

Nontrivial Deformation of a Trivial Bundle^{*}

Piotr M. HAJAC^{†‡} and Bartosz ZIELIŃSKI[§]

[†] *Institut Matematyczny, Polska Akademia Nauk, ul. Śniadeckich 8, 00-956 Warszawa, Poland*

E-mail: pmh@impan.pl

URL: <http://www.impan.pl/~pmh/>

[‡] *Katedra Metod Matematycznych Fizyki, Uniwersytet Warszawski, ul. Hoża 74, 00-682 Warszawa, Poland*

[§] *Department of Computer Science, Faculty of Physics and Applied Informatics, University of Łódź, Pomorska 149/153 90-236 Łódź, Poland*

E-mail: bzielinski@uni.lodz.pl

Received October 29, 2013, in final form March 03, 2014; Published online March 27, 2014

<http://dx.doi.org/10.3842/SIGMA.2014.031>

Abstract. The $SU(2)$ -prolongation of the Hopf fibration $S^3 \rightarrow S^2$ is a trivializable principal $SU(2)$ -bundle. We present a noncommutative deformation of this bundle to a quantum principal $SU_q(2)$ -bundle that is *not* trivializable. On the other hand, we show that the $SU_q(2)$ -bundle is *piecewise* trivializable with respect to the closed covering of S^2 by two hemispheres intersecting at the equator.

Key words: quantum prolongations of principal bundles; piecewise trivializable quantum principal bundles

2010 Mathematics Subject Classification: 58B32

Dedicated to Marc A. Rieffel on the occasion of his 75th birthday

1 Introduction and preliminaries

The goal of this paper is to show how a noncommutative deformation can turn a trivializable principal bundle into a nontrivializable quantum principal bundle. This is a peculiar phenomenon because noncommutative deformations usually preserve basic topological features of deformed objects, e.g. K -groups.

On the other hand, this paper exemplifies the general theory of piecewise trivial principal comodule algebras developed in [7, 9]. Therefore we follow the notation, conventions and general setup employed therein. To make our exposition self-contained and easy to read, we often recall basic concepts and definitions.

Let $\pi : X \rightarrow M$ be a principal G -bundle over M , and G' be a subgroup of G . A G' -reduction of $X \rightarrow M$ is a subbundle $X' \subseteq X$ over M that is a principal G' -bundle over M via the restriction of the G -action on X . Many important structures on manifolds can be formulated as reductions of their frame bundles. For instance, an orientation, a volume form and a metric on a manifold M correspond to reductions of the frame bundle FM to a $GL_+(n, \mathbb{R})$, $SL(n, \mathbb{R})$ and $O(n, \mathbb{R})$ -bundle, respectively. See [10] for more details.

An operation inverse to a reduction of a principal bundle is a prolongation of a principal bundle. Let $\pi : X \rightarrow M$ be a principal G' -bundle over M , and let G' be a subgroup of G . Define $X \times_{G'} G := (X \times G)/\sim$, where $(x, g) \sim (xh, h^{-1}g)$, for all $x \in X$, $g \in G$ and $h \in G'$.

^{*}This paper is a contribution to the Special Issue on Noncommutative Geometry and Quantum Groups in honor of Marc A. Rieffel. The full collection is available at <http://www.emis.de/journals/SIGMA/Rieffel.html>

Then

$$\hat{\pi} : X \times_{G'} G \longrightarrow M, \quad [x, g] \longmapsto \pi(x),$$

is a G -bundle called the G -prolongation of X , with the G -action given by $[x, g]h := [x, gh]$. The bundle $X \rightarrow M$ is a G' -reduction of $X \times_{G'} G \rightarrow M$.

An interesting special case is when $X = G$ and $M = G/G'$, that is the homogenous bundle case. It is easy to see that $G \times_{G'} G \rightarrow G/G'$ is always a trivializable bundle. Indeed, the following G -equivariant bundle maps provide an explicit isomorphism and its inverse:

$$\begin{aligned} f : G \times_{G'} G &\longrightarrow G/G' \times G, & [g_1, g_2] &\longmapsto ([g_1], g_1 g_2), \\ f^{-1} : G/G' \times G &\longrightarrow G \times_{G'} G, & ([g], h) &\longmapsto [g, g^{-1} h]. \end{aligned}$$

A quantum-group version of the trivializability of $G \times_{G'} G \rightarrow G/G'$ can be easily checked mimicking the classical argument. In particular, the $SU_q(2)$ -prolongation

$$SU_q(2) \times_{U(1)} SU_q(2) \longrightarrow S_q^2$$

of the standard quantum Hopf fibration is trivializable [5, p. 1104]. However, as the main result of this paper, we show that the $SU_q(2)$ -prolongation

$$SU(2) \times_{U(1)} SU_q(2) \longrightarrow S^2$$

of the classical Hopf fibration is *not* trivializable.

1.1 Notation

We work over the field \mathbb{C} of complex numbers. The unadorned tensor product stands for the tensor product over this field. The comultiplication, counit and the antipode of a Hopf algebra H are denoted by Δ , ε and S , respectively. Our standing assumption is that S is invertible. A right H -comodule algebra P is a unital associative algebra equipped with an H -coaction $\Delta_P : P \rightarrow P \otimes H$ that is an algebra homomorphism. For a comodule algebra P , we call

$$P^{\text{co}H} := \{p \in P \mid \Delta_P(p) = p \otimes 1\}$$

the subalgebra of coaction-invariant elements in P . A left coaction on V is denoted by ${}_V\Delta$. For comultiplications and coactions, we often employ the Heynemann–Sweedler notation with the summation symbol suppressed:

$$\Delta(h) =: h_{(1)} \otimes h_{(2)}, \quad \Delta_P(p) =: p_{(0)} \otimes p_{(1)}, \quad {}_V\Delta(v) =: v_{(-1)} \otimes v_{(0)}.$$

1.2 Reductions and prolongations of principal comodule algebras

Definition 1 ([4]). Let H be a Hopf algebra, P be a right H -comodule algebra and let $B := P^{\text{co}H}$ be the coaction-invariant subalgebra. The comodule algebra P is called *principal* iff:

- 1) $P \otimes_B P \ni p \otimes q \mapsto \text{can}(p \otimes q) := pq_{(0)} \otimes q_{(1)} \in P \otimes H$ is bijective,
- 2) there exists a left B -linear right H -colinear splitting of the multiplication map $B \otimes P \rightarrow P$,
- 3) the antipode of H is bijective.

Here (1) is the Hopf–Galois (freeness) condition, (2) means equivariant projectivity of P , and (3) ensures a left-right symmetry of the definition (everything can be re-written for left comodule algebras).

A particular class of principal comodule algebras is distinguished by the existence of a cleaving map. A cleaving map is defined as a unital right H -colinear convolution-invertible map $j : H \rightarrow P$. Comodule algebras admitting a cleaving map are called *cleft*. One can show that a cleaving map is automatically injective. However, in general, they are not algebra homomorphisms.

If $j : H \rightarrow P$ is a right H -colinear algebra homomorphism, then it is automatically convolution-invertible and unital. A cleft comodule algebra admitting a cleaving map that is an algebra homomorphism is called a *smash product*. All commutative smash products reduce to the tensor algebra $P^{\text{co}H} \otimes H$, so that smash products play the role of trivial bundles. Here a cleaving map is simply given by $j(h) := 1 \otimes h$. A cleaving map defines a left action of H on $P^{\text{co}H}$ making it a left H -module algebra: $h \triangleright p := j(h_{(1)})p j^{-1}(h_{(2)})$. Conversely, if B is a left H -module algebra, one can construct a smash product $B \rtimes H$ by equipping the vector space $B \otimes H$ with the multiplication

$$(a \otimes h)(b \otimes k) := a(h_{(1)} \triangleright b) \otimes h_{(2)} k, \quad a, b \in B, \quad h, k \in H,$$

and coaction $\Delta_{B \rtimes H} := \text{id} \otimes \Delta$. Then again a cleaving map is simply given by $j(h) := 1 \otimes h$.

Definition 2 ([6, 8, 12]). Let P be a principal H -comodule algebra and J be a Hopf ideal of H such that H/J is a principal left H/J -comodule algebra. We say that an ideal I of P is a *J -reduction* of P if and only if the following conditions are satisfied:

- 1) I is an H/J -subcomodule of P ,
- 2) P/I with the induced coaction is a principal H/J -comodule algebra,
- 3) $(P/I)^{\text{co}H/J} = P^{\text{co}H}$.

Loosely speaking, J plays the role of the ideal of functions vanishing on a subgroup and I the ideal of functions vanishing on a subbundle. Thus H/J works as the algebra of the reducing subgroup, and P/I as the algebra of the reduced bundle. The coaction-invariant subalgebra $P^{\text{co}H}$ remains intact – the base space of a subbundle coincides with the base space of the bundle.

If M is a right comodule over a coalgebra C and N is a left C -comodule, then we define their *cotensor product* as

$$M \square_C N := \{t \in M \otimes N \mid (\Delta_M \otimes \text{id})(t) = (\text{id} \otimes \Delta_N)(t)\}.$$

In particular, for a principal H' -comodule algebra P and a Hopf algebra epimorphism $H \xrightarrow{\pi} H'$ making H a left H' -comodule in the obvious way, one proves that the cotensor product $P \square_{H'} H$ is a principal H -comodule algebra with the H -coaction defined by $\text{id} \otimes \Delta$. We call the principal comodule algebra $P \square_{H'} H$ the *H -prolongation* of P .

1.3 Piecewise triviality

Definition 3 (cf. [7, Definition 3.6]). A family of surjective algebra homomorphisms $\{\pi_i : P \rightarrow P_i\}_{i \in \{1, \dots, N\}}$, $N \in \mathbb{N} \setminus \{0\}$, is called a *covering* iff

- 1) $\bigcap_{i \in \{1, \dots, N\}} \text{Ker } \pi_i = 0$,
- 2) The family of ideals $(\text{Ker } \pi_i)_{i \in \{1, \dots, N\}}$ generates a distributive lattice with $+$ and \cap as meet and join respectively.

We recall now (cf. [7, Definition 3.8]) a quantum version of the notion of piecewise triviality of principal bundles (like local triviality, but with respect to closed subsets).

Definition 4. An H -comodule algebra P is called *piecewise trivial* iff there exists a family $\{\pi_i : P \rightarrow P_i\}_{i \in \{1, \dots, N\}}$, $N \in \mathbb{N} \setminus \{0\}$, of surjective H -colinear maps such that:

- 1) the restrictions $\pi_i|_{P^{\text{co}H}} : P^{\text{co}H} \rightarrow P_i^{\text{co}H}$ form a covering,
- 2) the P_i 's are smash products ($P_i \cong P_i^{\text{co}H} \rtimes H$ as H comodule algebras).

Assume also that the antipode of H is bijective. Then, as smash products are principal, it follows from [7, Theorem 3.3] that piecewise trivial comodule algebras are automatically principal. To emphasize this fact and stay in touch with the classical terminology, we frequently use the phrase ‘‘piecewise trivial principal comodule algebra’’. Note also that the consequence of principality of P is that $\{\pi_i : P \rightarrow P_i\}_{i \in \{1, \dots, N\}}$ is a covering of P (see [9]).

Definition 5 ([9]). Let $\{\pi_i : P \rightarrow P_i\}_{i \in \{1, \dots, N\}}$, $N \in \mathbb{N} \setminus \{0\}$, be a covering by right H -colinear maps of a principal right H -comodule algebra P such that the restrictions $\pi_i|_{P^{\text{co}H}} : P^{\text{co}H} \rightarrow P_i^{\text{co}H}$ also form a covering. A *piecewise trivialization* of P with respect to the covering $\{\pi_i : P \rightarrow P_i\}_{i \in \{1, \dots, N\}}$ is a family $\{j_i : H \rightarrow P_i\}_{i \in \{1, \dots, N\}}$ of right H -colinear algebra homomorphisms (cleaving maps).

It is clear that a principal comodule algebra is piecewise trivial if and only if it admits a piecewise trivialization (see the preceding section).

1.4 The Peter–Weyl comodule algebra

The Peter–Weyl comodule algebra (see [1] and references therein) extends the notion of regular functions in the C^* -algebra of a compact quantum group (linear combinations of matrix coefficients of the finite-dimensional corepresentations) to unital C^* -algebras equipped with a compact quantum group action.

Definition 6 (cf. [11]). For a unital C^* -algebra A and a compact quantum group (H, Δ) , we say that an injective unital $*$ -homomorphism $\delta : A \rightarrow A \otimes_{\min} H$ is a *coaction* if and only if

- 1) $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \Delta) \circ \delta$ (coassociativity),
- 2) $\{\delta(a)(1 \otimes h) \mid a \in A, h \in H\}^{\text{cls}} = A \otimes_{\min} H$ (counitality).

Here \otimes_{\min} denotes the spatial tensor product of C^* -algebras and $\{\cdot\}^{\text{cls}}$ stands for the closed linear span of a subset of a Banach space. We say that a compact quantum group *acts* on a unital C^* -algebra if there is a coaction in the aforementioned sense.

Next, we denote by $\mathcal{O}(H)$ the dense Hopf $*$ -subalgebra of H spanned by the matrix coefficients of finite-dimensional corepresentations. We define the *Peter–Weyl subalgebra* of A [1] as

$$\mathcal{P}_H(A) := \{a \in A \mid \delta(a) \in A \otimes \mathcal{O}(H)\}.$$

One shows that it is an $\mathcal{O}(H)$ -comodule algebra which is a dense $*$ -subalgebra of A [11, 13].

The Peter–Weyl comodule algebra of functions on a compact Hausdorff space with an action of a compact group is principal if and only if the action is free [1, 2]. In other words, the Galois condition of Hopf–Galois theory holds if and only if we have a compact principal bundle.

2 The $SU_q(2)$ -prolongation of the classical Hopf fibration

To fix the notation, let us recall definitions of the Hopf algebras $\mathcal{O}(U(1))$ and $\mathcal{O}(SU_q(2))$, and the Peter–Weyl comodule algebra $\mathcal{P}_{C(U(1))}(C(S^3))$ of functions on the classical sphere S^3 . For details on the latter algebra we refer the reader to [3].

Recall that the $*$ -algebra $\mathcal{O}(U(1))$ of polynomial functions on $U(1)$ is generated by the unitary element $u : U(1) \ni x \mapsto x \in \mathbb{C}$, and can be equivalently defined as the algebra of Laurent polynomials in u subject to the relation $u^{-1} = u^*$. The Hopf algebra structure is given by $\Delta(u) := u \otimes u$, $\varepsilon(u) := 1$ and $S(u) := u^{-1}$.

The algebra of polynomial functions on $SU_q(2)$ [14] is generated as a $*$ -algebra by α and γ satisfying relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1, \quad (1)$$

where $0 < q \leq 1$. The Hopf algebra structure comes from the matrix

$$U := \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}, \quad \text{i.e.} \quad \Delta(U_{ij}) := \sum_k U_{ik} \otimes U_{kj}, \quad S(U_{ij}) := U_{ji}^*, \quad \varepsilon(U_{ij}) := \delta_{ij}.$$

The Hopf $*$ -algebra epimorphism

$$\pi : \mathcal{O}(SU_q(2)) \longrightarrow \mathcal{O}(U(1)), \quad \pi(\alpha) := u, \quad \pi(\gamma) := 0, \quad (2)$$

makes $\mathcal{O}(SU_q(2))$ into a left and right $\mathcal{O}(U(1))$ -comodule algebra via the left and right coactions $(\pi \otimes \text{id}) \circ \Delta$ and $(\text{id} \otimes \pi) \circ \Delta$ respectively. For $q = 1$ the Hopf algebra $\mathcal{O}(SU_q(2))$ is commutative, and we denote its generators by a and c rather than α and γ .

The Peter–Weyl comodule algebra $\mathcal{P}_{C(U(1))}(C(S^3))$ is the subalgebra of $C(SU(2))$ that is the *algebraic* direct sum of the modules of continuous sections of the complex line bundles L_n , $n \in \mathbb{Z}$, associated to the Hopf fibration:

$$\mathcal{P}_{C(U(1))}(C(S^3)) = \bigoplus_{n \in \mathbb{Z}} \Gamma(L_n).$$

We have the following proper inclusions of function algebras:

$$\mathcal{O}(SU(2)) \subsetneq \mathcal{P}_{C(U(1))}(C(S^3)) \subsetneq C(S^3).$$

Next, recall that $a, c : S^3 \rightarrow \mathbb{C}$ are coordinate functions on S^3 satisfying $|a|^2 + |c|^2 = 1$. The diagonal action of $U(1)$ on S^3 yielding the Hopf fibration dualizes to the $\mathcal{O}(U(1))$ -comodule algebra structure on $\mathcal{P}_{C(U(1))}(C(S^3))$ given by $a \mapsto a \otimes u$, $c \mapsto c \otimes u$.

Now we will describe the piecewise trivial structure of $\mathcal{P}_{C(U(1))}(C(S^3))$. For brevity, we define

$$\omega := \sqrt{\frac{2}{1 + ||a|^2 - |c|^2|}}.$$

Note that ω is an element of the coaction-invariant subalgebra $\mathcal{P}_{C(U(1))}(C(S^3))^{\text{co}\mathcal{O}(U(1))} = \Gamma(L_0)$, which we identify with $C(S^2)$. Let us also define the following ideals $I_a, I_c \subseteq C(S^2)$:

$$I_a := \{f \in C(S^2) \mid f(x) = 0 \text{ for all } x \in S^2 \text{ such that } |a|^2(x) \leq 1/2\},$$

$$I_c := \{f \in C(S^2) \mid f(x) = 0 \text{ for all } x \in S^2 \text{ such that } |a|^2(x) \geq 1/2\}.$$

It is well known (cf. [3]) that the canonical surjections $C(S^2) \rightarrow C(S^2)/I_i \cong C(D)$, $i = a, c$, where D is the unit disk, form a covering, and that $(1 - \omega^2|a|^2) \in I_a$, $(1 - \omega^2|c|^2) \in I_c$. We also know [3, equation (3.4.57)] that

$$(1 - \omega^2|a|^2)(1 - \omega^2|c|^2) = 0.$$

The covering of $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))$ can now be given by the canonical surjections in terms of I_a and I_c (cf. [3]):

$$\begin{aligned}\pi_a &: \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \longrightarrow \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))/(I_a\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))), \\ \pi_c &: \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \longrightarrow \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))/(I_c\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))).\end{aligned}$$

Indeed, since $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))$ is a principal $\mathcal{O}(\mathcal{U}(1))$ -comodule algebra with the coaction-invariant subalgebra $C(S^2)$, it follows from [7, Proposition 3.4] that the maps π_i form a covering.

A trivialization associated with the above covering is given by the following cleaving maps, which are clearly algebra homomorphisms:

$$\begin{aligned}j_a &: \mathcal{O}(\mathcal{U}(1)) \longrightarrow \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))/(I_a\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))), & u^n &\longmapsto \pi_a(\omega a)^n, \\ j_c &: \mathcal{O}(\mathcal{U}(1)) \longrightarrow \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))/(I_c\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))), & u^n &\longmapsto \pi_c(\omega c)^n.\end{aligned}$$

One can argue (cf. [3]) that

$$\begin{aligned}f_a &: \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))/(I_a\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))) \xrightarrow{\cong} C(D) \otimes \mathcal{O}(\mathcal{U}(1)), \\ f_c &: \mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))/(I_c\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))) \xrightarrow{\cong} C(D) \otimes \mathcal{O}(\mathcal{U}(1)), \\ f_i &: x \longmapsto \pi_i(x_{(0)})j_i(Sx_{(1)}) \otimes x_{(2)}, \quad i = a, c.\end{aligned}\tag{3}$$

To see this, first note that $\pi_a(C(S^2)) \cong C(D) \cong \pi_c(C(S^2))$. Then, for any $n \in \mathbb{Z}$,

$$f_i(\pi_i(\Gamma(L_n))) = \pi_i(C(S^2)) \otimes u^n = C(D) \otimes u^n, \quad i = a, c,$$

whence $f_i(\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))) = C(D) \otimes \mathcal{O}(\mathcal{U}(1))$. Indeed, $f_i(\pi_i(\Gamma(L_n))) \subseteq \pi_i(C(S^2)) \otimes u^n$. On the other hand, consider an arbitrary element $y \in C(D)$. Then there exist elements $y_a, y_c \in C(S^2)$ such that $y = \pi_a(y_a) = \pi_c(y_c)$. Hence

$$y \otimes u^n = f_z(\pi_z(y_z z^n \omega^n)), \quad z = a, c.$$

Here we adopt the convention that $z^{-|n|} := (z^*)^{|n|}$. Summarizing, $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))$ is a piecewise trivial principal comodule algebra [3].

Since $\mathcal{O}(\mathrm{SU}_q(2))$ is a left principal $\mathcal{O}(\mathcal{U}(1))$ -comodule algebra, by [9, Lemma 1.13] the cotensor product $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \square_{\mathcal{O}(\mathcal{U}(1))} \mathcal{O}(\mathrm{SU}_q(2))$ is a piecewise trivial principal comodule algebra. Explicitly, the covering and trivializations inherited from $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))$ make it piecewise trivial via the formulas:

$$\hat{\pi}_i := \pi_i \otimes \mathrm{id}, \quad \hat{j}_i := (j_i \circ \pi \otimes \mathrm{id}) \circ \Delta_{\mathcal{O}(\mathrm{SU}_q(2))}, \quad i = a, c.$$

Using these formulas and the isomorphisms (3), one can check that the trivializable pieces of the comodule algebra $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \square_{\mathcal{O}(\mathcal{U}(1))} \mathcal{O}(\mathrm{SU}_q(2))$ are isomorphic to $C(D) \otimes \mathcal{O}(\mathrm{SU}_q(2))$ (cf. [9, equation (1.8)]).

Furthermore, as the comodule algebra $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \square_{\mathcal{O}(\mathcal{U}(1))} \mathcal{O}(\mathrm{SU}_q(2))$ is a cotensor product, combining [9, Lemma 1.13] with [9, Theorem 1.5] yields that $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3))$ is a piecewise trivial $(\mathrm{Ker} \pi)$ -reduction (see (2)) of $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \square_{\mathcal{O}(\mathcal{U}(1))} \mathcal{O}(\mathrm{SU}_q(2))$.

Theorem 1 (main result). *The comodule algebra $\mathcal{P}_{C(\mathcal{U}(1))}(C(S^3)) \square_{\mathcal{O}(\mathcal{U}(1))} \mathcal{O}(\mathrm{SU}_q(2))$ is not isomorphic to any smash product $C(S^2) \rtimes \mathcal{O}(\mathrm{SU}_q(2))$ comodule algebra.*

Proof. Suppose that there exists a cleaving map

$$\mathcal{O}(\mathrm{SU}_q(2)) \longrightarrow \mathcal{P}_{C(U(1))}(C(S^3)) \square_{\mathcal{O}(U(1))} \mathcal{O}(\mathrm{SU}_q(2))$$

that is an algebra homomorphism. It is tantamount to the existence of a $U(1)$ -equivariant algebra homomorphism $f: \mathcal{O}(\mathrm{SU}_q(2)) \rightarrow \mathcal{P}_{C(U(1))}(C(S^3))$ [5, Proposition 4.1]. Let α and γ denote generators of $\mathcal{O}(\mathrm{SU}_q(2))$, and a, c their classical counterparts. Since $f([\alpha, \alpha^*]) = 0$, it follows from (1) that $f(\gamma) = 0$ and $f(\alpha)f(\alpha)^* = 1$.

On the other hand, by the $U(1)$ -equivariance, $f(\alpha) = f_1 a + f_2 c$, for some $f_1, f_2 \in C(S^2)$. Furthermore, any continuous section of the Hopf line bundle L_1 can be written as $g_1 a + g_2 c$ for some $g_1, g_2 \in C(S^2)$. We can rewrite it as $(g_1 a + g_2 c)f(\alpha)^* f(\alpha)$. Since $(g_1 a + g_2 c)f(\alpha)^* \in C(S^2)$, we conclude that $f(\alpha)$ spans $\Gamma(L_1)$ as a left $C(S^2)$ -module. Also, if $g f(\alpha) = 0$ for some $g \in C(S^2)$, then $g = g f(\alpha) f(\alpha)^* = 0$. Hence $f(\alpha)$ is a basis of $\Gamma(L_1)$ contradicting its nonfreeness. ■

Acknowledgements

The authors are grateful to Tomasz Brzeziński for discussions, and to the referees for careful proofreading of the manuscript. This work was partially supported by the NCN-grant 2011/01/B/ST1/06474.

References

- [1] Baum P.F., De Commer K., Hajac P.M., Free actions of compact quantum groups of unital C^* -algebras, [arXiv:1304.2812](#).
- [2] Baum P.F., Hajac P.M., Local proof of algebraic characterization of free actions, [arXiv:1402.3024](#).
- [3] Baum P.F., Hajac P.M., Matthes R., Szymański W., Noncommutative geometry approach to principal and associated bundles, in Quantum Symmetry in Noncommutative Geometry, to appear, [math.DG/0701033](#).
- [4] Brzeziński T., Hajac P.M., The Chern–Galois character, *C. R. Math. Acad. Sci. Paris* **338** (2004), 113–116, [math.KT/0306436](#).
- [5] Brzeziński T., Zieliński B., Quantum principal bundles over quantum real projective spaces, *J. Geom. Phys.* **62** (2012), 1097–1107, [arXiv:1105.5897](#).
- [6] Günther R., Crossed products for pointed Hopf algebras, *Comm. Algebra* **27** (1999), 4389–4410.
- [7] Hajac P.M., Krähmer U., Matthes R., Zieliński B., Piecewise principal comodule algebras, *J. Noncommut. Geom.* **5** (2011), 591–614, [arXiv:0707.1344](#).
- [8] Hajac P.M., Matthes R., Sołtan P.M., Szymański W., Zieliński B., Hopf–Galois extensions and C^* algebras, in Quantum Symmetry in Noncommutative Geometry, to appear.
- [9] Hajac P.M., Rudnik J., Zieliński B., Reductions of piecewise trivial comodule algebras, [arXiv:1101.0201](#).
- [10] Kobayashi S., Nomizu K., Foundations of differential geometry. Vol. I, Interscience Publishers, New York – London, 1963.
- [11] Podleś P., Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups, *Comm. Math. Phys.* **170** (1995), 1–20, [hep-th/9402069](#).
- [12] Schauenburg P., Galois objects over generalized Drinfeld doubles, with an application to $u_q(\mathfrak{sl}_2)$, *J. Algebra* **217** (1999), 584–598.
- [13] Sołtan P.M., On actions of compact quantum groups, *Illinois J. Math.* **55** (2011), 953–962, [arXiv:1003.5526](#).
- [14] Woronowicz S.L., Twisted $SU(2)$ group. An example of a non-commutative differential calculus, *Publ. Res. Inst. Math. Sci.* **23** (1987), 117–181.