An Asymmetric Noncommutative Torus

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Abstract. We introduce a family of spectral triples that describe the curved noncommutative two-torus. The relevant family of new Dirac operators is given by rescaling one of two terms in the flat Dirac operator. We compute the dressed scalar curvature and show that the Gauss–Bonnet theorem holds (which is not covered by the general result of Connes and Moscovici).

Key words: noncommutative geometry; Gauss–Bonnet; spectral triple

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1   Introduction

In the seminal works [2, 7] Connes and Tretkoff initiated the investigation of curvature aspects on the noncommutative two torus $\mathbb{T}_\theta^2$ and have shown the analogue of the Gauss–Bonnet theorem for the conformally rescaled Dirac operator D and the related spin Laplacian corresponding to the standard conformal structure. In [10] these studies were extended to arbitrary conformal structure. The scalar curvature itself was defined and computed in [6] and independently in [11].

The methods used in these papers build on Connes' pseudodifferential calculus [7] and heat kernel short-time asymptotic expansion. The novelty therein is the employment of *twisted* (or *non-unimodular*) spectral triples [5], and as a consequence non-tracial weights and the modular operator. For some related papers see [1, 12, 13].

In the present paper we introduce a family of *usual* spectral triples in the sense of [4], known also as K-cycles [3, Definition IV.2.11], that describe a curved $\mathbb{T}_\theta^2$. The relevant new Dirac operators correspond to the rescaling one of two terms in the flat Dirac operator by a positive element k from the opposite coordinate algebra in the commutant. This assures that the commutators of D with the elements of the algebra are bounded. Thus, to the best of our knowledge, this provides the first instance of usual spectral triples which describe non flat (curved) noncommutative geometry. It should also be stressed that partial rescalings by k from the algebra itself go outside the realm of spectral triples (even twisted), with geometric meaning (to the best of our knowledge) not yet fully clarified (in this respect see however, e.g., [15]).

Unlike the classical case the new family of Dirac operators is not conformally related to the standard family of "flat" (equivariant) Dirac operators. Since they are not an overall rescaling of any of the so far known Dirac operators one cannot apply the general argument used by Connes

and Moscovici [6, Theorem 2.1] to establish the Gauss–Bonnet theorem. Thus it is the first instance when one needs to effectively compute it, without certainty that it will be satisfied.

This has been shown *perturbatively* up to second order in [8] for small modifications of the flat Dirac operator depending on four elements in the opposite algebra. In the present paper we perform the *exact* computation, which shows that the Gauss–Bonnet theorem indeed extends to a much larger class of spectral triples.

Moreover, we compute the expression for the dressed scalar curvature as an element of the opposite algebra of $\mathbb{T}^2_\theta$. In the classical case it corresponds to the product of the scalar curvature by the coefficient of the metric volume form with respect to the Lebesgue volume. It should be mentioned that such a dressing is actually tacitly present in all the previous computations.

The results obtained here are possible due to a recent neat generalisation by Lesch [14] of the "rearrangement lemma", which is an important technical tool in [7].

The paper is organised as follows: in Section 2 we briefly present the asymmetric rescaling of the classical flat 2-torus and its curvature. Next we introduce a quantum (noncommutative) analogue of the associated Dirac operator, and the corresponding new family of spectral triples on the noncommutative torus. In Section 3, which is the main part of the paper, we compute the value of the respective zeta function of the new Dirac operators at $z = 0$ following the heat trace techniques and pseudodifferential calculus elaborated in [6, 7], and an adaptation of the Lesch rearrangement lemma [14]. In Section 4 we comment on some algebra computations of the (undressed) curvature.

2 Dirac operator

Let $\mathbb{T}^2$ be the classical torus with coordinates $(x, y) \in [0, 2\pi]^2$, equipped with the metric
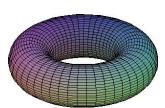
$$dx^2 + k^{-2}(x, y)dy^2, \tag{2.1}$$

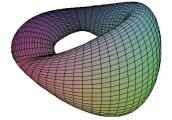where k is a positive function. For example such an "asymmetric torus" metric with

$$k^{-1} = c + \cos y$$

comes as the induced metric on the usual realisation of $\mathbb{T}^2$ as an embedded surface in $\mathbb{R}^3$ with the following parametrisation

$$X = (c + \cos y) \cos x, \qquad Y = (c + \cos y) \sin x, \qquad Z = \sin y.$$

Torus embedded in $\mathbb{R}^3$.   Asymmetric torus in $\mathbb{R}^3$.

The scalar curvature of the torus with the metric (2.1) reads

$$R = 2k^{-1}\partial_x^2(k) - 4k^{-2}(\partial_x(k))^2. \tag{2.2}$$

In the commutative case such metric is, of course, conformally equivalent to some flat metric on the torus even though the explicit formula for the curvature depends on the chosen coordinate

system. However, when passing to the noncommutative torus we are entering a new unexplored land, where one does not know what is a *metric* and what is the exact meaning of *conformally equivalent*. As in the approach of Connes the natural object is the Dirac operator rather than the metric itself. For this reason we propose a new Dirac operator, which generalises to the noncommutative situation the classical case of asymmetric torus.

We start with the commutative case of Dirac operator on $L^2(\mathbb{T}^2, k^{-1}dxdy) \otimes \mathbb{C}^2$ for the metric (2.1)

$$\tilde{D} = -i\sigma^1 \left(\partial_x - \tfrac{1}{2}k^{-1}\partial_x(k) \right) - i\sigma^2 k \ \partial_y,$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Using the multiplication by $\sqrt{k}$ we obtain a unitarily equivalent self-adjoint Dirac operator

$$\tilde{D} = -i\sigma^1\partial_x - i\sigma^2 \left(k\partial_y + \tfrac{1}{2}\partial_y(k) \right) \tag{2.3}$$

on the dense domain $H^{1,2}(T)$ in the Hilbert space $L^2(T, dxdy) \otimes \mathbb{C}^2$.

Next we pass to the noncommutative torus $\mathbb{T}^2_\theta$ [3], about which we collected the needed information in Appendix A. In particular we shall use therein the usual trace $\mathfrak{t}$, the usual self-adjoint derivations δ_1, δ_2, and the standard real structure (Tomita–Takesaki conjugation) J on the coordinate algebra $A(\mathbb{T}^2_\theta)$. Then we introduce the following new family of spectral triples, which is an immediate noncommutative analogue of (2.3).

Definition 2.1. For $k \in JA(\mathbb{T}^2_\theta)J^{-1}$ we set

$$D_k = \sigma^1\delta_1 + \sigma^2 \left(k \, \delta_2 + \tfrac{1}{2}\delta_2(k) \right)$$

acting on $\mathcal{H} = L^2(\mathbb{T}^2_\theta, \mathfrak{t}) \otimes \mathbb{C}^2$, obtaining a family of spectral triples $(\mathbb{T}^2_\theta, \mathcal{H}, D_k)$.

The form of the operator D_k resembles the classical Dirac operator D (2.3), however with the partial derivatives replaced by $i\delta_1$, $i\delta_2$. Since we take k in the algebra $JA(\mathbb{T}^2_\theta)J^{-1}$ commuting with $A(\mathbb{T}^2_\theta)$ due to the property (A.1), D_k has bounded commutators with $a \in A(\mathbb{T}^2_\theta)$. Next the self-adjointness and the compact resolvent property follow from the fact that D_k is a perturbation of the flat Dirac operator D (corresponding to $k = 1$) by a (self-adjoint) bounded operator. Thus it defines a spectral triple in the usual sense.

Furthermore, D_k is a differential operator in the sense of [7] and we can extract the associated (dressed) scalar curvature following the methods introduced and applied to the global conformal rescaling of the noncommutative [6, 7]. The idea is to look for the functional:

$$f \mapsto a_2(D_k; f),$$

where f is an element of $JA(\mathbb{T}^2_\theta)J^{-1}$ (or $A(\mathbb{T}^2_\theta)$) and $a_2(D_k; f)$ is the second heat trace coefficient of $\mathrm{Tr}(fe^{-tD_k^2})$. Since the computations which use the pseudodifferential operators allow us to express the results in terms of $\tau(f\widetilde{R})$ for an element $\widetilde{R}$ of the algebra $JA(\mathbb{T}^2_\theta)J^{-1}$, one can interpret it as the *dressed scalar curvature*, which in the classical case equals $\widetilde{R} = \sqrt{g}R$.

The noncommutative case is, in fact, much richer, since the chiral heat trace functional $\mathrm{Tr}(f\gamma e^{-tD_k^2})$ does not vanish identically. Using similar arguments as above we can introduce the *dressed chiral scalar curvature*, $\widetilde{R}_\gamma$ as an element of $JA(\mathbb{T}^2_\theta)J^{-1}$ such that the heat trace coefficient $a_2(D_k, f\gamma)$ is expressed as $\tau(f\widetilde{R}_\gamma)$.

3   The curvature

The square of D reads

$$(D_k)^2 = (\delta_1)^2 + k^2(\delta_2)^2 + \left(\tfrac{3}{2}k\delta_2(k) + \tfrac{1}{2}\delta_2(k)k + \sigma^1\sigma^2\delta_1(k)\right)\delta_2$$
$$+ \tfrac{1}{4}(\delta_2(k))^2 + \tfrac{1}{2}\sigma^1\sigma^2\delta_{12}(k) + \tfrac{1}{2}k\delta_{22}(k),$$

and its symbol is

$$\sigma((D_k)^2) = a_0 + a_1 + a_2,$$

where

$$a_0 = \xi_1^2 + k^2\xi_2^2, \qquad a_1 = \left(\tfrac{3}{2}k\delta_2(k) + \tfrac{1}{2}\delta_2(k)k + \sigma^1\sigma^2\delta_1(k)\right)\xi_2,$$
$$a_2 = \tfrac{1}{4}(\delta_2(k))^2 + \tfrac{1}{2}\sigma^1\sigma^2\delta_{12}(k) + \tfrac{1}{2}k\delta_{22}(k).$$

As was demonstrated first in [7] the value $\zeta(0)$ at the origin of the zeta function of the operator $(D_k)^2$ is given by

$$\zeta(0) = -\int \mathfrak{t}(b_2(\xi))d\xi,$$

where $b_2(\xi)$ is a symbol of order -4 of the pseudodifferential operator $((D_k)^2 + 1)^{-1}$. Here we assume that (D_k) has no zero eigenvalues, which is satisfied for generic k. Otherwise, in the case that the Dirac operator has 0 as an eigenvalue, a constant term equal to $\dim(\ker(D_k))$ is present. The term on the right hand side is interpreted as the quantum analogue of the integrated scalar curvature dressed by the square root of the determinant of the metric tensor.

The symbol $b_2(\xi)$ can be computed by pseudodifferential calculus of symbols from the symbol $a_2(\xi) + a_1(\xi) + a_0(\xi)$ of $(D_k)^2$ as follows

$$b_0 = (a_2 + 1)^{-1}, \qquad b_1 = -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2)b_0 + \partial_2(b_0)\delta_2(a_2)b_0),$$
$$b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1)b_0 + \partial_2(b_0)\delta_2(a_1)b_0 + \partial_1(b_1)\delta_1(a_2)b_0$$
$$+ \partial_2(b_1)\delta_2(a_2)b_0 + \tfrac{1}{2}\partial_{11}(b_0)\delta_1^2(a_2)b_0 + \tfrac{1}{2}\partial_{22}(b_0)\delta_2^2(a_2)b_0 + \partial_{12}(b_0)\delta_{12}(a_2)b_0). \qquad (3.1)$$

Since to obtain the curvature (or $\zeta(0)$) we need to integrate with respect to ξ_1, ξ_2, we notice that all the terms which contain odd powers of these variables shall vanish. Therefore, we can neglect them and keep for computations only the relevant parts with even powers. They read then as follows

$$b_2^{\text{even}} = A + B + C,$$

where

$$A = -2kb_0^2\delta_1(k)kb_0\delta_1(k)b_0\xi_2^4 + 4kb_0^2\delta_1(k)kb_0^2\delta_1(k)b_0\xi_1^2\xi_2^4 - 2kb_0^2\delta_1(k)b_0\delta_1(k)kb_0\xi_2^4$$
$$+ 4kb_0^2\delta_1(k)b_0^2\delta_1(k)kb_0\xi_1^2\xi_2^4 + 8kb_0^3\delta_1(k)kb_0\delta_1(k)b_0\xi_1^2\xi_2^4 + 8kb_0^3\delta_1(k)b_0\delta_1(k)kb_0\xi_1^2\xi_2^4$$
$$- b_0\delta_1(k)b_0\delta_1(k)b_0\xi_2^2 + 2b_0^2\delta_1(k)\delta_1(k)b_0\xi_2^2 - 2b_0^2\delta_1(k)kb_0\delta_1(k)kb_0\xi_2^4$$
$$+ 4b_0^2\delta_1(k)kb_0^2\delta_1(k)kb_0\xi_1^2\xi_2^4 - 2b_0^2\delta_1(k)k^2b_0\delta_1(k)b_0\xi_2^4 + 4b_0^2\delta_1(k)k^2b_0^2\delta_1(k)b_0\xi_1^2\xi_2^4$$
$$- 8b_0^3\delta_1(k)\delta_1(k)b_0\xi_1^2\xi_2^2 + 8b_0^3\delta_1(k)kb_0\delta_1(k)kb_0\xi_1^2\xi_2^4 + 8b_0^3\delta_1(k)k^2b_0\delta_1(k)b_0\xi_1^2\xi_2^4,$$
$$B = \tfrac{15}{4}kb_0\delta_2(k)kb_0\delta_2(k)b_0\xi_2^2 - 3kb_0\delta_2(k)k^2b_0^2\delta_2(k)kb_0\xi_2^4 - 3kb_0\delta_2(k)k^3b_0^2\delta_2(k)b_0\xi_2^4$$
$$+ \tfrac{9}{4}kb_0\delta_2(k)b_0\delta_2(k)kb_0\xi_2^2 + 6k^2b_0^2\delta_2(k)\delta_2(k)b_0\xi_2^2 - 8k^2b_0^2\delta_2(k)kb_0\delta_2(k)kb_0\xi_2^4$$

$$- 10k^2 b_0^2 \delta_2(k) k^2 b_0 \delta_2(k) b_0 \xi_2^4 + 4k^2 b_0^2 \delta_2(k) k^3 b_0^2 \delta_2(k) k b_0 \xi_2^6 + 4k^2 b_0^2 \delta_2(k) k^4 b_0^2 \delta_2(k) b_0 \xi_2^6$$
$$- 12k^3 b_0^2 \delta_2(k) k b_0 \delta_2(k) b_0 \xi_2^4 + 4k^3 b_0^2 \delta_2(k) k^2 b_0^2 \delta_2(k) k b_0 \xi_2^6 + 4k^3 b_0^2 \delta_2(k) k^3 b_0^2 \delta_2(k) b_0 \xi_2^6$$
$$- 10k^3 b_0^2 \delta_2(k) b_0 \delta_2(k) k b_0 \xi_2^4 - 8k^4 b_0^3 \delta_2(k) \delta_2(k) b_0 \xi_2^4 + 8k^4 b_0^3 \delta_2(k) k b_0 \delta_2(k) k b_0 \xi_2^6$$
$$+ 8k^4 b_0^3 \delta_2(k) k^2 b_0 \delta_2(k) b_0 \xi_2^6 + 8k^5 b_0^3 \delta_2(k) k b_0 \delta_2(k) b_0 \xi_2^6 + 8k^5 b_0^3 \delta_2(k) b_0 \delta_2(k) k b_0 \xi_2^6$$
$$- \tfrac{1}{4} b_0 \delta_2(k) \delta_2(k) b_0 + \tfrac{3}{4} b_0 \delta_2(k) k b_0 \delta_2(k) k b_0 \xi_2^2 + \tfrac{5}{4} b_0 \delta_2(k) k^2 b_0 \delta_2(k) b_0 \xi_2^2$$
$$- b_0 \delta_2(k) k^3 b_0^2 \delta_2(k) k b_0 \xi_2^4 - b_0 \delta_2(k) k^4 b_0^2 \delta_2(k) b_0 \xi_2^4,$$

and

$$C = k b_0^2 \delta_{11}(k) b_0 \xi_2^2 - 4k b_0^3 \delta_{11}(k) b_0 \xi_1^2 \xi_2^2 + b_0^2 \delta_{11}(k) k b_0 \xi_2^2 - 4 b_0^3 \delta_{11}(k) k b_0 \xi_1^2 \xi_2^2 - \tfrac{1}{2} k b_0 \delta_{22}(k) b_0$$
$$+ 2k^2 b_0^2 \delta_{22}(k) k b_0 \xi_2^2 + 4k^3 b_0^2 \delta_{22}(k) b_0 \xi_2^2 - 4k^4 b_0^3 \delta_{22}(k) k b_0 \xi_2^4 - 4k^5 b_0^3 \delta_{22}(k) b_0 \xi_2^4.$$

Similarly for the chiral part $\widetilde{R}_\gamma$ of the curvature, we have

$$b_{\gamma 2}^{\mathrm{even}} = A_\gamma + B_\gamma + C_\gamma,$$

where

$$A_\gamma = -2k^2 b_0^2 \delta_1(k) k b_0 \delta_2(k) b_0 i \xi_2^4 - 2k^2 b_0^2 \delta_1(k) b_0 \delta_2(k) k b_0 i \xi_2^4 + \tfrac{5}{2} b_0 \delta_1(k) k b_0 \delta_2(k) b_0 i \xi_2^2$$
$$- 2 b_0 \delta_1(k) k^2 b_0^2 \delta_2(k) k b_0 i \xi_2^4 - 2 b_0 \delta_1(k) k^3 b_0^2 \delta_2(k) b_0 i \xi_2^4 + \tfrac{3}{2} b_0 \delta_1(k) b_0 \delta_2(k) k b_0 i \xi_2^2,$$
$$B_\gamma = \tfrac{3}{2} k b_0 \delta_2(k) b_0 \delta_1(k) b_0 i \xi_2^2 - 2k^2 b_0^2 \delta_2(k) k b_0 \delta_1(k) b_0 i \xi_2^4$$
$$- 2k^3 b_0^2 \delta_2(k) b_0 \delta_1(k) b_0 i \xi_2^4 + \tfrac{1}{2} b_0 \delta_2(k) k b_0 \delta_1(k) b_0 i \xi_2^2,$$

and

$$C_\gamma = 2k^2 b_0^2 \delta_{12}(k) b_0 i \xi_2^2 - \tfrac{1}{2} b_0 \delta_{12}(k) b_0 i.$$

3.1 The classical limit

At this point we can check the classical (commutative) value of our expressions for $\theta = 0$. The symbol b_2 becomes

$$b_2(\xi) = 48 b_0^5 k^6 \delta_2(k)^2 \xi_2^6 + 48 b_0^5 k^2 \delta_1(k)^2 \xi_1^2 \xi_2^4 - 8 b_0^4 k^5 \delta_{22}(k) \xi_2^4 - 56 b_0^4 k^4 \delta_2(k)^2 \xi_2^4$$
$$- 8 b_0^4 k^2 \delta_1(k)^2 \xi_2^4 - 8 b_0^4 k \delta_{11}(k) \xi_1^2 \xi_2^2 - 8 b_0^4 \delta_1(k)^2 \xi_1^2 \xi_2^2 + 6 b_0^3 k^3 \delta_{22}(k) \xi_2^2$$
$$+ 14 b_0^3 k^2 \delta_2(k)^2 \xi_2^2 + 2 b_0^3 k \delta_{11}(k) \xi_2^2 + b_0^3 \delta_1(k)^2 \xi_2^2 - \tfrac{1}{2} b_0^2 k \delta_{22}(k) - \tfrac{1}{4} b_0^2 \delta_2(k)^2,$$

which after integration gives

$$\int b_2 d\xi_1 d\xi_2 = -\frac{\pi}{3} \frac{(\delta_1(k))^2}{k^3} + \frac{\pi}{6} \frac{\delta_{11}(k)}{k^2}.$$

Taking into account that we compute the Gilkey–Seeley–deWitt coefficients of the heat kernel asymptotic for the square of the Dirac operator and not the Laplace operator itself, and assuming as above that D_k has no zero eigenvalues, we have

$$\zeta(0) = \frac{1}{48\pi} \int \sqrt{g} R.$$

The assumption that 0 is not in the spectrum of D_k, is satisfied for generic k. Otherwise, in the case that D_k has 0 as an eigenvalue, there is a constant term contribution equal to $\dim(\ker D_k)$.

Moreover, since $\mathfrak{t} = \frac{1}{4\pi^2}\int dx dy$ for $\theta = 0$, taking into account the appropriate rescaling of the volume form and putting it all together we obtain

$$\sqrt{g}R = 48\pi\frac{1}{4\pi^2}\left(-\frac{\pi}{3}\frac{(\partial_1(k))^2}{k^3} + \frac{\pi}{6}\frac{\partial_{11}(k)}{k^2}\right) = \left(2k^{-2}\partial_{11}(k) - 4k^{-3}(\partial_1(k))^2\right),$$

which agrees with the classical formula (2.2). Similarly, the symbol $b_{\gamma 2}$ becomes

$$b_{\gamma 2}(\xi) = -12b_0^4 k^3\delta_1(k)\delta_2(k)i\xi_2^4 + 2b_0^3 k^2\delta_{12}(k)i\xi_2^2 + 6b_0^3 k\delta_1(k)\delta_2(k)i\xi_2^2 - \tfrac{1}{2}b_0^2\delta_{12}(k)i,$$

which after integration yields

$$\int d\xi_1 d\xi_2 b_{\gamma 2} = 0$$

and thus

$$\sqrt{g}R_\gamma = 0.$$

Before we can proceed with the noncommutative computation let us recall the general framework of computations as shown recently by Lesch [14].

3.2   Rearrangement lemma

In [14] Lesch proved the following formula (with a slightly modified notation)

$$\int_0^\infty f_0\big(uk^2\big)a_1 f_1\big(uk^2\big)a_2\cdots a_p f_p\big(uk^2\big)du$$
$$= k^{-2m}F\big(\Delta_2^{(1)}, \Delta_2^{(1)}\Delta_2^{(2)}, \ldots, \Delta_2^{(1)}\cdots\Delta_2^{(p)}\big)(a_1\otimes a_2\otimes\cdots\otimes a_p),$$

where m is the p-fold multiplication, the function F of p variables reads

$$F(s_1,\ldots,s_p) = \int_0^\infty f_0(u)f_1(us_1)\cdots f_p(us_p)du$$

and $\Delta_2^{(j)}$ signifies the square of the modular operator Δ, $\Delta(a) = k^{-1}ak$, acting on the j-th factor a_j in the p-fold tensor product product $a_1\otimes a_2\otimes\cdots\otimes a_p$.

In our case we need to adapt the formula to a slightly different setting, when we integrate over two variables ξ_1 and ξ_2. All the integrals we have are of the form

$$\mathcal{J} = \int_{-\infty}^\infty d\xi_1 \int_{-\infty}^\infty d\xi_2 k^{n_1}b_0^{m_1}(\xi_1,\xi_2)Xk^{n_2}b_0^{m_2}(\xi_1,\xi_2)Yk^{n_3}b_0^{m_3}(\xi_1,\xi_2)\xi_1^{2k_1}\xi_2^{2k_2},$$

where X, Y are some derivatives of k and

$$b_0(\xi_1,\xi_2) = \frac{1}{1+\xi_1^2+k^2\xi^2}.$$

Extending the result of Lesch we see that

$$\mathcal{J} = F\big(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)}\big)(X\cdot Y),$$

where after change of variables we obtain

$$F(s,t) = 2\int_0^\infty dv \int_0^\infty du\, k^{n_1+n_2+n_3-1-2k_2}\frac{u^{k_2-\frac{1}{2}}v^{2k_1}}{(1+v^2+u)^{m_1}}\frac{s^{n_2}}{(1+v^2+us^2)^{m_2}}\frac{t^{n_3}}{(1+v^2+ut^2)^{m_3}}.$$

In case $Y = 1$ the resulting function depends only on s.

3.3   The curvature and its trace

In order to compute explicitly the expressions for the curvature we shall use the following lemma.

Lemma 3.1. *Under the trace an entire function F of two variables satisfies*

$$\mathfrak{t}\big(F\big(\Delta^{(1)},\Delta^{(1)}\Delta^{(2)}\big)(X\cdot Y)\big) = \mathfrak{t}\big(F\big(\Delta^{(1)},\mathrm{id}\big)(XY)\big) = \mathfrak{t}\big(F\big(\Delta^{(1)},1\big)(X)Y\big).$$

and in case of one variable

$$\mathfrak{t}\big(F\big(\Delta^{(1)}\big)(X)\big) = \mathfrak{t}(F(1)X).$$

Proof. We have

$$F(s,t) = \sum_{n,m\geq 0} f_{nm}s^n t^m,$$

so

$$\mathfrak{t}\big(F\big(\Delta^{(1)},\Delta^{(1)}\Delta^{(2)}\big)(X\cdot Y)\big) = \sum_{n,m\geq 0} f_{nm}\mathfrak{t}\big(\Delta^{n+m}(X)\Delta^m(Y)\big)$$

$$= \sum_{n,m\geq 0} f_{nm}\mathfrak{t}\big(\Delta^m\big(\Delta^n(X)Y\big)\big) = \sum_{n,m\geq 0} f_{nm}\mathfrak{t}\big(\Delta^n(X)Y\big) = \mathfrak{t}\big(F\big(\Delta^{(1)},1\big)(X\cdot Y)\big).$$

The other identity is a simple consequence of the above one.   ∎

3.4   Curvature and chiral curvature

We can present the first main result on the dressed scalar curvature of the asymmetric noncommutative torus:

Proposition 3.2. *The dressed scalar curvature for the asymmetric torus is*

$$\widetilde{R} = F_{11}\big(\Delta^{(1)},\Delta^{(1)}\Delta^{(2)}\big)\big(\delta_1(k)\cdot\delta_1(k)\big) + F_{22}\big(\Delta^{(1)},\Delta^{(1)}\Delta^{(2)}\big)\big(\delta_2(k),\delta_2(k)\big)$$
$$+ F_{11}'\big(\Delta^{(1)}\big)\big(\delta_1(k)^2\big) + F_{22}'\big(\Delta^{(1)}\big)\big(\delta_2(k)^2\big) + F_1\big(\Delta^{(1)}\big)\big(\delta_{11}(k)\big) + F_2\big(\Delta^{(1)}\big)\big(\delta_{22}(k)\big),$$

where

$$F_{11}(s,t) = -\frac{2\pi}{3k^3}\frac{\big(2s^2+4st+4s+3+8t+3t^2\big)}{(t+1)^3(s+1)(s+t)}, \qquad F_{22}(s,t) = \frac{\pi}{2k}\frac{(t^2-6t+1)}{(t+1)^3},$$

$$F_{11}'(s) = \frac{4\pi}{3k^3}\frac{1}{(s+1)^3}, \qquad F_{22}'(s) = -\frac{\pi}{2k}\frac{(s^2-6s+1)}{(s+1)^3},$$

and

$$F_1(s) = \frac{2\pi}{3k^2}\frac{1}{(s+1)^2}, \qquad F_2(s) = 0.$$

The trace of $\widetilde{R}$ vanishes.

Proof. First of all, observe that

$$F_{22}(s,1) + F_{22}'(1) = 0, \qquad F_2(1) = 0,$$

so all terms containing $\delta_2(k)$ and $\delta_{22}(k)$ vanish.

For the terms containing $\delta_1(k)$ we have

$$F_{11}(s,1) + F'_{11}(1) = -\frac{\pi}{3k^3}\frac{s+3}{(s+1)^2}.$$

Then using the identity

$$\mathfrak{t}\left(k^{-2}\delta_{11}(k)\right) = 2\mathfrak{t}\left(k^{-2}\delta_1(k)k^{-1}\delta_1(k)\right) = 2\mathfrak{t}\left(k^{-3}\Delta^{-1}(\delta_1(k))\delta_1(k)\right),$$

which follows directly from the Leibniz rule and the fact that the trace is closed, we can rewrite the trace of the sum of the remaining terms F_{11}, F'_{11} and F_1 as

$$\mathfrak{t}\left(k^{-3}H(\Delta)(\delta_1(k))\delta_1(k)\right),$$

where

$$H(s) = \frac{\pi}{3k^3}\frac{1-s}{s(s+1)^2}.$$

Next, we observe that for any A and B and an entire function H

$$\mathfrak{t}\left(k^{-3}H(\Delta)(A)B\right) = \mathfrak{t}\left(H(\Delta)(\Delta^3(A))k^{-3}B\right) = \mathfrak{t}\left(k^{-3}BH(\Delta)(\Delta^3(A))\right),$$

and

$$\mathfrak{t}\left(k^{-3}H(\Delta)(A)B\right) = \mathfrak{t}\left(k^{-3}AH(\Delta^{-1})(B)\right).$$

Now if $A = B$ then both expressions on the right-hand side are identical. In our case, however

$$H(s)s^3 = \frac{\pi}{3k^3}\frac{s^2(1-s)}{(s+1)^2} \qquad \text{and} \qquad H(s^{-1}) = \frac{\pi}{3k^3}\frac{s^2(s-1)}{(s+1)^2},$$

and therefore since

$$H(s)s^3 = -H(s^{-1}),$$

the trace of the above expression must vanish, hence, the Gauss–Bonnet theorem holds.   ∎

Next we give the second main result on the dressed chiral curvature of the asymmetric noncommutative torus:

Proposition 3.3. *The dressed chiral curvature for the asymmetric torus is*

$$\widetilde{R}_\gamma = G_{12}\left(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)}\right)(\delta_1(k), \delta_2(k))$$
$$+ G_{21}\left(\Delta^{(1)}, \Delta^{(1)}\Delta^{(2)}\right)(\delta_2(k), \delta_1(k)) + G\left(\Delta^{(1)}\right)(\delta_{12}(k)),$$

where

$$G_{12}(s,t) = \frac{\pi}{k^2}\frac{(t-1)}{(t+1)^2(s+1)}, \qquad G_{21}(s,t) = \frac{\pi}{k^2}\frac{(t-1)}{(t+1)^2(s+t)}, \qquad G(s) = -\frac{\pi}{k}\frac{(s-1)}{(s+1)^2}.$$

The trace of R_γ vanishes.

Proof. By computation. Then the last statement follows from

$$G_{12}(s,1) = G_{21}(s,1) = G(1) = 0.$$ ∎

4   Final comments

We have introduced a new class of spectral triples that describe a curved geometry of the noncommutative torus. The associated Dirac operators are obtained through a partial rescaling of the flat Dirac operator by elements from the opposite coordinate algebra. It turns out that the dressed scalar curvature does not vanish, but the Gauss–Bonnet theorem holds for that family.

Let us stress that even though in the classical limit they arise from the metric, which is conformally equivalent to the flat one, this is an open problem in the noncommutative situation. An interesting question is if and in which sense this can be established also in the noncommutative case.

It becomes now more evident that the class of admissible Dirac operators on the noncommutative torus is certainly bigger than the one-parameter family of "flat metric" (equivariant) Dirac operators. It is therefore necessary to study the conditions and the general setup of such construction, and we conjecture that the class should contain all operators proposed in [8] and their fluctuations.

Although in this paper we have concentrated on the 2-dimensional case, it is natural to consider generalisations of the result to 3 and more dimensions. In particular it is interesting to study the curvature and minimality of yet another class Dirac operators introduced in [9], which are similar to the ones encountered here.

Finally, we can comment on the purely algebraic computation by Rosenberg [16] of the curvature and Gauss–Bonnet term for the conformally rescaled metric on the noncommutative torus using the noncommutative notion of Levi-Civita connection. His method can be extended to the case we consider and a direct computation yields the Gauss–Bonnet functional

$$2\mathfrak{t}\left(-\tfrac{9}{4}k^{-2}\delta_1(k)k^{-1}\delta_1(k) + \tfrac{1}{4}k^{-3}\delta_1(k)\delta_1(k) + k^{-2}\delta_{11}(k)\right),$$

which however does not vanish, and the Gauss–Bonnet theorem does not hold.

Instead there is an even simpler possibility that yields an algebraic expression for a candidate for the scalar curvature. Namely, we note that the formula (2.1) in [8] for the classical scalar curvature in terms of the orthonormal basis of vector fields (*moving frame* or *two-bein*) for the partially rescaled metric, makes a perfect sense also for the noncommutative torus (there is no ordering ambiguity). Taking the two-bein as $(\delta_1, k\delta_2)$ one obtains

$$\mathfrak{t}\left(-4k^{-2}\delta_1(k)k^{-1}\delta_1(k) + 2k^{-2}\delta_{11}(k)\right),$$

which vanishes due to the properties of the trace.

A   Appendix

The unital $*$-algebra $\mathcal{A}_\theta$ of smooth functions on the noncommutative 2-torus consists of power series of the form

$$a = \sum_{m,n\in\mathbb{Z}^2} a_{mn}U_1^m U_2^n,$$

where the (double) sequence of coefficients $\{a_{mn}\in\mathbb{C}, (m,n)\in\mathbb{Z}^2\}$ decreases rapidly at 'infinity'

$$\|a\|_k := \sup_{m,n}\left\{\left(1+m^2+n^2\right)^k|a_{mn}|^2\right\} < \infty, \qquad \forall\, k\in\mathbb{N}.$$

The $*$-algebra structure is specified by the commutation relations of the unitary generators U_1, U_2

$$U_1U_2 = e^{2\pi i\theta}U_2U_1,$$

where θ is a real parameter.

On A_θ there is a (unique if θ is irrational) normalised faithful positive definite tracial state $\mathfrak{t}\colon \mathcal{A}_\theta \to \mathbb{C}$; it is given by

$$\mathfrak{t}(a) = \mathfrak{t}\left(\sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, U_1^m U_2^n\right) := a_{00}.$$

We call $\mathfrak{t}$ *trace*, for brevity. It is invariant under the action of the commutative torus $\mathbb{T}^2 = U(1) \times U(1)$ on $\mathcal{A}_\theta$ by automorphisms, $U_1 \mapsto z_1 U_1$ and $U_2 \mapsto z_2 U_2$, where $|z_1| = 1 = |z_2|$. The infinitesimal action is determined by two commuting derivations δ_1, δ_2 acting by

$$\delta_\mu(U_\nu) = 2\pi i \delta_\mu^\nu U_\nu, \qquad \mu, \nu = 1, 2.$$

The invariance means just that $\mathfrak{t}(\delta_\mu a) = 0$, for $\mu = 1, 2$ and any $a \in \mathcal{A}_\theta$.

There is a sesquilinear form on $\mathcal{A}_\theta$

$$(a, b) = \mathfrak{t}(a^* b)$$

and a norm

$$\|a\| := \sqrt{\mathfrak{t}(a^* a)}.$$

The completion of $\mathcal{A}_\theta$ in this norm yields a Hilbert space $H_{\mathrm{o}} := L^2(\mathcal{A}_\theta, \mathfrak{t})$, which carries a $*$-representation of $\mathcal{A}_\theta$ by left multiplication operators

$$\pi(a)\colon\ b \mapsto ab.$$

Moreover π is the GNS representation w.r.t. the state $\mathfrak{t}$, the cyclic and separating vector is $1 \in \mathcal{A}_\theta \subset H_{\mathrm{o}}$. The operator norm of $\pi(a)$ is $\|a\|$. The associated *Tomita conjugation* provides an antiunitary operator J_{o} on H_{o},

$$J_{\mathrm{o}}(b) := b^*.$$

Clearly $J_{\mathrm{o}}^2 = 1$ and

$$\pi^{\mathrm{op}}(a) := J_{\mathrm{o}} \pi(a^*) J_{\mathrm{o}}\colon\ b \mapsto ba$$

(right multiplication by a) is a representation of the opposite algebra of $\mathcal{A}_\theta$. Since the left and right multiplications commute, we have

$$[\pi(a), J_{\mathrm{o}} \pi(b^*) J_{\mathrm{o}}] = 0, \qquad \forall\, a, b. \tag{A.1}$$

(In the paper we are omitting the symbol π.)

For Dirac spinors (Hilbert space) we take the trivial module over A_θ of rank two

$$H = H_{\mathrm{o}} \otimes \mathbb{C}^2 \simeq H_{\mathrm{o}} \oplus H_{\mathrm{o}} =: H_+ \oplus H_-.$$

The inner product is $\langle (a, a'), (b, b') \rangle = \mathfrak{t}(a^* b + a' b'^*)$. The *charge conjugation* operator on H is given by

$$J := -i J_{\mathrm{o}} \otimes (\sigma_2 \circ \text{c.c.}).$$

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