On 1-Harmonic Functions*

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Abstract. Characterizations of entire subsolutions for the 1-harmonic equation of a constant 1-tension field are given with applications in geometry via transformation group theory. In particular, we prove that every level hypersurface of such a subsolution is calibrated and hence is area-minimizing over $\mathbb{R}$; and every 7-dimensional $SO(2) \times SO(6)$-invariant absolutely area-minimizing integral current in $\mathbb{R}^8$ is real analytic. The assumption on the $SO(2)\times SO(6)$-invariance cannot be removed, due to the first counter-example in $\mathbb{R}^8$, proved by Bombieri, De Girogi and Giusti.

Key words: 1-harmonic function; 1-tension field; absolutely area-minimizing integral current

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1 Introduction

The study of 1-harmonic functions, or more generally that of p-harmonic maps is an area of an active research that is related with many branches of mathematics. For instance, in a celebrated paper of Bombieri, De Girogi and Giusti [3], a 1-harmonic function has been constructed to provide a counter-example for interior regularity of the solution to the co-dimension one Plateau problem in $\mathbb{R}^n$ for $n > 7$. Recall a C^1 functions $f : \mathbb{R}^n \to \mathbb{R}$ is said to be 1-*harmonic* if it is a weak solution of 1-harmonic equation

$$\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) = 0\,, \tag{1.1}$$

where $|\nabla f|$ is the length of the gradient ∇f of f, and for a C^2 function f without a critical point, $\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right)$ is said to be the 1-*tension field* of f.

In this paper, characterizations of entire subsolutions for the 1-harmonic equation of a constant 1-tension field are given in various aspects, and their relationships with calibration geometry are established (cf. Theorem 2, Corollary 3). As applications, we prove via transformation group theory (cf. [9, 10, 13, 2, 21]) that the cone over $S^1 \times S^5$ is not minimizing in $\mathbb{R}^8$ but is stable; that any 7-dimensional $SO(2)\times SO(6)$-invariant absolutely area-minimizing integral current in $\mathbb{R}^8$ is real analytic; and that the only 7-dimensional $SO(3)\times SO(5)$-invariant minimizing integral current with singularities in $\mathbb{R}^8$ is the cone over $S^2\times S^4$, and is minimizing over $\mathbb{R}$ (cf. Theorems 3–5). These results improved an early partial proof by numerical computation done by Plinio Simoes [17] in his Berkeley thesis. The assumption on the $SO(2) \times SO(6)$-invariance cannot be removed, due to the first counter-example of Bombieri, De Girogi and Giusti that the cone over $S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}}) \subset S^7(1)$ is area-minimizing in $\mathbb{R}^8$. It should be pointed out that Fang-Hua Lin [14] proved that the cone over $S^1\times S^5$ is one-sided area-minimizing and is stable by a different method. By constructing 1-harmonic functions on hyperbolic space H^n, $H^n \times H^n$,

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$H^n \times SO(n,1)$ and many other associated spaces, S.P. Wang and the author [19] show the Bernstein Conjecture in these spaces to be false in all dimensions. In particular, these constructions give the *first* set of examples of complete, smooth, embedded, minimal (hyper-)surfaces in hyperbolic space H^n in all dimensions (cf. also Remark 3(ii)).

2 Fundamentals in geometric measure theory

For our subsequent development, we recall some fundamental facts, definitions, and notations, for which the reference is Federer's book [5] and paper [7].

Let N denote an n-dimensional Riemannian manifold and denote by $\mathcal{R}_p^{\mathrm{loc}}(N)$ the set of p-dimensional, locally rectifiable currents (of Federer and Fleming, cf. [8]) on N. For $S \in \mathcal{R}_p^{\mathrm{loc}}(N)$, denote the mass of S by $\mathbf{M}(S)$, and the *boundary* of S by ∂S, and is given by $(\partial S)(w) = S(dw)$, where w is a smooth p-form and d is the exterior differentiation. From a calculus of variational viewpoint, we make the following

Definition 1. A current $T \in \mathcal{R}_k^{\mathrm{loc}}(N)$ is said to be *stationary* if $\frac{d}{dt}\mathbf{M}(\phi_{t*}^V(T))|_{t=0}$ for all vector fields V on N with compact support where ϕ_t^V is the flow associated with V, and *stable* if for every vector fields V on N with compact support, there exists an $\epsilon > 0$ such that $\mathbf{M}(T) \le \mathbf{M}(\phi_{t*}^V(T))$ for $|t| < \epsilon$.

We are primarily interested in minimizing currents.

Definition 2. A current $T \in \mathcal{R}_k^{\mathrm{loc}}(N)$ is *homologically (resp. absolutely) area-minimizing* over $\mathbb{Z}$ if for all compact sets $K \subset M$, we have $\mathbf{M}(\phi_K T) \le \mathbf{M}((\phi_K T) + S)$ for all $S \in \mathcal{R}_k^{\mathrm{loc}}(N)$ having compact support and being the boundary of some current in $\mathcal{R}_{k+1}^{\mathrm{loc}}(N)$ with compact support (resp. the empty boundary)(here ϕ_K denotes the characteristic function on K).

Using a dimension reduction technique, Federer proves that the support of an area-minimizing integral current T [8] minus another compact set S whose Hausdorff dimension does not exceed $n-8$ is an $(n-1)$-dimensional analytic manifold [6]. Hence, if $n \le 7$, then S $= \varnothing$. If $n = 8$, S consists of at most isolated points [5, 5.4.16]. This result is optimal by the counter-example due to Bombieri–De Giorgi–Giusti [3] that $\{x \in \mathbb{R}^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\}$ is an area-minimizing cone over the product of $(m-1)$-spheres $\left\{x \in \mathbb{R}^{2m} : x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2 = \frac{1}{2}\right\}$ in $\mathbb{R}^{2m}$ for $m \ge 4$.

The union of the groups $\mathcal{F}_{m,K}(U) = \{R + \partial T : R \in \mathcal{R}_{m,K}(U), T \in \mathcal{R}_{m+1,K}(U)\}$ corresponding to all compact $K \subset U$ is the group $\mathcal{F}_m(U)$ of m-dimensional *integral flat chains in an open subset* U of $\mathbb{R}^n$. We denote the group of m-dimensional *integral flat chains, cycles and boundaries* by $\mathcal{F}_m(A) = \mathcal{F}_m(\mathbb{R}^n) \cap \{S : \operatorname{spt} S \subset A\}$, $\mathcal{Z}_m(A,B) = \mathcal{F}_m(A) \cap \{S : \partial S \subset \mathcal{F}_m(B) \text{ or } m = 0\}$, and $\mathcal{B}_m(A,B) = \{R + \partial T : R \in \mathcal{F}_m(B), T \in \mathcal{F}_{m+1}(A)\}$ respectively. Similarly, we define and denote $\mathbf{F}_m(A)$, $\mathbf{Z}_m(A,B)$ and $\mathbf{B}_m(A,B)$ the vector space of m-dimensional *real flat chains, cycles and boundaries* respectively, where $B \subset A$ are compact Lipschitz neighborhood retract in U.

For every positive convex parametric integrand ψ, and every compact subset K of A, we define $\mathcal{Z}_{m,K}(A,B) = \mathcal{Z}_m(A,B) \cap \{R : \operatorname{spt} R \subset K\}$, $\mathcal{B}_{m,K}(A,B) = \mathcal{B}_m(A,B) \cap \{R : \operatorname{spt} R \subset K\}$, $\mathbf{Z}_{m,K}(A,B) = \mathbf{Z}_m(A,B) \cap \{R : \operatorname{spt} R \subset K\}$, and $\mathbf{B}_{m,K}(A,B) = \mathbf{B}_m(A,B) \cap \{R : \operatorname{spt} R \subset K\}$, and make the following

Definition 3. An m-dimensional rectifiable current Q (resp. Q') is said to be *absolutely (resp. homologically) ψ-minimizing in K with respect to (A,B) over $\mathbb{Z}$* if

$$\int_Q \psi = \inf\left\{\int_S \psi : S \in \mathcal{F}_{m,K}(U), Q - S \in \mathcal{Z}_{m,K}(A,B)\right\}$$

$$\left(\text{resp.} \quad \int_{Q'} \psi = \inf\left\{\int_S \psi : S \in \mathcal{B}_{m,K}(U), Q' - S \in \mathcal{B}_{m,K}(A,B)\right\}\right).$$

Definition 4. An m-dimensional real flat chain Q (resp. Q') is said to be *absolutely (resp. homologically) ψ-minimizing in K with respect to* (A, B) *over* $\mathbb{R}$ if

$$\int_Q \psi = \inf\left\{\int_S \psi : S \in \mathbf{F}_{m,K}(U), Q - S \in \mathbf{Z}_{m,K}(A, B)\right\}$$

$$\left(\text{resp.} \quad \int_{Q'} \psi = \inf\left\{\int_S \psi : S \in \mathbf{B}_{m,K}(U), Q' - S \in \mathbf{B}_{m,K}(A, B)\right\}\right).$$

We will make comparisons between real and integral absolute (resp. homological) minimizing currents in the subsequent Sections 3, 4, and 5.

3 Characterizations of subsolutions for 1-harmonic equation of constant 1-tension field

We connect an entire subsolution of this sort, with a calibration. Recall a calibration is a closed form with comass 1.

Lemma 1. *Let M be a complete noncompact Riemannian manifold. For any $x_0 \in M$ and any pair of positive numbers s, t with $s < t$, there exists a rotationally symmetric Lipschitz continuous function $\psi(x) = \psi(x; s, t)$ and a constant $C_1 > 0$ (independent of x_0, s, t) with the properties:*

$$(i) \quad \psi \equiv 1 \ \text{ on } \ B(x_0; s), \ \text{ and } \ \psi \equiv 0 \ \text{ off } \ B(x_0; t);$$

$$(ii) \quad |\nabla\psi| \le \frac{C_1}{t - s}, \quad a.e. \ \text{ on } \ M. \tag{3.1}$$

Proof. (cf. Andreotti and Vesentini [1], Yau [22], Karp [11]). ■

Theorem 1. *Let Ω be a domain in $\mathbb{R}^n$ containing a ball $B(x_0, r)$ of radius r, centered at x_0, and $g : \Omega \to \mathbb{R}$ be a continuous function with $g \ge 0$, and $c = \inf_{x \in B(x_0, \frac{r}{2})} g(x)$. Let $f : \Omega \to \mathbb{R}$ be a C^1 weak solution of*

$$\operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) = g(x) \qquad on \ \ \Omega, \tag{3.2}$$

then the infimum c satisfies

$$0 \le c \le \frac{C_1 2^n}{r},$$

where C_1 is as in (3.1).

Proof. Let $\psi \ge 0$ be as in Lemma 1, in which $M = \mathbb{R}^n$, $t = r$, $s = \frac{r}{2}$. Choose ψ to be a test function in the distribution sense of (3.2). Then via the assumption on g, and Cauchy–Schwarz inequality we have:

$$\begin{aligned} \int_{B(x_0, \frac{r}{2})} c\psi(x)dx &\le \int_{B(x_0, \frac{r}{2})} g(x)\psi(x)dx \\ &\le \int_{B(x_0, r)} g(x)\psi(x)dx = -\int_{B(x_0, r)} \frac{\nabla f}{|\nabla f|} \cdot \nabla\psi dx \le \int_{B(x_0, r)} |\nabla\psi| dx. \end{aligned}$$

Hence,

$$c\mathrm{Vol}\left(B\left(x_0, \frac{r}{2}\right)\right) \le \frac{C_1}{r}\mathrm{Vol}(B(x_0, r))$$

yields the desired. ■

Corollary 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 weak subsolution of 1-harmonic equation (1.1) with constant 1-tension field c, i.e. $0 \le \operatorname{div}\left(\frac{\nabla f}{|\nabla f|}\right) = c$ in the distribution sense. Then f is a 1-harmonic function.*

Corollary 2. *There does not exist a C^1 weak subsolution $f : \mathbb{R}^n \to \mathbb{R}$ of equation (3.2) with $\lim_{r\to\infty} \inf_{x\in B(x_0,r)} g(x) > 0$, for any $x_0 \in \mathbb{R}^n$.*

Let $A \subset R^n$ be an open set. We denote $BV_{\text{loc}}(A) = \{f \in L^1_{\text{loc}}(A)$: the distributional derivatives $D_i f$ of f are (locally) measures$\}= \{f \in L^1_{\text{loc}}(A) : \operatorname{supp}\phi_n \subset K \subset A,\ \phi_n \to 0$ uniformly, imply $\left(\frac{\partial}{\partial x_i} f\right)\phi_n \to 0\}$. Let $Df = (D_1 f, \dots, D_n f)$ denote the gradient of f in the sense of distributions and $|Df|$ the scalar measure defined by $\int_K |Df| = \sup \int_K \sum_i \epsilon_i(x) D_i f$, where the supremum is taken over all sets $\{\epsilon_i(x),\ i = 1, \dots, n\}$ of $C^\infty(K)$ functions which satisfy $\sum \epsilon_i^2(x) \le 1$.

Definition 5. A function $f \in BV_{\text{loc}}(A)$ *has least gradient* in A if for every $g \in BV_{\text{loc}}(A)$, with compact support $K \subset A$ we have

$$\int_K |Df| \le \int_K |D(f+g)|. \tag{3.3}$$

Definition 6. Let E be a set in $\mathbb{R}^n$ and ϕ_E its characteristic function. *E has an oriented boundary of least area* with respect to A, if (i) $\phi_E \in BV_{\text{loc}}(A)$ and (ii) for each $g \in BV_{\text{loc}}(A)$ with compact support $K \subset A$ we have $\int_K |D\phi_E| \le \int_K |D(\phi_E + g)|$.

Theorem 2. *Let $f \in H^{1,1}_{\text{loc}}(\mathbb{R}^n)$, and $\nabla f(x) \ne 0$ for every x in $\mathbb{R}^n$. Let $E_\lambda = \{x : f(x) \ge \lambda\}$, and $S_\lambda = \{x : f(x) = \lambda\}$. We denote the set of integers by $\mathbb{Z}$. Then the following thirteen statements (1)–(13) are equivalent and each of them implies the fourteenth statement (14).*

1. *$f : \mathbb{R}^n \to \mathbb{R}$ is a C^1 weak subsolution of (1.1) with constant 1-tension field.*
2. *f is a C^1 weak solution of (1.1) on $\mathbb{R}^n$.*
3. *f is a C^1 1-harmonic function on $\mathbb{R}^n$.*
4. *For each $(a, t_0) = (a_1, \dots, a_{n-1}, t_0) \in S_\lambda$, there exists a neighborhood $\mathcal{D}$ of a in $\mathbb{R}^{n-1}$, and a unique real analytic function $\eta : \mathcal{D} \to \mathbb{R}$ such that $\eta(a) = t_0$, $f(x_1, \dots, x_{n-1}, \eta(x_1, \dots, x_{n-1})) = \lambda$ and $\operatorname{div}\left(\frac{\nabla \eta}{\sqrt{1+|\nabla\eta|^2}}\right) = 0$ on $\mathcal{D}$.*
5. *Each level hypersurface S_λ is minimal in $\mathbb{R}^n$.*
6. *$\frac{*df}{|df|}$ is a globally defined "weakly" closed form with comass 1.*
7. *f is a function of least gradient in $\mathbb{R}^n$.*
8. *Each E_λ, $\lambda \in \mathbb{R}$ has an oriented boundary of least area with respect to $\mathbb{R}^n$.*
9. *Each level hypersurface S_λ is absolutely area-minimizing in $\mathbb{R}^n$ over $\mathbb{Z}$.*
10. *Each level hypersurface S_λ is absolutely area-minimizing in $\mathbb{R}^n$ over $\mathbb{R}$.*
11. *Each level hypersurface S_λ is homologically area-minimizing in $\mathbb{R}^n$ over $\mathbb{R}$.*
12. *Each level hypersurface S_λ is homologically area-minimizing in $\mathbb{R}^n$ over $\mathbb{Z}$.*
13. *Each level hypersurface S_λ is stable in $\mathbb{R}^n$.*
14. *If $f \in C^2(\mathbb{R}^n)$, then $\frac{*df}{|df|}$ is closed and the restriction $\left.\frac{*df}{|df|}\right|_{S_\lambda}$ is its volume form, hence each S_λ is real absolutely area-minimizing in $\mathbb{R}^n$ over $\mathbb{R}$.*

Corollary 3. *Every level hypersurface of a C^2 subsolution of 1-harmonic equation on $\mathbb{R}^{n+1}$ with constant 1-tension field is calibrated and hence is area-minimizing over $\mathbb{R}$.*

Proof. (1) ⇔ (2) ⇔ (3) : This follows immediately from Corollary 1.

(2) ⇔ (4) : (⇒) Let $f(x_1,\dots,x_{n-1},t)=\eta(x_1,\dots,x_{n-1})-t$. The assertion follows from the implicit function theorem and

$$0=\int\frac{\sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}\frac{\partial\varphi}{\partial x_i}}{|\nabla f|}+\int\frac{\frac{\partial f}{\partial t}}{|\nabla f|}\frac{\partial\varphi}{\partial t}=\int\sum_{i=1}^{n-1}\frac{\frac{\partial\eta}{\partial x_i}}{\sqrt{1+|\nabla\eta|^2}}\frac{\partial\varphi}{\partial x_i} \tag{3.4}$$

for all $\varphi\in C_0^\infty(\mathcal{D}\times\mathbb{R})$. The regularity of solutions of minimal surface equation implies that η is real analytic and completes the proof. (⇐) This follows immediately from (3.3).

(4) ⇔ (5) : This is due to the fact that the graph of a solution to the minimal surface equation on $\mathcal{D}$ is a minimal hypersurface in $\mathcal{D}\times\mathbb{R}$.

(2) ⇔ (6) : This follows from the following: For every $\phi\in C_0^\infty(A)$,

$$\begin{aligned}\int_A\frac{*df}{|df|}\wedge d\phi&=\int_A\sum_{i,j=1}^{n}(-1)^{i-1}\frac{\frac{\partial f}{\partial x_i}}{|\nabla f|}dx^1\wedge\cdots\wedge dx^{i-1}\wedge dx^{i+1}\wedge\cdots\wedge dx^n\wedge\frac{\partial\phi}{\partial x_j}dx^j\\&=\int_A\sum_{i=1}^{n}(-1)^{n-1}\frac{\frac{\partial f}{\partial x_i}\frac{\partial\phi}{\partial x_i}}{|\nabla f|}dx^1\wedge\cdots\wedge dx^i\wedge\cdots\wedge dx^n.\end{aligned}$$

(2) ⇒ (7): let us first assume that $g\in C_0^1(A)$. Let $h(t)=\int|D(f+tg)|$. Then

$$h'(t)=\int\frac{\left(\sum_{i=1}^{n}\frac{\partial(f+tg)}{\partial x_i}\frac{\partial g}{\partial x_i}\right)}{\left(\sum_{i=1}^{n}\left(\frac{\partial(f+tg)}{\partial x_i}\right)^2\right)^{\frac12}}.$$

Hence $h'(0)=0$ by assumption. Furthermore,

$$h''(t)=\int\frac{\left(\sum_{i=1}^{n}(\frac{\partial g}{\partial x_i})^2\right)\left(\sum_{i=1}^{n}(\frac{\partial(f+tg)}{\partial x_i})^2\right)-\left(\sum_{i=1}^{n}\frac{\partial(f+tg)}{\partial x_i}\frac{\partial g}{\partial x_i}\right)^2}{\left[\sum_{i=1}^{n}(\frac{\partial(f+tg)}{\partial x_i})^2\right]^{\frac32}}\ge 0,$$

by the Cauchy–Schwarz inequality. Therefore $\int|Df|=h(0)\le h(1)=\int|D(f+g)|$. If $g\in BV_{\text{loc}}(A)$ with compact support K and let $Dg=G_1+G_2$ where G_1 is completely continuous and G_2 is the singular part of Dg with support N_g of measure zero. Then we have $\int_K|D(f+g)|=\int_K|Df+G_1|+\int_K|G_2|$ because $f\in H^{1,1}_{\text{loc}}(A)$. Let $g_\varepsilon=g*\psi_\varepsilon$ where ψ_ε is a mollifier. Then $g\in C_0^1(A)$ and $\int_{K_\epsilon}|Df|\le\int_{K_\epsilon}|D(f+g_\epsilon)|\le\int_{K_\epsilon}|Df+G_1*\Psi_\epsilon|+\int_A|G_2*\Psi_\epsilon|$, where $K_\epsilon=\{x\in A:\operatorname{dist}(x,K)<\epsilon\}$. Letting $\epsilon\to 0$ completes the proof (cf. [3]).

(7) ⇒ (8) : This follows from Coarea formula for BV functions [15], $\int_K|Df|=\int_{-\infty}^{\infty}(\int_K|D\phi_\lambda|)d\lambda$ together with two observations:

(i) If f_1 and f_2 satisfy (3.3), so does $\sup(f_1,f_2)$.

(ii) If $f_i\in BV_{\text{loc}}(A)$, $f_i\to f$ in L^1_{loc} and each f_i satisfies (3.3), so does also $f\in BV_{\text{loc}}(A)$ and satisfies (3.3).

For detailed proof see [16].

(8) $\Rightarrow$ (9) : Let $\phi_\lambda = \phi_{E_\lambda}$. Since for every x in $\mathbb{R}^n$, $\nabla f(x) \neq 0$, $\partial E_\lambda = S_\lambda$ for $S_\lambda \neq \varnothing$. It follows from a theorem of Miranda [15] that on any compact set K in $\mathbb{R}^n$, the Hausdorff $(n-1)$-measure

$$\mathcal{H}^{n-1}(K \cap S_\lambda) = \int_K |D\phi_\lambda| \leq \int_K |D(\phi_\lambda + g)| = \mathcal{H}^{n-1}(K \cap T)$$

for all sets T with $\partial(K \cap T) = \partial(K \cap S_\lambda)$.

(9) $\Rightarrow$ (10) : It follows from Theorem 6.

(10) $\Rightarrow$ (11) $\Rightarrow$ (12) : Since absolute area-minimization over $\mathbb{R} \Rightarrow$ homological area-minimization over $\mathbb{R} \Rightarrow$ homological area-minimization over $\mathbb{Z}$.

(12) $\Rightarrow$ (13) $\Rightarrow$ (5) : Since homological minimization over $\mathbb{Z} \Rightarrow$ stability $\Rightarrow$ minimality. This completes the proof of (1) $\Leftrightarrow \cdots \Leftrightarrow$ (13).

(2) $\Rightarrow$ (14) : If $f \in C^2(A)$ then by (3.4) $\frac{*df}{|df|}$ is closed. Now let $e_1, \ldots, e_{n-1}$ be an orthonormal basis for the tangent space of S_λ at x_0 and ν a unit normal vector at x_0. We denote by tilde "$\sim$" the canonical isomorphism between a tangent space and its dual space. To show $\frac{*df}{|df|}$ has comass 1, note for any $(n-1)$-vector field ξ,

$$\begin{aligned}\frac{*df}{|df|}(\xi) &= \left(* \frac{\widetilde{\nabla f}}{|\nabla f|} \right)(\xi) \qquad \left(\text{because } \frac{df}{|df|}(X) = \frac{Xf}{|\nabla f|} = \left\langle \frac{\nabla f}{|\nabla f|}, X \right\rangle \right) \\ &= (*\tilde{\nu})(\xi) = (e_1 \wedge \widetilde{\cdots \wedge} e_{n-1})(\xi) = \langle e_1 \wedge \cdots \wedge e_{n-1}, \xi \rangle.\end{aligned}$$

In particular $\frac{*df}{|df|}(e_1 \wedge \cdots \wedge e_{n-1}) = 1$, $\frac{*df}{|df|}(\xi) \leq 1$ and $\left.\frac{*df}{|df|}\right|_{S_\lambda}$ = volume element of S_λ. By the formalism of Stokes theorem, for any integral current T with $\partial T = \partial(S_\lambda \cap B_r)$

$$\begin{aligned}M(S_\lambda \cap B_r) &= (S_\lambda \cap B_r)\left(\frac{*df}{|df|}\right) = T\left(\frac{*df}{|df|}\right) \\ &= \int \frac{*df}{|df|}\left(\overrightarrow{T_x}\right) d||T||(x) \leq \int d||T|| = M(T),\end{aligned}$$

where $\overrightarrow{T}$ is the field of oriented unit tangent planes to T. ■

Remark 1. In Theorem 2, if one replace $\mathbb{R}^n$ with an open subset A in $\mathbb{R}^n$, then assertions (2)$\Leftrightarrow \cdots \Leftrightarrow$ (13) $\Rightarrow$ (14) remain to be true.

Remark 2. Concerning the assertion (2) $\Rightarrow$ (7), a stronger theorem can be found in [3]: Let $A \subset R^n$ be an open set and let $f \in H^{1,1}_{\text{loc}}(A)$. Suppose that (i) $\mathcal{H}_n(\{x \in A : |\nabla f| = 0\}) = 0$, (ii) $\mathcal{H}_{n-1}(N) = 0$ where N is a closet set in A, (iii) $\int_{A-N} |\nabla f|^{-1} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial \phi}{\partial x_i} dx = 0$ for every $\phi \in C_0^1(A-N)$. Then f has least gradient with respect to A.

Remark 3. (i) The assertion (7) $\Rightarrow$ (9) is due to Miranda.

(ii) Connecting the assertions (5), (6), and (12) on Riemannian manifolds, S.P. Wang and the author [19] prove that if each level hypersurface of a smooth function $f : M \to \mathbb{R}$ on an oriented *Riemannian manifold* M with nowhere vanishing ∇f, is minimal, then there exists a closed form with comass 1 on M and hence each level hypersurface is homologically area-minimizing over $\mathbb{R}$.

Corollary 4. *Let A be an open subset in $\mathbb{R}^n$, N be a closed subset in A with $\mathcal{H}_{n-1}(N) = 0$. Then the graph of any weak solution of the minimal surface equation $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\frac{\partial f}{\partial x_i}}{\sqrt{1+|\nabla f|^2}}\right) = 0$ on $A - N$ is in fact absolutely area-minimizing in $A \times \mathbb{R} \subset \mathbb{R}^{n+1}$ over $\mathbb{R}$.*

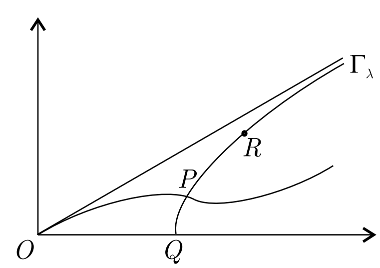
Proof. Applying (3.4) in which "$f(x_1,\ldots,x_{n-1},t)=\eta(x_1,\ldots,x_{n-1})-t$" is replaced with "$F(x_1,\ldots,x_n,t)=f(x_1,\ldots,x_n)-t$", and Remark 2, we have that F is a C^1 1-harmonic function in A. By Theorem 2, the zero level set $S_0=\{(x_1,\ldots,x_n,t):t=f(x_1,\ldots,x_n)\}$ is absolutely area-minimizing in $A\times\mathbb{R}\subset\mathbb{R}^{n+1}$ over $\mathbb{R}$. ■

4 Further applications

A natural question arises: Are Bombieri–De Giorgi–Giusti and Lawson cones the only $SO(m)\times SO(n)$-invariant singular absolutely area-minimizing integral currents in Euclidean space $\mathbb{R}^{m+n+2}$? The answer is affirmative. Combining the theory of 1-harmonic functions developed, and the techniques of transformation groups in [10, 13, 2], and [21], evolved from the ideas in [9], one obtains the following:

Theorem 3. *The cone $C(S^m\times S^n)$ over $S^m\times S^n$ is the unique singular absolutely area-minimizing hypersurface in the class of $SO(m+1)\times SO(n+1)$-invariant integral currents in $\mathbb{R}^{m+n+2}$ over $\mathbb{R}$ for $m+n>7$ or $m+n=6$, $|m-n|\le 2$. (It is known that the cone is not even stable otherwise.)*

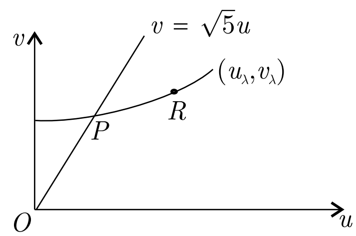
Proof. Assume $m=n$. Let Lie group $G=SO(n+1)\times SO(n+1)$ acting on manifold $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ in the standard way, i.e. assigning $\big((A,B),(x,y)\big)\in G\times\mathbb{R}^{2n+2}$ to $(A\cdot x,B\cdot y)\in\mathbb{R}^{2n+2}$, where "$\cdot$" is the matrix multiplication. Then the collection X of principle orbits is given by $X=\{(x,y)\in\mathbb{R}^{2n+2}:|x||y|\neq 0\}$, where "$|\cdot|$" is the length of "$\cdot$" in $\mathbb{R}^{n+1}$. The orbit space which is stratified, can be represented as $\mathbb{R}^{2n+2}/G=\{(u,v)\in\mathbb{R}^2:u,v\ge 0\}=X\cup\{(u,v)\in\mathbb{R}^2:u=0,v>0\}\cup\{(u,v)\in\mathbb{R}^2:u>0,v=0\}\cup\{(0,0)\}$. The canonical metric on $\mathbb{R}^{2n+2}/G$ (compatible with the fibration over each stratum) is the usual flat one $ds_0^2=du^2+dv^2$. The canonical projection $\pi:\mathbb{R}^{2n+2}\to\mathbb{R}^{2n+2}/G$ is given by $\pi(x,y)=(|x|,|y|)$, and let $X/G=\pi(X)$. Then the length of a curve σ in $(X/G,ds_0^2)$ is the length of any orthogonal trajectory through the corresponding orbits in X, and $2n$-dimensional volume of $\pi^{-1}((u,v))$ (which is diffeomorphic to $S^n\times S^n$) is proportional to u^nv^n, for $(u,v)\in X/G$. Thus if we choose the metric $ds^2=u^{2n}v^{2n}(du^2+dv^2)$ on $\mathbb{R}^{2n+2}/G$, then by Fubini's theorem, the length of a curve σ in $(\mathbb{R}^{2n+2}/G,ds^2)$ is equal to $(2n+1)$-dimensional volume of hypersurface $\pi^{-1}\sigma$ (with possible singularities) in $\mathbb{R}^{2n+2}$, up to a constant factor. It follows that σ is a length minimizing geodesic "downstairs" (in $(\mathbb{R}^{2n+2}/G,ds^2)$), if and only if $\pi^{-1}\sigma$ is area-minimizing *in the class of G-invariant* $(2n+1)$-dimensional currents "upstairs" (in $(\mathbb{R}^{2n+2},dx_1^2+\cdots+dx_{2n+2}^2)$), or equivalently, $\pi^{-1}\sigma$ is area-minimizing in $(\mathbb{R}^{2n+2},dx_1^2+\cdots+dx_{2n+2}^2)$ *in general* (cf. [13], [2, p. 174, 6.4] and [21]). Furthermore, if a length minimizing geodesic σ meets the boundary $\{(u,v)\in\mathbb{R}^2:u=0,v>0\}\cup\{(u,v)\in\mathbb{R}^2:u>0,v=0\}$, it meets the boundary orthogonally by the first variational formula for the arc-length functional, and the corresponding $\pi^{-1}\sigma$ is a regular, embedded and analytic hypersurface in $\mathbb{R}^{2n+2}$. If σ meets the vertex $\{(0,0)\}$, then $\pi^{-1}\sigma$ is singular. Therefore, it suffices to show that any curve in $\mathbb{R}^{2n+2}/G$, other than the diagonal ray emanating from the origin is not absolutely length minimizing with respect to the metric $ds^2=u^{2n}v^{2n}(du^2+dv^2)$.



Now let $\Gamma = \{(u_0(t), v_0(t))\}$ be the geodesic through $(1,0)$ in $(\mathbb{R}^{2n+2}/G, ds^2)$, and $\Gamma_\lambda = \{(\lambda u_0(t), \lambda v_0(t))\}$, $\lambda > 0$. In [3], a 1-harmonic function was constructed in such a way that the lift of family $\{\Gamma_\lambda\}$ of these homothetic geodesics are level hypersurfaces in $(\mathbb{R}^{2n+2}, dx_1^2 + \cdots + dx_{2n+2}^2)$. Hence Γ_λ is absolutely length minimizing in $(\mathbb{R}^{2n+2}/G, ds^2)$ (cf. also Theorem 2, Remark 2). Now suppose Theorem 3 were not true. Then there would exist a curve $QP \subset \Gamma_\lambda$ transverse to a length minimizing curve OP. It follows that the length $l(OP)$ of OP would satisfy $l(OP) = l(QP)$. Consider the curve OPR where R is on the curve Γ_λ, and $l(OPR) = l(QPR)$. Then the curve OPR would be a geodesic, and hence smooth at P. This is a contradiction. Similarly, one can show the remaining case $m \neq n$. ■

Theorem 4. *The cone $C(S^1 \times S^5)$ over $S^1 \times S^5$ is not absolutely area-minimizing, although it is stable.*

Proof. Suppose, on the contrary, that the cone were absolutely area-minimizing. Then consider Lie group $G = SO(2) \times SO(6)$ acting on manifold $\mathbb{R}^2 \times \mathbb{R}^6$ in the standard way. By the previous argument, this would imply the line segment $\overline{OP}$ were length-minimizing in $(\mathbb{R}^8/G, ds^2)$, where $ds^2 = u^2 v^6 (du^2 + dv^2)$. On the other hand, based on the study of Simoes' thesis [17], [13] and [21], the level curve (u_λ, v_λ) in the u, v-plane is absolutely length-minimizing. Argue as before, the curve OPR would be smooth at P. This is a contradiction. The stability of the cone follows from Simons' work [18]. ■



Theorem 5. *Any 7-dimensional $SO(2) \times SO(6)$-invariant absolutely area- minimizing integral current in $\mathbb{R}^8$ is real analytic.*

Proof. By the argument given in the proof of Theorem 3, it suffices to show that any curve in $\mathbb{R}^{2n+2}/G$, from the origin is not absolutely length minimizing with respect to the metric $ds^2 = u^2 v^6 (du^2 + dv^2)$. By Theorem 4, the diagonal ray emanating from the origin is not length minimizing. Similarly, if there were an absolutely length minimizing curve starting from the origin lying above $v = \sqrt{5}u$, then this would lead to an irregularity of a geodesic, a contradiction. ■

5 Comparison theorem

It is known that each level hypersurface of a function of least gradient defined on an open subset $A \subset \mathbb{R}^n$ is absolutely area-minimizing in A over $\mathbb{Z}$. It is tempting to ask it if is absolutely area-minimizing in A over $\mathbb{R}$. This motivates our discussion on comparison between real and integral absolute (or homological) minima. In general they are distinct. Examples are given by Almgren [7, 5.11], Federer [7] and Lawson [12]. Furthermore, in the case of 1-dimensional (or co-dimension 1) integral flat chains, Federer [7] has shown that real and integral homological (or absolute) minimizing are the same.

Let $\overline{M}$ be a locally Lipschitz neighborhood retract in $\mathbb{R}^n$ (i.e. there exists a locally Lipschitz map which retracts a neighborhood of $\overline{M}$ onto $\overline{M}$), M be an open subset of $\overline{M}$, and A be an open subset of $\mathbb{R}^n$. Using the assumption on vanishing topology, an exhaustion of M by an increasing sequence of compact set $K_i \subset M$, we obtain the following:

Theorem 6. (1) *Let T^{n-1} denote a codimension* 1 *integral absolutely area-minimizing rectifiable current in M with homology group $H_{n-1}(\overline{M}) = 0$. Then T^{n-1} is absolutely area-minimizing in M if and only if T^{n-1} is absolutely area-minimizing in A; and if and only if T^{n-1} is real absolutely area-minimizing in A.* (2) *Let $H_1(\overline{M}) = 0$. T^1 is a homologically area-minimizing rectifiable current of degree* 1 *of M if and only if T^1 is real homologically area-minimizing in M.*

We have the following immediate

Corollary 5. *The level hypersurface of a function of least gradient in an open subset A of $\mathbb{R}^n$ is absolutely area-minimizing over $\mathbb{R}$.*

Corollary 6. *Let N be a closed set in $A \subset \mathbb{R}^n$ with $H_{n-1}(N) = 0$. The graph of any weak solution of the minimal surface equation $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\frac{\partial f}{\partial x_i}}{\sqrt{1+|\nabla f|^2}}\right) = 0$ on $A - N$ is in fact absolutely area-minimizing in $A \times \mathbb{R} \subset \mathbb{R}^{N+1}$ over $\mathbb{R}$.*

Corollary 7. *All the examples we find in [21] are absolutely area-minimizing over $\mathbb{R}$.*

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