

$$a_{m,0} = \alpha_2 \frac{(1-2g-r_2)_m}{(g-r_2)_m}\, {}_4F_3\left( \begin{matrix} r_2-r_1,\ g,\ 2g+r_2, 1-g+r_2-m; \\ 1-g+r_2-r_1, 1+g+r_2, 2g+r_2-m \end{matrix} \,\Big|\, 1 \right). \tag{5.17}$$

To calculate the correct normalisation coefficients $\alpha_1$, $\alpha_2$ for (5.16), (5.17) we take the limit $x_1 \to 0$, $x_2 \to 1$ in (5.4). Then only the term $c_{0,0}$ contributes to the rhs of (5.4) and we use (3.5), (5.9) to calculate $a_{0,0}$.

From the other side the hypergeometric series in (5.16), (5.17) becomes Saalschützian ${}_3F_2$ (i.e. $1 + a_1 + a_2 + a_3 = b_1 + b_2$) and can be evaluated using the so called Saalschütz's theorem [2, 19]

$$ {}_3F_2\left( \begin{array}{c} a, b, -n; \\ c, 1+a+b-c-n \end{array} \right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}. \tag{5.18} $$

Then we obtain

$$ \alpha_1 = \frac{(r_1 - r_2)!}{r_1!} \frac{(3g)_{r_1} (2g)_{r_2}}{(2g)_{r_1 - r_2} (1-g)_{r_2}}, \tag{5.19} $$

$$ \alpha_2 = \frac{r_2!}{r_1!} \frac{(1+g)_{r_1} (g)_{r_1 - r_2}}{(1+g)_{r_2} (2g)_{r_1 - r_2}} \frac{(3g)_{r_1} (2g)_{r_2}}{(2g)_{r_1} (g)_{r_2}}. \tag{5.20} $$

Now we can substitute (5.16)–(5.17) into (5.11). Expanding hypergeometric functions into series we can interchange the order of summations and calculate the sum over $l$ in (5.11). Substituting the result into (5.9) we finally come to the following explicit expressions for $c_{m,n}$

$$ \begin{aligned} c_{m,n} = {} & \frac{(r_1 - r_2)!}{r_1!} \frac{(3g)_{r_1} (2g)_{r_2}}{(2g)_{r_1 - r_2} (1-g)_{r_2}} \frac{(-r_1)_{m+n} (g - r_2)_{m+n} (1-g)_m}{m! n! (1 - 2g - r_1)_m (1 - g - r_2)_m (3g)_n} \\ & \times {}_4F_3\left( \begin{array}{c} -r_2,\ g,\ -g - r_1, 1 - 2g - m - n; \\ 1 - g - r_2, 1 - 2g - r_1, g - m \end{array} \Big| \, 1 \right) \end{aligned} \tag{5.21} $$

or

$$ \begin{aligned} c_{m,n} = {} & \frac{r_2!}{r_1!} \frac{(1+g)_{r_1} (g)_{r_1 - r_2}}{(1+g)_{r_2} (2g)_{r_1 - r_2}} \frac{(3g)_{r_1} (2g)_{r_2}}{(2g)_{r_1} (g)_{r_2}} \frac{(1 - 2g - r_2)_m (-r_1)_{m+n} (2g)_{m+n}}{(1 - 2g - r_1)_m (1 - g - r_2)_m (3g)_n} \\ & \times \frac{1}{m! n!} {}_4F_3\left( \begin{array}{c} r_2 - r_1,\ g,\ 2g + r_2, 1 - g + r_2 - m - n; \\ 1 - g + r_2 - r_1, 1 + g + r_2, 2g + r_2 - m \end{array} \Big| \, 1 \right). \end{aligned} \tag{5.22} $$

As we mentioned above the coefficients of the expansion of the separating polynomial $f_\lambda(x)$ in $x$ are given by $c_{m,0}$, i.e. expressed in terms of hypergeometric functions ${}_4F_3$. It is quite surprising that the coefficients $c_{m,n}$ of the expansion (5.4) are still given in terms of ${}_4F_3$ by formulas (5.21), (5.22) and do not involve higher hypergeometric functions. This happens probably because the basis $p_{mn}$ is very special.

## 6 A representation for the $A_2$ Jack polynomials

Now we are ready to give explicit formulas for the $A_2$ Jack polynomials. Applying the operator $\hat{\mathcal{S}}_3^{-1}$ to (5.3) and using (5.1), (5.2), (5.4) we obtain two representations

$$ \begin{aligned} P^{(1/g)}_{(\lambda_1,\lambda_2,\lambda_3)}(x_1, x_2, x_3) = {} & \frac{(g)_{\lambda_{23}} (2g)_{\lambda_{13}}}{(g)_{\lambda_{12}} (1-g)_{\lambda_{23}}} \frac{(\lambda_{12})!}{(\lambda_{13})!} (x_1 x_2)^{\lambda_3} x_3^{\lambda_1 + \lambda_2 - \lambda_3} \\ & \times \sum_{0 \le m+n \le \lambda_{13}} \frac{1}{m! n!} \frac{(-\lambda_{13})_{m+n} (g - \lambda_{23})_{m+n} (1-g)_m}{(1 - 2g - \lambda_{13})_m (1 - g - \lambda_{23})_m (2g)_n} \\ & \times {}_4F_3\left( \begin{array}{c} -\lambda_{23},\ g,\ -g - \lambda_{13}, 1 - 2g - m - n; \\ 1 - g - \lambda_{23}, 1 - 2g - \lambda_{13}, g - m \end{array} \Big| \, 1 \right) \left[ \frac{x_1 x_2}{x_3^2} \right]^m \left[ \left(1 - \frac{x_1}{x_3}\right) \left(1 - \frac{x_2}{x_3}\right) \right]^n \end{aligned} \tag{6.1} $$

and

$$
\begin{aligned}
P^{(1/g)}_{(\lambda_1,\lambda_2,\lambda_3)}(x_1,x_2,x_3) &= \frac{(g)_{\lambda_{13}+1}}{(g)_{\lambda_{23}+1}}\frac{(\lambda_{23})!}{(\lambda_{13})!}\,(x_1x_2)^{\lambda_3}x_3^{\lambda_1+\lambda_2-\lambda_3} \\
&\times \sum_{0\le m+n\le\lambda_{13}} \frac{1}{m!n!}\,{}_4F_3\left(\begin{matrix} -\lambda_{12},\ g,\ 2g+\lambda_{23}, 1-g+\lambda_{23}-m-n; \\ 1+g+\lambda_{23}, 1-g-\lambda_{12}, 2g+\lambda_{23}-m \end{matrix}\,\Big|\,1\right) \\
&\times \frac{(-\lambda_{13})_{m+n}(1-2g-\lambda_{23})_m(2g)_{m+n}}{(1-2g-\lambda_{13})_m(1-g-\lambda_{23})_m(2g)_n}\left[\frac{x_1x_2}{x_3^2}\right]^m\left[\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_3}\right)\right]^n.
\end{aligned}
\tag{6.2}
$$

Formulas (6.1), (6.2) simplify when $\lambda_2=\lambda_3$ and $\lambda_1=\lambda_2$ accordingly.

First consider the case of one-row partitions $(\lambda,0,0)$. It is convenient to set $x_3=1$. Then we obtain from (6.1)

$$
P^{(1/g)}_{\lambda,0,0}(x_1,x_2,1) = \frac{(2g)_\lambda}{(g)_\lambda}\sum_{0\le m+n\le\lambda}\frac{(-\lambda)_{m+n}(g)_{m+n}}{(1-2g-\lambda)_m(2g)_n}\frac{(x_1x_2)^m[(1-x_1)(1-x_2)]^n}{m!n!}. \tag{6.3}
$$

Expanding the last term we can transform this formula to another basis

$$
P^{(1/g)}_{\lambda,0,0}(x_1,x_2,1) = \sum_{0\le i+2j\le\lambda}\frac{(-1)^{i+j}}{i!j!}\frac{(-\lambda)_{i+2j}\,(g)_{i+j}(g)_{\lambda-i-2j}}{(g)_\lambda}(x_1+x_2)^i(x_1x_2)^j. \tag{6.4}
$$

Finally restoring dependence on $x_3$ we shall rewrite (6.4) in the basis of elementary symmetric functions $e_1=x_1+x_2+x_3$, $e_2=x_1x_2+x_1x_3+x_2x_3$ and $e_3=x_1x_2x_3$.

$$
P^{(1/g)}_{\lambda,0,0}(x_1,x_2,x_3) = \frac{\lambda!}{(g)_\lambda}(-1)^\lambda\sum_{i+2j+3k=\lambda}(-1)^{i+j+k}\frac{(g)_{i+j+k}}{i!j!k!}\,e_1^ie_2^je_3^k. \tag{6.5}
$$

In fact, (6.5) corresponds to the first fundamental weight $\omega_1$ of $A_2$ and is a special case of the Proposition 2.2 from [20] for arbitrary number of variables

$$
P^{(1/g)}_{(r)}(x) = \frac{r!}{(g)_r}\sum_{|\mu|=r}e_\mu\frac{(g)_{l(\mu)}}{\prod\limits_{i\ge1}m_i!}(-1)^{r-l(\mu)} \tag{6.6}
$$

for all partitions $\mu$ such that

$$
\mu=(1^{m_1}2^{m_2}\cdots),\qquad |\mu|=r,\qquad l(\mu)=\sum m_i,\qquad e_\mu=e_1^{m_1}e_2^{m_2}\cdots\,. \tag{6.7}
$$

Now from (6.2) we can similarly calculate $P^{(1/g)}_{\lambda,\lambda,0}(x_1,x_2,x_3)$ which corresponds to the second fundamental weight $\omega_2$ of $A_2$

$$
P^{(1/g)}_{(\lambda,\lambda,0)}(x_1,x_2,x_3) = \frac{\lambda!}{(g)_\lambda}\sum_{|\mu|=2\lambda}e_\mu\frac{(g)_{\lambda-m_3}(-1)^{m_3}}{\left(\lambda-\sum\limits_{i=1}^3 m_i\right)!\prod\limits_{i=1}^2 m_i!} \tag{6.8}
$$

for all partitions $\mu=(1^{m_1}2^{m_2}3^{m_3})$, $|\mu|=2r$.

It seems that (6.8) has the following generalisation to $n$ variables

$$
P^{(1/g)}_{(\lambda^{n-1},0)}(x_1,\ldots,x_n) = \frac{\lambda!}{(g)_\lambda}\sum_{|\mu|=(n-1)\lambda}e_\mu\frac{(g)_{\lambda-m_n}(-1)^{m_n}}{\left(\lambda-\sum\limits_{i=1}^n m_i\right)!\prod\limits_{i=1}^{n-1} m_i!} \tag{6.9}
$$

for all partitions $\mu = (1^{m_1} \cdots n^{m_n})$, $|\mu| = (n-1)\lambda$. We are not aware of any appearances of (6.9) in the literature.

Note that formulas for the $A_2$ Jack polynomials were already obtained earlier in [15]. They were generalised in [11] for partitions of length 3 with any number of variables and further in [12] for the general $A_n$ case.

The approach used in [15, 11] is based on the inversion of the Pieri formulas. For the $A_2$ case formulas of [15] can be written as

$$P_{(\lambda_1,\lambda_2,0)} = \sum_{i=0}^{\min(\lambda_1-\lambda_2,\lambda_2)} \beta^i_{\lambda_1,\lambda_2} P_{(\lambda_1-\lambda_2-i,0,0)} P_{(\lambda_2-i,\lambda_2-i,0)}, \tag{6.10}$$

$$P_{(\lambda_1,\lambda_2,0)} = \sum_{i=0}^{\lambda_2} \gamma^i_{\lambda_1,\lambda_2} P_{(\lambda_1+i,0,0)} P_{(\lambda_2-i,0,0)}, \tag{6.11}$$

where $\beta^i_{\lambda_1,\lambda_2}$ and $\gamma^i_{\lambda_1,\lambda_2}$ are explicitly given factorised coefficients. Note also that (6.11) is a specialisation of the result [4] for partitions of length 2 and any number of variables.

It is interesting to compare (6.10), (6.11) with our formulas (6.1), (6.2). Using the expressions (6.5) and (6.8) we can rewrite (6.10), (6.11) in terms of quintuple sum in the basis of elementary symmetric functions. However, our formulas (6.1), (6.2) contain only triple sums in a different basis.

We tried hard to obtain directly (6.1), (6.2) from (6.10), (6.11) with no success. The corresponding transformations can be reduced to multiple sum identities involving a large number of Pochhammer symbols. It is absolutely unclear how to prove them.

Finally, it would be interesting to rewrite formulas (6.1), (6.2) in the basis $e_1$, $e_2$, $e_3$ like (6.5) for a general partition $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. However, it involves sums of truncated hypergeometric functions ${}_4F_3$ and we did not succeed at this stage to obtain any compact expression for general partitions of length 3.

## 7 Conclusions

In this letter we applied the separation of variables method to construct a formula for the $A_2$ Jack polynomials based on the results of [7]. We explicitly evaluated the coefficients of the expansion of the $A_2$ Jack polynomials in a special basis.

It is a difficult problem to show that our formulas (6.1), (6.2) can be reduced to results of [15, 11]. In fact, their structure is completely different. Nevertheless, our approach can be useful to find a simpler representation for Jack and Macdonald polynomials.

### Acknowledgments

I greatly benefited in my understanding of the quantum SoV from very stimulating discussions with Vadim Kuznetsov and Evgeny Sklyanin. A warm atmosphere at Leeds University, critical and creative talks with Vadim are things which will be greatly missed.

## References

[1] Calogero F., Solution of a three-body problem in one dimension, *J. Math. Phys.* **10** (1969), 2191–2196.

[2] Erdélyi A. (Editor), Higher transcendental functions, Vol. I, McGraw-Hill Book Company, 1953.

[3] Jack H., A class of symmetric functions with a parameter, *Proc. Royal Soc. Edinburgh (A)* **69** (1970), 1–18.

[4] Jing N.H., Józefiak T., A formula for two-row Macdonald functions, *Duke Math. J.* **67** (1992), 377–385.

[5] Koornwinder T.H., Sprinkhuizen-Kuyper I.G., Generalized power series expansions for a class of orthogonal polynomials in two variables, *SIAM J. Math. Anal.* **9** (1978), 457–483.

[6] Kuznetsov V.B., Mangazeev V.V., Sklyanin E.K., $Q$-operator and factorized separation chain for Jack polynomials, *Indag. Math.* **14** (2003), 451–482, math.CA/0306242.

[7] Kuznetsov V.B., Sklyanin E.K., Separation of variables in the $A_2$ type Jack polynomials, *RIMS Kokyuroku* **919** (1995), 27–34, solv-int/9508002.

[8] Kuznetsov V.B., Sklyanin E.K., On Bäcklund transformations for many-body systems, *J. Phys. A: Math. Gen.* **31** (1998), 2241–2251, solv-int/9711010.

[9] Kuznetsov V.B., Sklyanin E.K., Factorisation of Macdonald polynomials, in Symmetries and Integrability of Difference Equations, *Lecture Note Series*, Vol. 255, London Math. Society, Cambridge University Press, 1999, 370–384, q-alg/9703013.

[10] Kuznetsov V.B., Sklyanin E.K., Separation of variables and integral relations for special functions, *The Ramanujan J.* **3** (1999), 5–35, q-alg/9705006.

[11] Lassalle M., Explicitation des polynômes de Jack et de Macdonald en longueur trois, *C.R. Acad. Sci. Paris Ser. I Math.* **333** (2001), 505–508.

[12] Lassalle M., Schlosser M., Inversion of the Pieri formula for Macdonald polynomials, *Adv. Math.*, **202** (2006), 289–325, math.CO/0402127.

[13] Macdonald I.G., Symmetric functions and Hall polynomials, 2nd ed., Oxford University Press, Oxford, 1995.

[14] Olshanetsky M.A., Perelomov A.M., Quantum integrable systems related to Lie algebras, *Phys. Rep.* **94** (1983), 313–404.

[15] Perelomov A.M., Ragoucy E., Zaugg P., Explicit solution of the quantum three-body Calogero–Sutherland model, *J. Phys. A: Math. Gen.* **31** (1998), L559–L565, hep-th/9805149.

[16] Oshima T., Sekiguchi H., Commuting families of differential operators invariant under the action of a Weyl group, *J. Math. Sci. Univ. Tokyo* **2** (1995), 1–75.

[17] Sklyanin E.K., Quantum inverse scattering method. Selected topics, in Quantum Group and Quantum Integrable Systems, Editor M.-L. Ge, *Nankai Lectures in Mathematical Physics*, World Scientific, Singapore, 1992, 63–97, hep-th/9211111.

[18] Sklyanin E.K., Separation of variables. New trends, *Progr. Theoret. Phys. Suppl.* **118** (1995), 35–60, solv-int/9504001.

[19] Slater L.J., Generalized hypergeometric functions, Cambridge University Press, 1966.

[20] Stanley R.P., Some combinatorial properties of Jack symmetric functions, *Adv. Math.* **77** (1989), 76–115.

[21] Sutherland B., Quantum many-body problem in one dimension, I, II, *J. Math. Phys.* **12** (1971), 246–250, 251–256.

[22] Watson G.N., The product of two hypergeometric functions, *Proc. London Math. Soc. (2)* **20** (1922), 191–195.