On the Limit from q-Racah Polynomials to Big q-Jacobi Polynomials*

Tom H. KOORNWINDER

Korteweg-de Vries Institute, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands
E-mail: *T.H.Koornwinder@uva.nl*
URL: http://www.science.uva.nl/~thk/

Received March 01, 2011; Published online April 21, 2011
doi:10.3842/SIGMA.2011.040

Abstract. A limit formula from q-Racah polynomials to big q-Jacobi polynomials is given which can be considered as a limit formula for orthogonal polynomials. This is extended to a multi-parameter limit with 3 parameters, also involving (q-)Hahn polynomials, little q-Jacobi polynomials and Jacobi polynomials. Also the limits from Askey–Wilson to Wilson polynomials and from q-Racah to Racah polynomials are given in a more conceptual way.

Key words: Askey scheme; q-Askey scheme; q-Racah polynomials; big q-Jacobi polynomials; multi-parameter limit

2010 Mathematics Subject Classification: 33D45; 33C45

Dedicated to Willard Miller on the occasion of his retirement

1 Introduction

The q-*Askey scheme* (see [3, p. 413]) consists of families of q-hypergeometric orthogonal polynomials connected by arrows denoting limit transitions. Askey–Wilson polynomials and q-Racah polynomials are on the top level. All other families in the scheme can be reached from these two families by (possibly successive) limit transitions. In particular, the scheme gives an arrow from the q-Racah polynomials to the big q-Jacobi polynomials. The explicit limit corresponding to this arrow is given in [3, (14.2.15)]. However, while the q-Racah polynomials approach this limit, they no longer form a (finite) system of orthogonal polynomials. It is the first aim of the present paper to give another limit from q-Racah to big q-Jacobi where the orthogonality property remains present while the limit is approached. I was motivated to look for such a limit by seeing a reference to [3, (14.2.15)] in Vinet & Zhedanov [6, end of § 5].

The q-Askey scheme is the q-analogue of the *Askey scheme* (see [3, p. 184]), which was first presented in [1]. The arrows in the Askey scheme represent limit transitions within that scheme, but there are also many limit transitions from families in the q-Askey scheme to families in the Askey scheme. The paper continues with the discussion of two such limits for $q \uparrow 1$: from Askey–Wilson to Wilson and from q-Racah to Racah. Different from their presentation in [3], these limits are given here such that a polynomial of degree n remains present in the limit transition.

The final section of this paper returns to the limit from q-Racah to big q-Jacobi and treats it as part of a multi-parameter limit (with 3 parameters). Thus the author's work in [5] to combine the limits in the Askey scheme (for $q = 1$) into multi-parameter limits, is extended to a small part of the (q-)Askey scheme.

*This paper is a contribution to the Special Issue "Symmetry, Separation, Super-integrability and Special Functions (S^4)". The full collection is available at http://www.emis.de/journals/SIGMA/S4.html

The book Koekoek, Lesky & Swarttouw [3] is the successor of the report Koekoek & Swarttouw [4], which can be alternatively used as a reference whenever the present paper refers to some formula in [3, Chapters 9 and 14]. For notation of q-hypergeometric series used in this paper the reader is referred to [2]. Throughout it will be assumed that $0 < q < 1$, that N is a positive integer and that $n \in \{0, 1, \ldots, N\}$ if N is present.

2 The limit formula

Big q-Jacobi polynomials, see [3, (14.5.1)], are defined as follows:

$$P_n(x; a, b, c; q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ qa, qc \end{matrix}; q, q\right).$$

A special value for $x = qc$ can be obtained by application of [2, (II.6)]:

$$P_n(qc; a, b, c; q) = (-1)^n q^{n(n+1)/2} a^n \frac{(qb; q)_n}{qa; q)_n}.$$

Another q-hypergeometric representation can be obtained by using [2, (III.12)]:

$$\frac{P_n(x; a, b, c; q)}{P_n(qc; a, b, c; q)} = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, qcx^{-1} \\ qb, qc \end{matrix}; q, a^{-1}x\right).$$

q-Racah polynomials, see [3, (14.2.1)], are defined as follows:

$$R_n\big(q^{-y} + q^{y-N}\delta; \alpha, \beta, q^{-N-1}, \delta \mid q\big) := {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+1}\alpha\beta, q^{-y}, q^{y-N}\delta \\ q\alpha, q\beta\delta, q^{-N} \end{matrix}; q, q\right) \tag{2.1}$$

$(n = 0, 1, \ldots, N)$.

They are indeed polynomials of degree n in x:

$$R_n\big(x; \alpha, \beta, q^{-N-1}, \delta \mid q\big) = \sum_{k=0}^{n} \frac{(q^{-n}, q^{n+1}\alpha\beta; q)_k\, q^k}{(q\alpha, q\beta\delta, q; q)_k} \prod_{j=0}^{k-1} \frac{1 - q^j x + q^{2j-N}\delta}{1 - q^{j-N}}.$$

Now observe that

$$\begin{aligned} &R_n\left(\frac{x}{q^{N+1}a}; b, a, q^{-N-1}, \frac{c}{a} \mid q\right) \\ &\quad = \sum_{k=0}^{n} \frac{(q^{-n}, q^{n+1}ab; q)_k\, q^k}{(qb, qc, q; q)_k} \prod_{j=0}^{k-1} \frac{1 - q^{j-N-1}a^{-1}x + q^{2j-N}a^{-1}c}{1 - q^{j-N}} \\ &\quad \longrightarrow \sum_{k=0}^{n} \frac{(q^{-n}, q^{n+1}ab; q)_k}{(qb, qc, q; q)_k} \big(a^{-1}x\big)^k \prod_{j=0}^{k-1} \big(1 - q^{j+1}cx^{-1}\big) \quad \text{as } N \to \infty. \end{aligned}$$

Thus we have proved our main result:

Theorem 1. *There is the following limit formula from q-Racah polynomials to big q-Jacobi polynomials:*

$$\lim_{N\to\infty} R_n\left(\frac{x}{q^{N+1}a}; b, a, q^{-N-1}, \frac{c}{a} \mid q\right) = \frac{P_n(x; a, b, c; q)}{P_n(qc; a, b, c; q)}. \tag{2.2}$$

Remark 1. Assume

$$0 < qa < 1, \qquad 0 \le qb < 1, \qquad c < 0. \tag{2.3}$$

Then the polynomials

$$x \mapsto R_n\left(\frac{x}{q^{N+1}a}; b, a, q^{-N-1}, \frac{c}{a} \mid q\right)$$

are orthogonal with respect to positive weights (see [3, (14.2.2)]) on the points

$$q^{N+1-y}a + q^{y+1}c \qquad (y = 0, 1, \ldots, N),$$

which, for certain M depending on N can be written as the union of the increasing sequence of nonpositive points

$$qc + q^{N+1}a,\ q^2c + q^N a,\ \ldots,\ q^M c + q^{N-M+2}a$$

and the decreasing sequence of nonnegative points

$$qa + q^{N+1}c,\ q^2a + q^N c,\ \ldots,\ q^{N-m+1}a + q^{M+1}c.$$

Formally, in the limit for $N \to \infty$ this tends to the union of the sequence of negative points $\{q^{k+1}c\}_{k=0,1,\ldots}$ and the sequence of positive points $\{q^{k+1}a\}_{k=0,1,\ldots}$. But indeed, we know that under the constraints (2.3) the big q-Jacobi polynomials are orthogonal with respect to positive weights on this set of points (see [3, (14.5.2)]). Thus the limit formula (2.2) is under the constraints (2.3) on the parameters really a limit formula for orthogonal polynomials.

Remark 2. The limit formula [3, (14.2.15)], which reads

$$P_n\big(q^{-y}; a, b, c; q\big) = \lim_{\delta \to 0} R_n\big(q^{-y} + c\delta q^{y+1}; a, b, c, \delta \mid q\big), \tag{2.4}$$

cannot be considered as a limit formula for orthogonal polynomials. Indeed, for the q-Racah polynomials on the right-hand side it is required that qa or $qb\delta$ or qc is equal to q^{-N} for some positive integer N (see [3, (14.2.1)]). Since $\delta \to 0$ and a, b, c remain fixed in (2.4), we must have qa or qc equal to q^{-N}. But then we arrive at a limit from q-Racah polynomials to q-Hahn polynomials (see [3, (14.2.16) or (14.2.18)]) rather than big q-Jacobi polynomials.

Remark 3. For $c = 0$ (2.2) specializes to a limit formula from q-Hahn polynomials to little q-Jacobi polynomials. For the left-hand side of (2.2) use that

$$R_n\left(\frac{x}{q^{N+1}a}; b, a, q^{-N-1}, 0 \mid q\right) = Q_n\left(\frac{x}{q^{N+1}a}; b, a, N; q\right), \tag{2.5}$$

see [3, (14.2.16)], where the Q_n are *q-Hahn polynomials* [3, (14.6.1)]. For the right-hand side of (2.2) use that

$$\frac{P_n(x; a, b, 0; q)}{P_n(0; a, b, 0; q)} = p_n\left(\frac{x}{qa}; b, a; q\right),$$

see [3, p. 442, Remarks, first formula], where the p_n are *little q-Jacobi polynomials* [3, (14.12.1)]. Thus for $c = 0$ (2.2) specializes to the limit formula

$$\lim_{N \to \infty} Q_n\left(\frac{x}{q^{N+1}a}; b, a, N; q\right) = p_n\left(\frac{x}{qa}; b, a; q\right), \tag{2.6}$$

which is also given in [3, (14.6.13)].

3 Limit from Askey–Wilson to Wilson

Consider *Askey–Wilson polynomials* (see [3, (14.1.1)]), putting $x = \cos\theta$:

$$p_n(x;a,b,c,d \mid q) := a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right)$$
$$= a^{-n}(ab,ac,ad;q)_n \sum_{k=0}^{n} \frac{(q^{-n}, abcdq^{n-1};q)_k\, q^k}{(ab,ac,ad,q;q)_k} \prod_{j=0}^{k} \big(1 - 2q^j ax + q^{2j}a^2\big). \tag{3.1}$$

Also consider *Wilson polynomials* (see [3, (9.1.1)]), putting $x = y^2$:

$$W_n(x;a,b,c,d) := (a+b,a+c,a+d)_n\, {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+iy, a-iy \\ a+b, a+c, a+d \end{matrix}; 1\right)$$
$$= (a+b,a+c,a+d)_n \sum_{k=0}^{n} \frac{(-n, n+a+b+c+d-1)_k}{(a+b,a+c,a+d)_k\, k!} \prod_{j=0}^{k} \big((a+j)^2 + x\big). \tag{3.2}$$

Rescale (3.1) as

$$(1-q)^{-3n} p_n\big(1 - \tfrac{1}{2}(1-q)^2 x; q^a, q^b, q^c, q^d \mid q\big) = \frac{(q^{a+b}, q^{a+c}, q^{a+d};q)_n}{q^{na}(1-q)^{3n}}$$
$$\times \sum_{k=0}^{n} \frac{(q^{-n}, q^{n+a+b+c+d-1};q)_k\, q^k (1-q)^2}{(q^{a+b}, q^{a+c}, q^{a+d}, q;q)_k} \prod_{j=0}^{k} \left(\frac{(1-q^{a+j})^2}{(1-q)^2} + q^{a+j}x\right). \tag{3.3}$$

From (3.3) and (3.2) we conclude that

$$\lim_{q\uparrow 1} (1-q)^{-3n} p_n\big(1 - \tfrac{1}{2}(1-q)^2 x; q^a, q^b, q^c, q^d \mid q\big) = W_n(x;a,b,c,d). \tag{3.4}$$

Remark 4. In [3, (14.1.21)] the following limit from Askey–Wilson polynomials to Wilson polynomials is given:

$$\lim_{q\uparrow 1} v(1-q)^{-3n} v p_n\big(\tfrac{1}{2}(q^{iy} + q^{-iy}); q^a, q^b, q^c, q^d \mid q\big) = W_n\big(y^2;a,b,c,d\big). \tag{3.5}$$

This limit follows immediately by comparing the (q-)hypergeometric expressions in (3.1) and (3.2). However, the limit (3.5) has the draw-back that the rescaled Askey–Wilson polynomial on the left no longer depends polynomially on y. Note that the limit (3.4) can be written more generally, by the same proof, as

$$\lim_{q\uparrow 1} (1-q)^{-3n} p_n\big(1 - \tfrac{1}{2}(1-q)^2 x + o\big((1-q)^2\big); q^a, q^b, q^c, q^d \mid q\big) = W_n(x;a,b,c,d). \tag{3.6}$$

Then (3.5) is a special case of (3.6), since

$$\tfrac{1}{2}(q^{iy} + q^{-iy}) = 1 - \tfrac{1}{2}(1-q)^2 y^2 + o\big((1-q)^2\big).$$

4 Limit from q-Racah to Racah

In (2.1) we introduced q-Racah polynomials. These are orthogonal with respect to positive weights if $0 < q\alpha < 1$, $0 < q\beta < 1$ and $\delta < q^N\alpha$, as can be read off from [3, (14.2.2)] and also from the requirement that $A_{n-1}C_n > 0$ for $n = 1, 2, \ldots, N$ in the normalized recurrence relation [3, (14.2.4)]. In order to keep positive weights in the limit from q-Racah polynomials to big q-Jacobi polynomials we needed $\delta < 0$, see Remark 1, or $\delta = 0$ in a degenerate case, see Remark 3. However, for the limit from q-Racah polynomials to Racah polynomials we will need $0 < \delta < q^N \alpha$. We can rewrite (2.1) as

$$R_n\big(x;\alpha,\beta,q^{-N-1},\delta \mid q\big) = \sum_{k=0}^{n} \frac{(q^{-n},q^{n+1}\alpha\beta;q)_k\, q^k}{(q\alpha,q\beta\delta,q^{-N},q;q)_k} \prod_{j=0}^{k-1}\big(1 - xq^j + q^{\delta-N+2j}\big). \tag{4.1}$$

Also consider *Racah polynomials* (see [3, (9.2.1)]), putting $x = y(y+\delta-N)$:

$$\begin{aligned} R_n(x;\alpha,\beta,-N-1,\delta) &:= {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -y, y+\delta-N \\ \alpha+1, \beta+\delta+1, -N \end{matrix}; 1\right) \\ &= \sum_{k=0}^{n} \frac{(-n,n+\alpha+\beta+1)_k}{(\alpha+1,\beta+\delta+1,-N)_k\, k!} \prod_{j=0}^{k}(-x + j(\delta-N+j)). \end{aligned} \tag{4.2}$$

These are orthogonal with respect to positive weights if $\alpha,\beta > -1$ and $\delta > N+\alpha$, see [3, (9.2.2)] or [3, (9.2.4)]. Rescale (4.1) as

$$\begin{aligned} &R_n\big((1-q)^2x + 1 + q^{\delta-N}; q^\alpha, q^\beta, q^{-N-1}, q^\delta \mid q\big) \\ &\quad = \sum_{k=0}^{n} \frac{(q^{-n},q^{n+\alpha+\beta+1};q)_k\,(1-q)^{2k}q^k}{(q^{\alpha+1},q^{\beta+\delta+1},q^{-N},q;q)_k} \prod_{j=0}^{k-1}\left(\frac{(1-q^j)(1-q^{\delta-N+j})}{(1-q)^2} - xq^j\right). \end{aligned} \tag{4.3}$$

From (4.3) and (4.2) we conclude that

$$\lim_{q\uparrow 1} R_n\big(1+q^{\delta-N}+(1-q)^2x; q^\alpha, q^\beta, q^{-N-1}, q^\delta \mid q\big) = R_n(x;\alpha,\beta,-N-1,\delta). \tag{4.4}$$

The orthogonal polynomials involved in this limit have positive weights if $\alpha,\beta > -1$ and $\delta > N+\alpha$.

Remark 5. In [3, (14.2.24)] the following limit from q-Racah polynomials to Racah polynomials is given:

$$\lim_{q\uparrow 1} R_n(q^{-y}+q^{y+\delta-N}; q^\alpha, q^\beta, q^{-N-1}, q^\delta \mid q) = R_n(y(y+\delta-N);\alpha,\beta,-N-1,\delta). \tag{4.5}$$

This limit follows immediately by comparing the (q-)hypergeometric expressions in (2.1) and (4.2). Just as for (3.5), the limit (4.5) has the draw-back that we no longer have polynomials in y on the left-hand side of (4.5). Note that the limit (4.4) can be written more generally, by the same proof, as

$$\begin{aligned} &\lim_{q\uparrow 1} R_n\big(1+q^{\delta-N}+(1-q)^2x + o((1-q)^2); q^\alpha, q^\beta, q^{-N-1}, q^\delta \mid q\big) \\ &\qquad = R_n(x;\alpha,\beta,-N-1,\delta). \end{aligned} \tag{4.6}$$

Then (4.5) is a special case of (4.6) since

$$y(y+\delta-N) = -\frac{(1-q^{\delta-N+y})((1-q^{-y})}{(1-q)^2} + o\big((1-q)^2\big)$$

and

$$1+q^{\delta-N}+(1-q)^2x = y(y+\delta-N) \qquad \text{for} \quad x = \frac{(1-q^{\delta-N+y})(1-q^{-y})}{(1-q)^2}.$$

Also note that the polynomials $x \mapsto R_n(1+q^{\delta-N}+(1-q)^2x)$ on the left-hand side of (4.4) are orthogonal with respect to weights on the points $-(1-q)^{-2}(1-q^{\delta-N+y})(1-q^{-y})$ $(y=0,1,\dots,N)$ by [3, (14.2.)]. In the limit for $q \uparrow 1$ this becomes an orthogonality on the points $y(y+\delta-N)$ $(y=0,1,\dots,N)$, as is indeed the case for Racah polynomials, see [3, (9.2.2)].

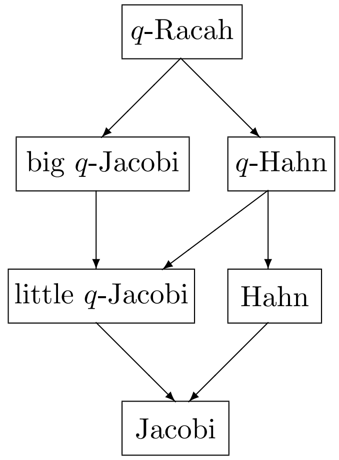

Figure 1. Part of (q-)Askey scheme.

5 A piece of (q-)Askey scheme below q-Racah

We earlier saw the limits (2.2) (q-Racah → big q-Jacobi), (2.5) (q-Racah → q-Hahn), (1) (big q-Jacobi → little q-Jacobi) and (2.6) (q-Hahn → little q-Jacobi). To these we can add limits from q-Hahn to Hahn (see [3, (14.6.18)])

$$\lim_{q\uparrow 1} Q_n(1+(1-q)x;\alpha,\beta,N;q) = Q_n(x;\alpha,\beta,N), \tag{5.1}$$

from little q-Jacobi to Jacobi (see [3, (14.12.15)])

$$\lim_{q\uparrow 1} p_n\big(x;q^\alpha,q^\beta;q\big) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)},$$

and from Hahn to Jacobi (see [3, (9.5.14)])

$$\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.$$

Note that in (5.1) the left-hand side of [3, (14.6.18)] was changed in order to keep polynomials in x while taking the limit. The validity of (5.1) is easily seen from [3, (14.6.1), (9.5.1)].

Fig. 1 combines these seven limits as a subgraph of the (q-)Askey scheme (see the graphs given in the beginning of Chapters 9 and 14 in [3]).

In [5] I combined the limits in the Askey scheme (i.e., for $q = 1$) into a small number of multi-parameter limits. This was done by renormalizing the Racah and Askey–Wilson polynomials on the top level of the scheme as families of orthogonal polynomials depending on four positive parameters such that these extend continuously for nonnegative parameter values, while (renormalized) families lower in the scheme are reached if one or more of the parameters become zero. At the end of [5] the obvious open problem was mentioned to extend this work to the q-Askey scheme including the limits for $q \uparrow 1$. Below I will work this out for the small part of the (q-)Askey scheme in Fig. 1.

Fix $\alpha,\beta > -1$ and renormalize the q-Racah polynomials as

$$\begin{aligned} p_n(x) = p_n\big(x;c,N^{-1},1-q\big) &:= \frac{q^{n(\beta+1)}\,(q^{\alpha+1},-q^{\beta+1}c,q^{-N};q)_n}{(q^{-N}-1)^n(q^{n+\alpha+\beta+1};q)_n} \\ &\quad\times R_n\big(1-q^{-N}c+q^{-\beta-1}(q^{-N}-1)x;q^\alpha,q^\beta,q^{-N-1},-c\mid q\big). \end{aligned} \tag{5.2}$$

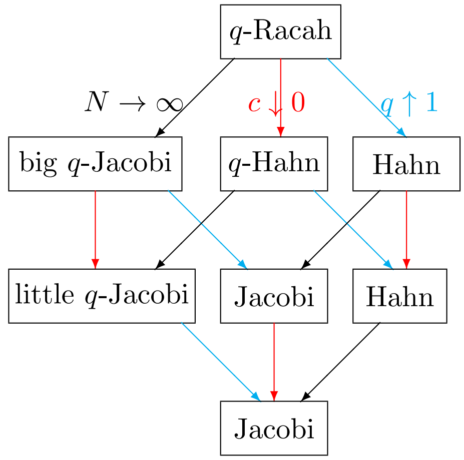

Figure 2. Part of (q-)Askey scheme with multi-parameter limits.

By the chosen coefficient on the right these are monic polynomials of degree n, see [3, (14.2.4)]. For the parameters in the arguments of p_n we require

$$c > 0, \qquad 0 < 1 - q < 1, \qquad N^{-1} \in \left\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\right\}. \tag{5.3}$$

We will see that the polynomials $p_n(x; c, N^{-1}, 1-q)$ remain continuous in $(c, N^{-1}, 1-q)$ if these three coordinates are also allowed to become zero.

For the demonstration we will use the same tool as in [5]. We will see that the coefficients in the three-term recurrence relation for the orthogonal polynomials (5.2) depend continuously on $(c, N^{-1}, 1-q)$ for values of these coordinates as in (5.3) or equal to zero.

It follows from [3, (14.2.4)] that p_n given by (5.2) satisfies the recurrence relation

$$xp_n(x) = p_{n+1}(x) + (A_n + C_n)p_n(x) + A_{n-1}C_n p_{n-1}(x) \tag{5.4}$$

with

$$A_n = q^{\beta+1}\big(1 + q^{n+\beta+1}c\big)\frac{(1-q^{n+\alpha+1})(1-q^{n+\alpha+\beta+1})}{(1-q^{2n+\alpha+\beta+1})(1-q^{2n+\alpha+\beta+2})}\,\frac{q^{n-N}-1}{q^{-N}-1}$$

and

$$C_n = q^{\beta+2}(c + q^{n+\alpha})\frac{(1-q^n)(1-q^{n+\beta})}{(1-q^{2n+\alpha+\beta})(1-q^{2n+\alpha+\beta+1})}\,\frac{q^{-N}-q^{n+\alpha+\beta}}{q^{-N}-1}.$$

Clearly, A_n and C_n are continuous in $(c, N^{-1}, 1-q)$ for $(N^{-1}, 1-q) \neq (0,0)$. In order to prove their continuity at $(N^{-1}, 1-q) = (0,0)$ we only have to consider the continuity there of the factors

$$\frac{q^{n-N}-1}{q^{-N}-1} = 1 - \frac{1-q^n}{1-q}\,\frac{1-q}{1-q^N}$$

and

$$\frac{q^{-N}-q^{n+\alpha+\beta}}{q^{-N}-1} = 1 + q^N\frac{1-q^{n+\alpha+\beta}}{1-q}\,\frac{1-q}{1-q^N}.$$

Their continuity follows from the limit

$$\lim_{q\uparrow 1;\, N\to\infty} \frac{1-q}{1-q^N} = 0,$$

which holds because

$$\frac{1-q}{1-q^N} = \frac{1}{1+q+\cdots+q^{N-1}} \le \frac{1}{1+q_0+\cdots+q_0^{N_0-1}} = \frac{1-q_0}{1-q_0^{N_0}} \quad \text{if } q_0 \le q < 1,\ N \ge N_0.$$

We can identify the cases where one or more of the parameters c, N^{-1}, $1-q$ in (5.2) are zero, with families situated below the q-Racah box in Fig. 1. This can be done by taking limits in (5.2) or by taking limits in the recurrence relation (5.4). Thus we see:

$$p_n(x;c,0,1-q) = \text{const} \cdot P_n\big(x - q^{\beta+1}c; q^\beta, q^\alpha, -q^{\beta+1}c; q\big) \quad \text{(big } q\text{-Jacobi)},$$
$$p_n\big(x;0,N^{-1},1-q\big) = \text{const} \cdot Q_n\big(1 + q^{-\beta-1}(q^{-N}-1)x; q^\alpha, q^\beta, N; q\big) \quad (q\text{-Hahn}),$$
$$p_n\big(x;c,N^{-1},0\big) = \text{const} \cdot Q_n(Nx;\alpha,\beta,N) \quad \text{(Hahn)},$$
$$p_n(x;0,0,1-q) = \text{const} \cdot p_n\big(x;q^\alpha,q^\beta;q\big) \quad \text{(little } q\text{-Jacobi)},$$
$$p_n(x;c,0,0) = \text{const} \cdot P_n^{(\alpha,\beta)}(1-2x) \quad \text{(Jacobi)}.$$

The various limits are collected in Fig. 2.

References

[1] Askey R., Wilson J., Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.* (1985), no. 319.

[2] Gasper G., Rahman M., Basic hypergeometric series, 2nd ed., *Encyclopedia of Mathematics and its Applications*, Vol. 96, Cambridge University Press, Cambridge, 2004.

[3] Koekoek R., Lesky P.A., Swarttouw R.F., Hypergeometric orthogonal polynomials and their q-analogues, *Springer Monographs in Mathematics*, Springer-Verlag, Berlin, 2010.

[4] Koekoek R., Swarttouw R.F., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1998, http://aw.twi.tudelft.nl/~koekoek/askey/.

[5] Koornwinder T.H., The Askey scheme as a four-manifold with corners, *Ramanujan J.* **20** (2009), 409–439, arXiv:0909.2822.

[6] Vinet L., Zhedanov A., A limit $q = -1$ for big q-Jacobi polynomials, *Trans. Amer. Math. Soc.*, to appear, arXiv:1011.1429.