WAVE PROPAGATION IN STRATIFIED MEDIUM: NEW MATRIX FORM OF THE WAVE DIFFERENCE EQUATION

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Results of research of possibility of transformation of a wave difference equation into a system of the first-order difference equations are presented. In contrast to the method used previously, an unknown grid function is split into two new auxiliary functions, which have definite properties. Several examples show that proposed approach can be used for solving different physical problems associated with the wave propagation in one dimension.

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INTRODUCTION

It is a well-known fact that second-order differential problems are very often encountered in the applications, especially among those derived from physics. A more frequent appearance of second-order problems is also true in difference equations.

The Helmholtz equation, which describes scalar wave propagation suited for electromagnetic wave propagating in a dielectric is the basic of the wave electromagnetic theory [1 - 12]. This differential equation has no analytical solution in the general case, except for a few special cases. It is usually considered in the framework of model, in which instead of the continuous variation of the permittivity $\varepsilon(\xi)$, the piecewise constant law is introduced. Together with the matrix methods this approach gives possibility to study different wave phenomena in inhomogeneous media [3, 6, 7, 9, 10]. Finite difference method is also used to analyze the electromagnetic waves in stratified medium (see, for example [13, 14]).

In this paper, a new method of modification of the difference wave equation is presented, in which, the difference equation is transformed into a system of the first-order difference equations. We also propose reverse scheme for finding the characteristics of some wave propagation problems. Numerical simulation shows that this method gives acceptable results in the study of some wave phenomena [15].

1. TRANSFORMATION OF A SECOND-ORDER LINEAR DIFFERENCE EQUATION

We shall deal with difference equation

$$y_{k+1} + y_{k-1} - \left(2 - h^2\varepsilon_k\right) y_k = 0, \tag{1}$$

that is a grid approximation of the wave differential equation

$$\frac{d^2 y}{d\xi^2} + \varepsilon(\xi) y = 0 \,. \tag{2}$$

It is common knowledge that a difference equation of order k may be transformed in a standard way to a system of k first-order difference equations [16 - 21]. For example, a second-order difference equation

$$y_{k+1} + a_k y_k + b_k y_{k-1} = 0 \tag{3}$$

can be rewritten as

$$Y_k = T_k Y_{k-1} + F_k \,, \tag{4}$$

where

$$Y_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix},\; x_k = y_{k+1},\; T_k = \begin{pmatrix} -a_k & -b_k \\ 1 & 0 \end{pmatrix},\; F_k = \begin{pmatrix} f_k \\ 0 \end{pmatrix}. \tag{5}$$

This is a pure mathematic transformation that does not take into account the possible physical meaning of the solution of the equation (1). It is desirable that the components of the vector Y_k represent some physical notions. For the case of wave propagation, very useful notions are "forward and backward waves" that form the general field. The problem of creating a special field distribution is also needed in such approach [22 - 27].

We shall seek a solution of difference equation (3) as [15]

$$y_k = y_k^{(1)} + y_k^{(2)} \,, \tag{6}$$

where $y_k^{(1)}, y_k^{(2)}$ are the new unknown grid functions.

By introducing two unknowns $y_k^{(1)}, y_k^{(2)}$ instead of the one y_k, we can impose an additional condition. Let us assume that

$$y_{k+1} = \rho_k^{(1)} y_k^{(1)} + \rho_k^{(2)} y_k^{(2)} \,, \tag{7}$$

where $\rho_k^{(1)}$ and $\rho_k^{(2)}$ ($\rho_k^{(1)} \neq \rho_k^{(2)}$) are the given numbers. We will call $\rho_k^{(1)}$ and $\rho_k^{(2)}$ as characteristic multipliers.

From (3), (6), and (7) it follows that we have the system of linear equations:

$$\begin{aligned} & y_{k+1}^{(1)} + y_{k+1}^{(2)} = \rho_k^{(1)} y_k^{(1)} + \rho_k^{(2)} y_k^{(2)}, \\ & \left(\rho_{k+1}^{(1)} + a_{k+1}\right) y_{k+1}^{(1)} + \left(\rho_{k+1}^{(2)} + a_{k+1}\right) y_{k+1}^{(2)} = b_{k+1}\left(y_k^{(1)} + y_k^{(2)}\right). \end{aligned} \tag{8}$$

This system can be easily transformed into the normal system of difference equations:

$$\begin{aligned} y_{k+1}^{(1)} &= T_{k,11} y_k^{(1)} + T_{k,12} y_k^{(2)}, \\ y_{k+1}^{(2)} &= T_{k,21} y_k^{(1)} + T_{k,22} y_k^{(2)}, \end{aligned} \tag{9}$$

or in matrix form:

$$\begin{pmatrix} y_{k+1}^{(1)} \\ y_{k+1}^{(2)} \end{pmatrix} = T_k \begin{pmatrix} y_k^{(1)} \\ y_k^{(2)} \end{pmatrix}. \tag{10}$$

Transfer matrix T_k has the following components

$$T_k = \begin{pmatrix} -\dfrac{\left[b_{k+1} + \left(\rho_{k+1}^{(2)} + a_{k+1}\right)\rho_k^{(1)}\right]}{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}\right)} & -\dfrac{\left[b_{k+1} + \left(\rho_{k+1}^{(2)} + a_{k+1}\right)\rho_k^{(2)}\right]}{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}\right)} \\ \dfrac{\left[b_{k+1} + \left(\rho_{k+1}^{(1)} + a_{k+1}\right)\rho_k^{(1)}\right]}{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}\right)} & \dfrac{\left[b_{k+1} + \left(\rho_{k+1}^{(1)} + a_{k+1}\right)\rho_k^{(2)}\right]}{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)}\right)} \end{pmatrix}. \tag{11}$$

If we know the matrix T_k elements, we can find the characteristic multipliers

$$\begin{aligned} \rho_k^{(1)} &= T_{k,11} + T_{k,21}, \\ \rho_k^{(2)} &= T_{k,22} + T_{k,12}. \end{aligned} \tag{12}$$

Now we describe several properties of the normal system of difference equations (10).

From (11) it follows that we can choose the sequences $\rho_k^{(1)}$ and $\rho_k^{(2)}$ in such way that matrix T_k will be diagonal one. It is realized by setting $T_{k,12} = 0$ and $T_{k,21} = 0$

$$T_{k,12} = -\frac{\left[b_{k+1} + \left(\rho_{k+1}^{(2)} + a_{k+1} \right) \rho_k^{(2)} \right]}{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)} \right)} = 0\,, \tag{13}$$

$$T_{k,21} = \frac{\left[b_{k+1} + \left(\rho_{k+1}^{(1)} + a_{k+1} \right) \rho_k^{(1)} \right]}{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)} \right)} = 0\,. \tag{14}$$

These conditions give the non-linear second-order rational difference equations (Riccati type[1] [18, 28 - 30]) for $\rho_k^{(1)}$ and $\rho_k^{(2)}$

$$\rho_{k+1}^{(1,2)} = -\frac{b_{k+1}}{\rho_k^{(1,2)}} - a_{k+1} = -\frac{b_{k+1} + \rho_k^{(1,2)} a_{k+1}}{\rho_k^{(1,2)}}\,. \tag{15}$$

If $\rho_k^{(1)} \neq \rho_k^{(2)}$, the system (9) takes the form:

$$\begin{aligned} y_{k+1}^{(1)} &= \rho_k^{(1)} y_k^{(1)}, \\ y_{k+1}^{(2)} &= \rho_k^{(2)} y_k^{(2)}. \end{aligned} \tag{16}$$

Solutions of these equations are

$$\begin{aligned} y_k^{(1)} &= \prod_{s=1}^{k-1} \rho_s^{(1)} y_1^{(1)},\ 2 \le k \\ y_k^{(2)} &= \prod_{s=2}^{k-1} \rho_s^{(2)} y_1^{(2)}. \end{aligned} \tag{17}$$

In this case $y_k^{(1)}, y_k^{(2)}$ are the linearly independent solutions [31] of the equation (3). Indeed,

$$y_{k+1}^{(1)} + a_k y_k^{(1)} + b_k y_{k-1}^{(1)} = 0\,, \tag{18}$$

$$y_{k+1}^{(2)} + a_k y_k^{(2)} + b_k y_{k-1}^{(2)} = 0\,. \tag{19}$$

and the determinant $\Delta_2\left(y_k^{(1)}, y_k^{(2)}\right)$ is not zero

$$\begin{aligned} \Delta_2(y_k^{(1)}, y_k^{(2)}) &= \begin{vmatrix} y_k^{(1)} & y_{k+1}^{(1)} \\ y_k^{(2)} & y_{k+1}^{(2)} \end{vmatrix} = \\ &= \begin{vmatrix} y_k^{(1)} & \rho_k^{(1)} y_k^{(1)} \\ y_k^{(2)} & \rho_k^{(2)} y_k^{(2)} \end{vmatrix} = y_k^{(1)} y_k^{(2)} \left(\rho_k^{(2)} - \rho_k^{(1)} \right) \neq 0. \end{aligned} \tag{18}$$

Characteristic multipliers $\rho_k^{(1)}$ and $\rho_k^{(2)}$ are the solutions of the nonlinear difference equation (15) with the initial values $\rho_1^{(1)} \neq \rho_1^{(2)}$. In the general case, these initial values can be chosen arbitrary.

For the homogeneous case, the equation (15) takes the form

$$\rho_{k+1}^{(1,2)} \rho_k^{(1,2)} + a\rho_k^{(1,2)} + b = 0\,, \tag{19}$$

and has two stationary points ρ_1, ρ_2 that are the solutions of the characteristic square equation

$$\rho^2 + a\rho + b = 0\,. \tag{20}$$

The solution of the equation (19) is [29]

$$\rho_k^{(1,2)} = \frac{\left(\rho_1^{(1,2)} - \rho_2 \right) \rho_1^k - \left(\rho_1^{(1,2)} - \rho_1 \right) \rho_2^k}{\left(\rho_1^{(1,2)} - \rho_2 \right) \rho_1^{k-1} - \left(\rho_1^{(1,2)} - \rho_1 \right) \rho_2^{k-1}}\,. \tag{21}$$

If we choose $\rho_1^{(1)} = \rho_1,\ \ \rho_1^{(2)} = \rho_2$, then

$$\begin{aligned} y_k^{(1)} &= \rho_1^{k-1} y_1^{(1)}, \\ y_k^{(2)} &= \rho_2^{k-1} y_1^{(2)}. \end{aligned} \tag{22}$$

This is a well-known result [16 - 21].

If we choose $\rho_1^{(1)} \neq \rho_1$ and $\rho_1^{(2)} \neq \rho_2$, the grid functions $y_k^{(1)}$ and $y_k^{(2)}$ are

$$\begin{aligned} y_k^{(1)} &= \frac{\left(\rho_1^{(1)} - \rho_2 \right) \rho_1^{k-1} - \left(\rho_1^{(1)} - \rho_1 \right) \rho_2^{k-1}}{\rho_1 - \rho_2} y_1^{(1)}, \\ y_k^{(2)} &= \frac{\left(\rho_1^{(2)} - \rho_2 \right) \rho_1^{k-1} - \left(\rho_1^{(2)} - \chi_1 \right) \rho_2^{k-1}}{\rho_1 - \rho_2} y_1^{(2)}, \end{aligned} \tag{23}$$

where $\rho_1^{(1)} \neq \rho_1^{(2)}$.

The sum of these grid functions ($y_k^{(1)} + y_k^{(2)}$) is a grid function that do not depend on $\rho_1^{(1)}$ and $\rho_1^{(2)}$.

We must note that the stationary point of the equation (15) can be unstable.

Diagonal systems of difference equations can be useful in many applications. One of them is finding conditions when the grid functions $y_k^{(1)}, y_k^{(2)}$ have the given properties.

Let's, for example, find the condition when b_k and a_k are not constants, but we want $\rho_k^{(1)}$ in (16) to be a constant $\rho_k^{(1)} = \rho_*$. From (15) we obtain that in this case b_{k+1} and a_{k+1} have to be linearly proportional

$$b_{k+1} = -\rho_* a_{k+1} - \rho_*^2\,. \tag{24}$$

In some cases, it is useful[2] to work with S-matrix (see, for example, [9, 10, 32])

$$\begin{pmatrix} y_k^{(2)} \\ y_{k+1}^{(1)} \end{pmatrix} = S_k \begin{pmatrix} y_k^{(1)} \\ y_{k+1}^{(2)} \end{pmatrix}. \tag{25}$$

Components of S-matrix for a second-order difference equation (3) are

$$S_k = \begin{pmatrix} -\dfrac{\left[b_{k+1} + \rho_k^{(1)} \left(\rho_{k+1}^{(1)} + a_{k+1} \right) \right]}{\left[b_{k+1} + \rho_k^{(2)} \left(\rho_{k+1}^{(1)} + a_{k+1} \right) \right]} & \dfrac{\left(\rho_{k+1}^{(1)} - \rho_{k+1}^{(2)} \right)}{\left[b_{k+1} + \rho_k^{(2)} \left(\rho_{k+1}^{(1)} + a_{k+1} \right) \right]} \\ \dfrac{\left(\rho_k^{(1)} - \rho_k^{(2)} \right) b_{k+1}}{\left[b_{k+1} + \rho_k^{(2)} \left(\rho_{k+1}^{(1)} + a_{k+1} \right) \right]} & -\dfrac{\left[b_{k+1} + \rho_k^{(2)} \left(\rho_{k+1}^{(2)} + a_{k+1} \right) \right]}{\left[b_{k+1} + \rho_k^{(2)} \left(\rho_{k+1}^{(1)} + a_{k+1} \right) \right]} \end{pmatrix}. \tag{26}$$

2. WAVE PROPAGATION IN ONE DIMENSION

2.1. WAVE PROPAGATION THROUGH A HOMOGENEOUS DIELECTRIC LAYER

The differential equation (2) is usually considered in the framework of model, in which instead of the continuous variation of the permittivity $\varepsilon(\xi)$, the piecewise constant law is introduced [3, 6, 7, 9, 10]. Matrix form that was proposed in the previous section in the case of $h \ll 1$ will be similar to the piecewise approach if $\rho_k^{(1)}, \rho_k^{(2)}$ are the solutions of the "local" characteristic equation

$$\rho_k^2 - \left(2 - h^2 \varepsilon_k \right) \rho_k + 1 = 0\,. \tag{27}$$

[1] Properties of the sequences defined by a difference equation of Riccati type are presented in [28, 29]

[2] It is known that S-matrix is useful when the solutions have exponential growth.

Indeed, in the frame of the piecewise approach we can find the transfer matrix (see, for example [3, 6]) and from (12) we obtain

$$\rho_k^{P(1,2)} = \exp(\pm i\sqrt{\varepsilon_k}h) . \qquad (28)$$

If $\sqrt{|\varepsilon_k|}h << 1$, $\rho_k^{P(1)}$ and $\rho_k^{P(2)}$ with an accuracy of $|\varepsilon_k| h^2$ satisfy the equation (27).

The most interesting question in the using of the proposed matrix form is the properties of the partial solutions $y_k^{(1)}, y_k^{(2)}$ in the case when matrix T_k is a diagonal.

Let's consider the example of the wave propagation through the homogeneous dielectric layer[3]

$$\varepsilon = \begin{cases} 1, & -\infty \le \xi \le \xi_1 \\ \varepsilon_2, & \xi_1 < \xi \le \xi_2 \\ 1, & \xi > \xi_2 \end{cases} . \qquad (29)$$

Using the standard "mode matching technique" [33], we obtain analytical expressions for the reflection and transmission coefficients

$$R_0 = \frac{(\varepsilon_2 - 1)2i\sin\left\{\sqrt{\varepsilon_2}(\xi_2 - \xi_1)\right\}\exp(i2\xi_1 + i\xi_2)}{\left[\left(\sqrt{\varepsilon_2}+1\right)^2\exp\left\{-i\sqrt{\varepsilon_2}(\xi_2-\xi_1)\right\} - \left(\sqrt{\varepsilon_2}-1\right)^2\exp\left\{i\sqrt{\varepsilon_2}(\xi_2-\xi_1)\right\}\right]},$$
$$T_0 = \frac{4\sqrt{\varepsilon_2}\exp(i\xi_1 - i\xi_2)}{\left[\left(\sqrt{\varepsilon_2}+1\right)^2\exp\left\{-i\sqrt{\varepsilon_2}(\xi_2-\xi_1)\right\} - \left(\sqrt{\varepsilon_2}-1\right)^2\exp\left\{i\sqrt{\varepsilon_2}(\xi_2-\xi_1)\right\}\right]}. \qquad (30)$$

Consider the case when the layer permittivity $\varepsilon_2 = 3 + i \times 0.03$ and $\xi_1 = 2\pi,\ \xi_2 = 11\times 2\pi$. For these parameters $|R_0| = 0.3274,\ |T_0| = 0.5307$ Grid approximation for the permittivity ($h = 2\pi/100$) is:

$$\varepsilon_k = \begin{cases} 1,\ k = -\infty,...,N_1 & N_1 = 100 \\ 3 + i\,0.03,\ k = N_1+1,...,N_2 & N_2 = 1100 \\ 1,\ k = N_2+1,...,\infty \end{cases} . \qquad (31)$$

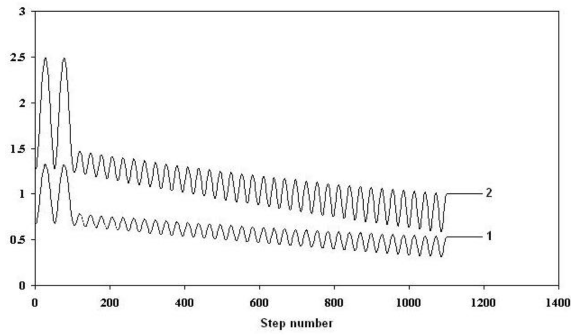

Fig. 1. Graph of the modulus of grid function $Y_k = y_k^{(1)} + y_k^{(2)}$, where $y_k^{(1)}, y_k^{(2)}$ are the solutions of the systems of the first-order difference equations (25) (S-matrix approach). Matrix T_k is not a diagonal – (1). Graph of modulus of the grid function $y_k^{(1)}$ for the reverse problem ($y_{N_3}^{(1)} = 1$) – (2)

Using the S-matrix formalism (25) with $\rho_k^{(1)}, \rho_k^{(2)}$ that are the solutions of the "local" characteristic equation (27), we can calculate the reflection and transmission coefficients and the values of the grid function $y_k = Y_k = y_k^{(1)} + y_k^{(2)}$ (for $k < N_1$ and $k > N_2$ $y_k^{(1)}$ is a forward wave and $y_k^{(2)}$ is a backward wave).

For $y_1^{(1)} = 1,\ y_1^{(2)} = R$ and $y_k^{(2)} = 0,\ k > N_2$, we obtained $\left|R^{(Y)}\right| = \left|S_{11}^{\Sigma}\right| = 0.3222,$ $\left|T^{(Y)}\right| = \left|S_{21}^{\Sigma}\right| = 0.5312.$ Comparison $R^{(Y)}$ and $T^{(Y)}$ with the exact values R_0 and T_0 shows good agreement. Graph of the grid function Y_k is presented in Fig. 1 (1).

It was already noted that the sequences $\rho_k^{(1)}$ and $\rho_k^{(2)}$ are the arbitrary ones. But in the frame of the S-matrix formalism it is convenient to use $\rho_k^{(1)}, \rho_k^{(2)}$ that are the solutions of the "local" characteristic equation (27) before and after the dielectric layer, and the arbitrary values insider the layer. In this case $R = S_{11}^{\Sigma}$ and $T = S_{21}^{\Sigma}$. For example, if we choose $\rho_k^{(1)} = 0.5$ and $\rho_k^{(2)} = 0.8$ inside the dielectric layer, we obtain $|R| = \left|S_{11}^{\Sigma}\right| = 0.3222,$ $|T| = \left|S_{21}^{\Sigma}\right| = 0.5312.$ Using the other values of $\rho_k^{(1)}$ and $\rho_k^{(2)}$ at constant h practically do not change R and T. This result proves that the choice of the values of $\rho_k^{(1)}, \rho_k^{(2)}$ does not effect on the numerical solutions of the difference equation (1).

If $\rho_k^{(1)}, \rho_k^{(2)}$ are the solutions of the Riccati equations (15), $y_k^{(1)}$ and $y_k^{(2)}$ are the linearly independent solutions. For finding these solutions we have to set the initial conditions $\rho_1^{(1)}, \rho_1^{(2)}$. It is convenient to choose $\rho_1^{(1)}, \rho_1^{(2)}$ that are the solutions of the "local" characteristic equation (27) with $\varepsilon_1 = 1$: $\rho_1^{(1)} = \rho_{vac}^{(1)},\ \rho_1^{(2)} = \rho_{vac}^{(2)}$. Results of calculation the solutions of the Riccati equations (15) $y_k^{(1)}$ and $y_k^{(2)}$ with the initial conditions $y_1^{(1)} = 1,\ y_1^{(2)} = 1$ are presented in Fig. 2.

Are these functions $y_k^{(1)}, y_k^{(2)}$ associated with the function Y_k ? Comparison of these grid functions shows that the following relation is fulfilled

$$Y_k = y_k^{(1)} + R^{(Y)} y_k^{(2)} . \qquad (32)$$

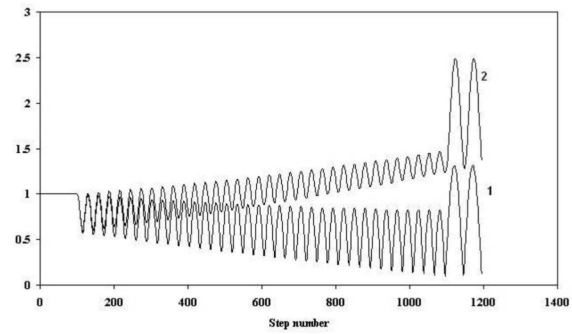

Fig. 2. Graphs of modulus of the grid functions $y_k^{(1)}$ (1) and $y_k^{(2)}$ (2) for the wave diffraction on the homogeneous dielectric layer. Matrix T_k is a diagonal

Up to now we have deal with the grid functions $y_k^{(1)}, y_k^{(2)}$ which fulfilled the initial conditions $y_1^{(1)} = 1,\ y_1^{(2)} = 1$ and were calculated on the basis of equations (16) with $\rho_k^{(1)}, \rho_k^{(2)}$ which are the solutions of

[3] Unlike the works [13, 14], we consider fields in the whole space, not only in the layer.

the Riccati equations (15) with the initial conditions $\rho_1^{(1)} = \rho_{vac}^{(1)}$, $\rho_1^{(2)} = \rho_{vac}^{(2)}$.

If we set the initial conditions[4] $y_{N_3}^{(1)} = 1$, $y_{N_3}^{(2)} = 1$ and $\rho_{N_3}^{(1)} = \rho_{vac}^{(1)}$, $\rho_{N_3}^{(2)} = \rho_{vac}^{(2)}$, and solve the equations (16) and the Riccati equations (15) in the reverse order

$$y_k^{(1)} = y_{k+1}^{(1)} / \rho_k^{(1)}, \quad y_k^{(2)} = y_{k+1}^{(2)} / \rho_k^{(2)}. \tag{33}$$

$$\rho_k^{(1,2)} = -\frac{b_{k+1}}{\left(\rho_{k+1}^{(1,2)} + a_{k+1}\right)}, \tag{34}$$

we obtain the new grid functions that are the independent solutions of the wave equation, too. After the layer, they describe the forward and backward waves. So, $y_k^{(2)}$ has no physical meaning and we have to deal with $y_k^{(1)}$ only. Graph of modulus of the grid function $y_k^{(1)}$ for the inverse problem are presented in Fig. 1 (2).

From physical point of view, before the layer $y_k^{(1)}$ must be the superposition of the forward and backward waves. Using simple decomposition, we can find amplitudes of the forward and backward waves and find the reflection and transmission coefficients $\left|R^{(rev)}\right| = 0.3219$, $\left|T^{(rev)}\right| = 0.5314$. Comparison $R^{(rev)}$ and $T^{(rev)}$ with the exact values R_0 and T_0 (or with $R^{(Y)}$, $T^{(Y)}$) shows good agreement. Setting the initial value $y_{N_3}^{(1)} = Y_{N_3} = 0.531 \times \exp(i \times 1.9865)$ gives the full coincidence of $y_k^{(1,rev)}$ and Y_k.

2.2. WAVE PROPAGATION THROUGH AN INHOMOGENEOUS DIELECTRIC LAYER

Let's consider wave reflection from an inhomogeneous layer. To obtain a rigorous solution of this problem, we must solve the equation (2). Such rigorous solutions in closed form are known only for a few forms of the function $\varepsilon(\xi)$ [1 - 6]. One function is

$$\varepsilon(\xi) = \frac{a}{(b+\xi)^2}. \tag{35}$$

In this case, electric field amplitude varies along the coordinate according the law [2]

$$E = C_1 (b+\xi)^{\chi_1} + C_2 (b+\xi)^{\chi_2}, \tag{36}$$

where C_1, C_2 are arbitrary constants and

$$\chi_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - a}. \tag{37}$$

Using the standard "mode matching technique", we obtain analytical expressions for the reflection and transmission coefficients

$$T_0 = 2\frac{\exp(-i\xi_2 + i\xi_1)}{\left[\begin{array}{l}\dfrac{\chi_2}{\chi_1-\chi_2}\dfrac{(b+\xi_1)^{\chi_1}}{(b+\xi_2)^{\chi_1}}\left[1-\dfrac{i\chi_1}{(b+\xi_1)}\right]\left[\dfrac{i}{\chi_2}(b+\xi_2)-1\right]- \\ \dfrac{\chi_1}{\chi_1-\chi_2}\dfrac{(b+\xi_1)^{\chi_2}}{(b+\xi_2)^{\chi_2}}\left[1-\dfrac{i\chi_2}{(b+\xi_1)}\right]\left[\dfrac{i}{\chi_1}(b+\xi_2)-1\right]\end{array}\right]}, \tag{38}$$

$$R_0 = \exp(i\xi_2 - i\xi_1)\frac{T}{2}\times \left[\begin{array}{l}\dfrac{\chi_2}{\chi_1-\chi_2}\dfrac{(b+\xi_1)^{\chi_1}}{(b+\xi_2)^{\chi_1}}\left[1+\dfrac{i\chi_1}{(b+\xi_1)}\right]\left[\dfrac{i}{\chi_2}(b+\xi_2)-1\right]- \\ \dfrac{\chi_1}{\chi_1-\chi_2}\dfrac{(b+\xi_1)^{\chi_2}}{(b+\xi_2)^{\chi_2}}\left[1+\dfrac{i\chi_2}{(b+\xi_1)}\right]\left[\dfrac{i}{\chi_1}(b+\xi_2)-1\right]\end{array}\right]. \tag{39}$$

Consider the case when $h = 2\pi/100$, $a = 32\pi^2$, $b = 2\pi$ and $\xi_1 = 2\pi$, $\xi_2 = 11 \times 2\pi$ ($\varepsilon(\xi_1) = 1.98$; $\varepsilon(\xi_2) = 0.0055$). For this parameters $|R_0| = 0.6876$, $|T_0| = 0.7260$ ($|R_0|^2 + |T_0|^2 = 1.0000$). Using the "reverse approach" (33) and (34), we get $\left|R^{(rev)}\right| = 0.6911$ $\left|T^{rev)}\right| = 0.7230$ ($\left|R^{(rev)}\right|^2 + \left|T^{(rev)}\right|^2 =$ 1.0003). Comparison $R^{(rev)}$ and $T^{(rev)}$ with the exact values R_0 and T_0 shows good agreement.

CONCLUSIONS

Results presented above show that we can transform the wave difference equation into a system of the first-order difference equations by splitting an unknown grid function into two new auxiliary functions. These functions are defined with an accuracy of two arbitrary grid functions. But the sum of these functions does not depend on the values of these auxiliary grid functions. Obtained numerical results coincide with the exact ones calculated on the bases of the exact formulas.

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[4] $N_3 > N_2$ is an arbitrary number. In our calculations we took $N_3 = 1200$.

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РАСПРОСТРАНЕНИЕ ВОЛН В СЛОИСТОЙ СРЕДЕ: НОВАЯ МАТРИЧНАЯ ФОРМА ВОЛНОВОГО РАЗНОСТНОГО УРАВНЕНИЯ

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Представлены результаты исследования возможности преобразования волнового разностного уравнения в систему разностных уравнений первого порядка. В отличие от существующего подхода, в предлагаемом методе сеточная функция представляется в виде суммы двух новых сеточных функций, которые обладают определенными свойствами. Рассмотренные примеры показывают, что предлагаемый метод может быть полезным при исследовании процессов распространения волн в одномерном случае.

РОЗПОВСЮДЖЕННЯ ХВИЛЬ У ШАРУВАТОМУ СЕРЕДОВИЩІ: НОВА МАТРИЧНА ФОРМА ХВИЛЬОВОГО РІЗНИЦЕВОГО РІВНЯННЯ

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Представлені результати дослідження можливості перетворення хвильового різницевого рівняння в систему різницевих рівнянь першого порядку. На відміну від існуючого підходу, в пропонованому методі сіткова функція представляється у вигляді суми двох нових сіткових функцій, які мають певні властивості. Розглянуті приклади показують, що пропонований метод може бути корисним при дослідженні процесів розповсюдження хвиль в одновимірному випадку.