Symmetry, Integrability and Geometry: Methods and Applications SIGMA **10** (2014), 050, 24 pages

Geometric Aspects of the Painlevé Equations PIII(D_6) and PIII(D_7)

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Received October 15, 2013, in final form April 10, 2014; Published online April 23, 2014
http://dx.doi.org/10.3842/SIGMA.2014.050

Abstract. The Riemann–Hilbert approach for the equations PIII(D_6) and PIII(D_7) is studied in detail, involving moduli spaces for connections and monodromy data, Okamoto–Painlevé varieties, the Painlevé property, special solutions and explicit Bäcklund transformations.

Key words: moduli space for linear connections; irregular singularities; Stokes matrices; monodromy spaces; isomonodromic deformations; Painlevé equations

2010 Mathematics Subject Classification: 14D20; 14D22; 34M55

1 Introduction

The aim of this paper is a study of the Painlevé equations PIII(D_6) and PIII(D_7) by means of isomonodromic families in a moduli space of connections of rank two on the projective line. This contrasts the work of Okamoto et al. on these third Painlevé equations, where the Hamiltonians are the main tool. However, there is a close relation between the two points of view, since the moduli spaces turn out to be the Okamoto–Painlevé varieties. We refer only to a few items of the extensive literature on Okamoto–Painlevé varieties. More details on Stokes matrices and the analytic classification of singularities can be found in [11].

A *rough sketch* of this Riemann–Hilbert method is as follows (see for some background [10] and see for details concerning PI, PII, PIV [8, 9, 12]). The starting point is a family $\mathbf{S}$ of differential modules M of dimension 2 over $\mathbb{C}(z)$ with prescribed singularities at fixed points of $\mathbb{P}^1$. The type of singularities gives rise to a monodromy set $\mathcal{R}$ built out of ordinary monodromy, Stokes matrices and 'links'. The map $\mathbf{S} \to \mathcal{R}$ associates to each module $M \in \mathbf{S}$ its monodromy data in $\mathcal{R}$. The fibers of $\mathbf{S} \to \mathcal{R}$ are parametrized by $T \cong \mathbb{C}^*$ and there results a bijection $\mathbf{S} \to \mathcal{R} \times T$. The set $\mathbf{S}$ has a priori no structure of algebraic variety. A moduli space $\mathcal{M}$ over $\mathbb{C}$, whose set of closed points consists of certain connections of rank two on the projective line, is constructed such that $\mathbf{S}$ coincides with $\mathcal{M}(\mathbb{C})$. There results an analytic Riemann–Hilbert morphism $\mathrm{RH} : \mathcal{M} \to \mathcal{R}$. The fibers of RH are the isomonodromic families which give rise to solutions of the corresponding Painlevé equation. The extended Riemann–Hilbert morphism $\mathrm{RH}^+ : \mathcal{M} \to \mathcal{R} \times T$ is an analytic isomorphism. From these constructions the Painlevé property for the corresponding Painlevé equation follows and the moduli space $\mathcal{M}$ is identified with an Okamoto–Painlevé space. Special properties of solutions of the Painlevé equations, such as special solutions, Bäcklund transformations etc., are derived from special points of $\mathcal{R}$ and the natural automorphisms of $\mathbf{S}$.

The above sketch needs many subtle refinements. One has to construct a geometric monodromy space $\tilde{\mathcal{R}}$ (depending on the parameters of the Painlevé equation) which is a geometric quotient of the monodromy data. The 'link' involves (multi)summation and in order to avoid the singular directions one has to replace T by its universal covering $\tilde{T} \cong \mathbb{C}$. In the construction of $\mathcal{M}$, one represents a differential module $M \in \mathbf{S}$ by a connection on a fixed vector bundle of rank two on $\mathbb{P}^1$. This works well for the case $(1,-,1/2)$, described in Section 2. In case $(1,-,1)$, described in Section 3, this excludes a certain set of reducible modules $M \in \mathbf{S}$. The moduli space of connections $\mathcal{M}$ is replaced by the topological covering $\tilde{\mathcal{M}} = \mathcal{M} \times_T \tilde{T}$. Now the main result is that the extended Riemann–Hilbert morphism $\mathrm{RH}^+ : \tilde{\mathcal{M}} \to \tilde{\mathcal{R}} \times \tilde{T}$ is a well defined analytic isomorphism.

Each family of linear differential modules and each Painlevé equation has its own story. For the family $(1,-,1/2)$, corresponding to PIII(D_7), the computations of $\mathcal{R}$, $\mathcal{M}$ and the Bäcklund transformations present no problems. For the resonant case (i.e., $\alpha = \pm 1$ and $\theta \in \mathbb{Z}$, see Sections 2.1, 2.2 and 2.5 for notation and the statement) there are algebraic solutions of PIII(D_7). The spaces $\mathcal{R}(\alpha)$ with $\alpha = \pm i$ have a special point corresponding to trivial Stokes matrices. This leads to a special solution q for PIII(D_7) and $\theta \in \frac{1}{2}+\mathbb{Z}$, which is transcendental according to [5]. According to [2], q is a univalent function of t and is a meromorphic at $t = 0$.

For the general case (i.e., $\alpha \neq \beta^{\pm 1}$) of the family $(1,-,1)$, corresponding to PIII(D_6), the computations of $\mathcal{R}$, $\mathcal{M}$ present no problems. The formulas for the Bäcklund transformations, derived from the automorphisms of $\mathbf{S}$, have denominators. These originate from the complicated cases $\alpha = \beta^{\pm 1}$ and/or $\alpha = \pm 1$ where reducible connections and/or resonance occur. Isomonodromy for reducible connections produces Riccati solutions and resonance is related to algebraic solutions.

2 The family $(1,-,1/2)$ and PIII(D_7)

In this section the set $\mathbf{S}$ consists of the (equivalence classes of the) pairs (M,t) of type $(1,-,1/2)$ (see [10] for the terminology), corresponding to the Painlevé equation PIII(D_7). The differential module M is given by $\dim M = 2$, the second exterior power of M is trivial, M has two singular points 0 and ∞. The Katz invariant $r(0) = 1$ and the generalized eigenvalues at 0 are normalized to $\pm\frac{t}{2}z^{-1}$ with $t \in T = \mathbb{C}^*$. The singular point ∞ has Katz invariant $r(\infty) = 1/2$ and generalized eigenvalues $\pm z^{1/2}$. Further (M,t) is equivalent to (M',t') if M is isomorphic to M' and $t = t'$.

The Riemann–Hilbert approach to PIII(D_7) in this section differs from [10] in several ways. The choice $(1,-,1/2)$ is made to obtain the classical formula for the Painlevé equation. Further we consider pairs (M,t) rather than modules M. This is needed in order to distinguish the two generalized eigenvalues at $z = 0$ and to obtain a good monodromy space $\mathcal{R}$. Finally, the definitions of the topological monodromy and the 'link' need special attention.

2.1 The construction of the monodromy space $\mathcal{R} \to \mathcal{P}$

This is rather subtle and we provide here the details. Given is some $(M,t) \in \mathbf{S}$ and we write δ_M for the differential operator on M. First we fix an isomorphism $\phi : \Lambda^2 M \to (\mathbb{C}(z), z\frac{d}{dz})$.

1. $\mathbb{C}((z)) \otimes M$ has a basis F_1, F_2 such that the operator δ_M has the matrix $\begin{pmatrix} -\omega & 0 \\ 0 & \omega \end{pmatrix}$ with $\omega = \frac{tz^{-1}+\theta}{2}$. We require that $\phi(F_1 \wedge F_2) = 1$. Then F_1, F_2 is unique up to a transformation $(F_1,F_2) \mapsto (\lambda F_1, \lambda^{-1}F_2)$. The solution space $V(0)$ at $z = 0$ has basis $f_1 = e^{-\frac{t}{2}z^{-1}}z^{\theta/2}F_1, f_2 = e^{\frac{t}{2}z^{-1}}z^{-\theta/2}F_2$. The formal monodromy and the two Stokes matrices have on the basis f_1, f_2 the matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}$, where $\alpha = e^{\pi i\theta}$. Their product (in this order) is the topological monodromy top_0 at $z = 0$.

2. $\mathbb{C}((z^{-1})) \otimes M$ has a basis E_1, E_2 such that the operator δ_M has the matrix $\begin{pmatrix} \frac{1}{4} & 1 \\ z & \frac{-1}{4} \end{pmatrix}$ with respect to this basis. Since this differential operator is irreducible, the basis E_1, E_2 is unique

up to a transformation $(E_1, E_2) \mapsto (\mu E_1, \mu E_2)$ with $\mu \in \mathbb{C}^*$. We require that $\phi(E_1 \wedge E_2) = 1$ and then $\mu \in \{1, -1\}$. The solution space $V(\infty)$ at $z = \infty$ has basis

$$e_1 = \frac{1}{\sqrt{-2}} z^{-1/4} e^{2z^{1/2}} \left(E_1 + \left(-\frac{1}{2} + z^{1/2} \right) E_2 \right),$$
$$e_2 = \frac{1}{\sqrt{-2}} z^{-1/4} e^{-2z^{1/2}} \left(E_1 + \left(-\frac{1}{2} - z^{1/2} \right) E_2 \right).$$

The formal monodromy and the Stokes matrix have on the basis e_1, e_2 the matrices $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$. Their product (in this order) is the topological monodromy top_∞ at $z = \infty$.

3. The link $L : V(0) \to V(\infty)$ is a linear map obtained from multisummation at $z = 0$, analytic continuation along a path from 0 to ∞ and the inverse of multisummation at $z = \infty$. The matrix $\begin{pmatrix} \ell_1 & \ell_2 \\ \ell_3 & \ell_4 \end{pmatrix}$ of L with respect to the bases e_1, e_2 and f_1, f_2 has determinant 1, due to the isomorphism $\phi : \Lambda^2 M \to (\mathbb{C}(z), z\frac{d}{dz})$ and the careful choices of the bases. The relation $\begin{pmatrix} \alpha & \alpha c_2 \\ \frac{c_1}{\alpha} & \frac{1+c_1 c_2}{\alpha} \end{pmatrix} = \mathrm{top}_0 = L^{-1} \circ \mathrm{top}_\infty \circ L$ yields $\alpha = -i\ell_{14} e + i\ell_{12} - i\ell_{34} \neq 0$ where $\ell_{ij} := \ell_i \ell_j$. One observes that α, c_1, c_2 are determined by L and top_∞. Thus the affine space, given by the above data and relations, has coordinate ring

$$\mathbb{C}\left[e, \ell_1, \ell_2, \ell_3, \ell_4, \frac{1}{-\ell_{14} e + \ell_{12} - \ell_{34}} \right] / (\ell_{14} - \ell_{23} - 1).$$

4. The group G generated by the base changes $(e_1, e_2) \mapsto (-e_1, -e_2)$ and $(f_1, f_2) \mapsto (\lambda f_1, \lambda^{-1} f_2)$, acts on this affine space. The monodromy space $\mathcal{R}$ is the categorical quotient by G and has coordinate ring

$$\mathbb{C}\left[e, \ell_{12}, \ell_{14}, \ell_{23}, \ell_{34}, \frac{1}{-\ell_{14} e + \ell_{12} - \ell_{34}} \right] / (\ell_{14} - \ell_{23} - 1, \ell_{12}\ell_{34} - \ell_{14}\ell_{23}).$$

This is in fact a *geometric quotient*. The morphism $\mathcal{R} \to \mathcal{P} = \mathbb{C}^*$ is given by $(e, \ell_{12}, \ell_{14}, \ell_{23}, \ell_{34}) \mapsto \alpha := -i\ell_{14} e + i\ell_{12} - i\ell_{34} \neq 0$. For a suitable linear change of the variables, the fibers $\mathcal{R}(\alpha)$ of $\mathcal{R} \to \mathcal{P}$ are nonsingular, affine cubic surfaces with three lines at infinity, given by the equation $x_1 x_2 x_3 + x_1^2 + x_2^2 + \alpha x_1 + x_2 = 0$. A detailed calculation resulting in this equation is presented in [10, Section 3.5].

Lemma 2.1. *The space $\mathcal{R}(\alpha)$ is simply connected.*

Proof. We remove from $\mathcal{R}(\alpha)$ the line $L := \{(0, 0, x_3) \,|\, x_3 \in \mathbb{C}\}$ and project onto $\mathbb{C}^2 \setminus S$ by $(x_1, x_2, x_3) \mapsto (x_1, x_2)$. Here S is the union of $\{(0, x_2) \,|\, x_2 \neq -1\}$ and $\{(x_1, 0) \,|\, x_1 \neq -\alpha\}$. If $x_1 x_2 \neq 0$, then the fiber is one point. If $x_1 x_2 = 0$, then the fiber is an affine line. Since $\mathbb{C}^2 \setminus S$ is simply connected, $\mathcal{R}(\alpha) \setminus L$ is simply connected. Then $\mathcal{R}(\alpha)$ is simply connected, too. ■

Remark on the differential Galois group. The differential Galois group of a module M, with $(M, t) \in \mathbf{S}$, can be considered as algebraic subgroup of $\mathrm{GL}(V(0))$. It is the smallest algebraic subgroup containing the local differential Galois group $G_0 \subset \mathrm{GL}(V(0))$ at $z = 0$ and $L^{-1} G_\infty L$, where $G_\infty \subset \mathrm{GL}(V(\infty))$ is the local differential Galois group at $z = \infty$. Now G_0 is generated (as algebraic group) by the exponential torus $\left\{ \begin{pmatrix} s_1 & 0 \\ 0 & s_1^{-1} \end{pmatrix} \middle| s_1 \in \mathbb{C}^* \right\}$, the formal monodromy $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ and the Stokes maps $\begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix}$. The group G_∞ is (as algebraic group) generated by the exponential torus $\left\{\begin{pmatrix} s_2 & 0 \\ 0 & s_2^{-1}\end{pmatrix}\middle| s_2\in\mathbb{C}^*\right\}$, the formal monodromy $\begin{pmatrix} 0 & -i \\ -i & 0\end{pmatrix}$ and the Stokes map $\begin{pmatrix} 1 & 0 \\ e & 1\end{pmatrix}$. This easily implies that the differential Galois group is $\mathrm{SL}(2,\mathbb{C})$. In particular, M is irreducible and the same holds for the differential module $\mathbb{C}(\sqrt[m]{z})\otimes M$ over $\mathbb{C}(\sqrt[m]{z})$ for any $m\geq 2$.

The construction needed to define the topological monodromy and the link.

For the definition of the link and the topological monodromies we have to choose nonsingular directions for the two multisummations and a path from 0 to ∞. At $z=\infty$ the singular direction does not depend on $t\in T$ and we can take a fixed nonsingular direction. However, at $z=0$, the singular directions for $t\in T=\mathbb{C}^*$, $t=|t|e^{i\phi}$ are ϕ and $\pi+\phi$ and they vary with t. Thus we cannot use a fixed path from 0 to ∞. In order to obtain a globally defined map $L:V(0)\to V(\infty)$ we replace $T=\mathbb{C}^*$ by its universal covering $\tilde{T}=\mathbb{C}\to T$, $\tilde{t}\mapsto e^{\tilde{t}}$. The elements of $\tilde{T}\cong\mathbb{R}_{>0}\times\mathbb{R}$ are written as $\tilde{t}=|t|e^{i\phi}$. Consider the path $\tilde{z}=re^{id(r)}$, $0<r<\infty$, with $d(r)=(\phi-\frac{\pi}{2})\frac{1}{1+r}+\frac{\pi}{2}\frac{r}{1+r}$ on the universal covering of $\mathbb{P}^1\setminus\{0,\infty\}$. Now L is defined by summation at $z=0$ in the direction $\phi-\frac{\pi}{2}$, followed by analytic continuation along the above path and finally the inverse of the summation at $z=\infty$ in the direction $\frac{\pi}{2}$.

Write $\tilde{\mathbf{S}}=\mathbf{S}\times_T\tilde{T}$. The elements of $\tilde{\mathbf{S}}$ are the pairs $(M,\tilde{t})$ with $(M,e^{\tilde{t}})\in\mathbf{S}$. For the elements in $\tilde{\mathbf{S}}$ the link and the monodromy at $z=0$ are defined as above. Since $\mathcal{R}$ is a geometric quotient, [10, Theorem 1.9] implies:

The above map $\tilde{\mathbf{S}}\to\mathcal{R}\times\tilde{T}$ is bijective.

Fix $\alpha\in\mathcal{P}=\mathbb{C}^*$. Let $\mathbf{S}(\alpha)$, $\tilde{\mathbf{S}}(\alpha)$ be the subsets of $\mathbf{S}$ and $\tilde{\mathbf{S}}$, consisting of the pairs (M,t) and $(M,\tilde{t})$ which have $\begin{pmatrix}\alpha & 0 \\ 0 & \frac{1}{\alpha}\end{pmatrix}$ as formal monodromy at $z=0$.

The map $\tilde{\mathbf{S}}(\alpha)\to\mathcal{R}(\alpha)\times\tilde{T}$ is bijective, since $\tilde{\mathbf{S}}\to\mathcal{R}\times\tilde{T}$ is bijective.

2.2 The construction of the moduli space $\mathcal{M}(\theta)$

Fix θ with $e^{\pi i\theta}=\alpha$. The moduli space $\mathcal{M}(\theta)$ is obtained by replacing each $(M,t)\in\mathbf{S}(\alpha)$ by a certain connection $(\mathcal{V},\nabla)$ on $\mathbb{P}^1$. This connection is uniquely determined by the data: Its generic fiber is M; $\nabla_{z\frac{d}{dz}}$ is formally equivalent at $z=0$ to $z\frac{d}{dz}+\begin{pmatrix}\omega & 0 \\ 0 & -\omega\end{pmatrix}$ with $\omega=\frac{tz^{-1}+\theta}{2}$ and is formally equivalent at $z=\infty$ to $z\frac{d}{dz}+\begin{pmatrix}-\frac{3}{4} & 1 \\ z & -\frac{1}{4}\end{pmatrix}$. It follows that $\Lambda^2(\mathcal{V},\nabla)$ is isomorphic to $(O(-1),d)$. Since $(\mathcal{V},\nabla)$ is irreducible one has that $\mathcal{V}\cong O\oplus O(-1)$ and the vector bundle $\mathcal{V}$ is identified with $Oe_1+O(-[\infty])e_2$.

Then $D:=\nabla_{z\frac{d}{dz}}:\mathcal{V}\to O([0]+[\infty])\otimes\mathcal{V}$ has with respect to e_1, e_2 the matrix $\begin{pmatrix}a & b \\ c & -a\end{pmatrix}$ with $a=a_{-1}z^{-1}+a_0+a_1z$, $c=c_{-1}z^{-1}+c_0$ and $b=b_{-1}z^{-1}+b_0+b_1z+b_2z^2$.

The condition at $z=0$ is $a^2+bc\in\left(\frac{tz^{-1}+\theta}{2}\right)^2+\mathbb{C}[[z]]$, equivalently

$$a_{-1}^2+b_{-1}c_{-1}=\frac{t^2}{4},\qquad 2a_{-1}a_0+b_{-1}c_0+b_0c_{-1}=\frac{t\theta}{2}.$$

The condition at $z=\infty$ is $a^2+a+bc=z+\mathbb{C}[[z^{-1}]]$, equivalently

$$a_1^2+b_2c_0=0,\qquad 2a_1a_0+a_1+b_2c_{-1}+b_1c_0=1.$$

The space, given by the above variables and relations has to be divided by the action of the group $\{e_1\mapsto e_1,\ e_2\mapsto\lambda e_2+(x_0+x_1z)e_1\}$ (with $\lambda\in\mathbb{C}^*$, $x_0,x_1\in\mathbb{C}$) of automorphisms of the vector bundle. Using the standard forms below one sees that this is a good geometric quotient.

A *standard form* for $c_{-1} \neq 0$ is $z\frac{d}{dz} + \begin{pmatrix} a_1 z & b \\ z^{-1} + c_0 & -a_1 z \end{pmatrix}$ with $b = b_{-1}z^{-1} + \cdots + b_2 z^2$ and equations

$$a_1^2 + b_2 c_0 = 0, \qquad a_1 + b_2 + b_1 c_0 = 1, \qquad b_{-1} = \frac{t^2}{4}, \qquad \frac{t^2}{4} c_0 + b_0 = \frac{t\theta}{2}.$$

A *standard form* for $c_0 \neq 0$ is $z\frac{d}{dz} + \begin{pmatrix} a_{-1}z^{-1} & b \\ c_{-1}z^{-1} + 1 & -a_{-1}z^{-1} \end{pmatrix}$ with equations

$$b_2 = 0, \qquad b_1 = 1, \qquad a_{-1}^2 + b_{-1}c_{-1} = \frac{t^2}{4}, \qquad b_{-1} + b_0 c_{-1} = \frac{t\theta}{2}.$$

By gluing the two standard forms, one obtains the nonsingular moduli space $\mathcal{M}(\theta)$. The map $\mathcal{M}(\theta) \to \mathbf{S}(\alpha)$, where $\alpha = e^{i\theta}$, is a bijection.

Observation. After scaling some variables one sees that $\mathcal{M}(\theta)$ is the union of two open affine spaces $U_1 \times T$ and $U_2 \times T$, where U_1 is given by the variables a_1, b_1, c_0 and the relation $a_1^2 + (1 - a_1 - b_1 c_0)c_0 = 0$, and U_2 is given by the variables a_{-1}, b_0, c_{-1} and the relation $a_{-1}^2 + (\frac{\theta}{2} - b_0 c_{-1})c_{-1} - \frac{1}{4} = 0$.

Let $U_{12} \subset U_1$ be defined by $c_0 \neq 0$ and $U_{21} \subset U_2$ by $c_{-1} \neq 0$. The gluing of $U_1 \times T$ and $U_2 \times T$ is defined by the isomorphism $U_{12} \times T \to U_{21} \times T$ obtained by suitable base changes in the group $\{e_1 \mapsto e_1,\ e_2 \mapsto \lambda e_2 + (x_0 + x_1 z)e_1\}$.

Using the two projections $U_1 \to \mathbb{C}$, $(a_1, b_1, c_0) \mapsto c_0$ and $U_2 \to \mathbb{C}$, $(a_{-1}, b_0, c_{-1}) \mapsto c_{-1}$ one finds that U_1 and U_2 are simply connected.

Define $M(\theta) := f^{-1}(1)$, where $f : \mathcal{M}(\theta) \to T$ is the canonical morphism. The space $M(\theta)$ is simply connected since it is the union of the two simply connected spaces U_1, U_2.

The universal covering of $\mathcal{M}(\theta)$ is $\tilde{\mathcal{M}}(\theta) = \mathcal{M}(\theta) \times_T \tilde{T}$. Indeed, it is the union of the two simply connected spaces $U_1 \times \tilde{T}$ and $U_2 \times \tilde{T}$. Using the explicitly defined link L one obtains a globally defined analytic morphism $\tilde{\mathcal{M}}(\theta) \to \mathcal{R}(\alpha) \times \tilde{T}$ which is bijective (and thus an analytic isomorphism, see [3]). Indeed, $\mathcal{M}(\theta) \to \mathbf{S}(\alpha)$ and $\tilde{\mathbf{S}}(\alpha) \to \mathcal{R}(\alpha) \times \tilde{T}$ are bijections.

Theorem 2.2. *Let $\theta \in \mathbb{C}$, $\alpha = e^{\pi i\theta}$. The extended Riemann–Hilbert map $\tilde{\mathcal{M}}(\theta) \to \mathcal{R}(\alpha) \times \tilde{T}$, with $\tilde{T} \cong \mathbb{C}$, is a well defined analytic isomorphism.*

Comment. The existence of an analytic isomorphism as in Theorem 2.2 is called the "geometric Painlevé property" in [1]. They prove this property for a number of Painlevé equations under a restriction on the parameters (loc. cit., Theorem 6.3). We prove it here for PIII(D_7) and in Sections 3.3.2 and 3.4.4 below for PIII(D_6) without any restriction.

2.3 Isomonodromy and the Okamoto-Painlevé space

The calculation is done on the 'chart' $c_0 \neq 0$ and q is supposed to be invertible. The data for the operator $z\frac{d}{dz} + A$ are

$$c_{-1} = -q, \qquad b_{-1} = q^{-1}\left(a_{-1}^2 - \frac{t^2}{4}\right), \qquad b_0 = -\frac{t\theta}{2}q^{-1} + q^{-2}\left(a_{-1}^2 - \frac{t^2}{4}\right),$$
$$b_1 = 1, \qquad b_2 = 0.$$

Now q and a_{-1} are functions of t and the family is isomonodromic if there is an operator $\frac{d}{dt} + B$ commuting with $z\frac{d}{dz} + A$. Equivalently $A' = \overset{\circ}{B} + [A, B]$, where $'$ denotes $\frac{d}{dt}$ and $\circ$ denotes $z\frac{d}{dz}$.

The Lie algebra $\mathfrak{sl}_2$ has standard basis $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. One writes $A = a_{-1}z^{-1}H + bE_1 + (-qz^{-1} + 1)E_2$ and $B = B_H H + B_1 E_1 + B_2 E_2$ with $B_* = \sum_{i=-1}^{2} B_{*,i} z^i$

for $* = H, 1, 2$ and $B_{*,i}$ only depending on t. Using the Lie algebra structure one obtains the equations:

$$a'_{-1}z^{-1} = \overset{\circ}{B}_H + B_2(z + b_0 + b_{-1}z^{-1}) - B_1(-qz^{-1} + 1), \qquad (H)$$

$$b'_0 + b'_{-1}z^{-1} = \overset{\circ}{B}_1 + 2B_1 a_{-1}z^{-1} - 2B_H(z + b_0 + b_1 z^{-1}), \qquad (E_1)$$

$$-q'z^{-1} = \overset{\circ}{B}_2 - 2B_2 a_{-1}z^{-1} + 2B_H(-qz^{-1} + 1). \qquad (E_2)$$

By Maple one obtains the system

$$q' = \frac{q + 2a_{-1}}{t}, \qquad a'_{-1} = \frac{-t^2 - \theta t q + 4a_{-1}^2 + 2qa_{-1} + 2q^3}{2tq}$$

and finally

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} - \frac{\theta}{t} + \frac{2q^2}{t^2} - \frac{1}{q}.$$

We note that the change $q = -Q$, $t = -T$ brings this equation in the form

$$\overset{**}{Q} = \frac{(\overset{*}{Q})^2}{Q} - \frac{\overset{*}{Q}}{T} - \frac{\theta}{T} - \frac{2Q^2}{T^2} - \frac{1}{Q} \qquad \text{with notation} \quad \overset{*}{-} = \frac{d-}{dT}.$$

This is the standard form for $\mathrm{PIII}'(\mathrm{D}_7)$ (see [6]). As in [8, 9, 12] one obtains:

Theorem 2.3. *The equation* $\mathrm{PIII}(\mathrm{D}_7)$ *has the Painlevé property. The analytic fibration* $\tilde{t} : \tilde{\mathcal{M}}(\theta) \to \tilde{T} = \mathbb{C}$ *with its foliation* $\{\mathrm{RH}^{-1}(r) \,|\, r \in \mathcal{R}(\alpha)\}$, *where* $\mathrm{RH} : \tilde{\mathcal{M}}(\theta) \to \mathcal{R}(\alpha)$ *is the Riemann–Hilbert map, is isomorphic to the Okamoto–Painlevé space for the equation* $\mathrm{PIII}(\mathrm{D}_7)$ *with parameter* θ.

Moreover, $M(\theta) \cong \mathcal{R}(\alpha)$ *is the space of initial values.*

2.4 Automorphisms of S and Bäcklund transforms

The automorphism s_1 of $\mathbf{S}$ is defined by $s_1(M, t) = (M, -t)$. The induced action on $\mathcal{R}$ leaves all data invariant except for interchanging the basis vectors f_1, f_2 of $V(0)$. As a consequence α is mapped to α^{-1}.

The automorphism s_2 of $\mathbf{S}$ is defined by $s_2(M, t) = (N \otimes M, -t)$. Here $N = \mathbb{C}(z)b$ is the differential module given by $\delta(b) = \frac{1}{2}b$. Starting with a local presentation $z\frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ with $\omega = \frac{tz^{-1}+\theta}{2}$ of M at $z = 0$ one obtains, after conjugation with $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$, the local presentation $z\frac{d}{dz} + \begin{pmatrix} -\tau & 0 \\ 0 & \tau \end{pmatrix}$, with $\tau = \frac{tz^{-1}-\theta+1}{2}$, of $N \otimes M$ at $z = 0$.

Starting with a local presentation $D := z\frac{d}{dz} + \begin{pmatrix} \frac{1}{4} & 1 \\ z & -\frac{1}{4} \end{pmatrix}$ of M at $z = \infty$ (say on the basis E_1, E_2, described in Section 2.1, part 2), one obtains the local presentation $z\frac{d}{dz} + \begin{pmatrix} \frac{3}{4} & 1 \\ z & \frac{1}{4} \end{pmatrix}$ of $N \otimes M$ at $z = \infty$. This is the matrix of D with respect to the basis zE_2, E_1. The induced action of s_2 on $\mathcal{R}$ maps α to $-\alpha^{-1}$, the formal monodromy at ∞ is multiplied by -1 and the Stokes data are essentially unchanged.

The group of automorphisms of $\mathbf{S}$, generated by s_1, s_2, has order 4. The Bäcklund transformations are the lifts of the elements of this group to isomorphisms (preserving isomonodromy) between various moduli spaces $\tilde{\mathcal{M}}(\theta)$.

$s_1^+ : \tilde{\mathcal{M}}(\theta) \to \tilde{\mathcal{M}}(-\theta)$ is the obvious lift of s_1, given by $\tilde{t} \mapsto \tilde{t} + \pi i$, $\theta \mapsto -\theta$. Further any solution $q(\tilde{t})$ of PIII(D$_7$) for the parameter θ is mapped to the solution $q(\tilde{t}+\pi i)$ for the parameter $-\theta$.

$s_2^+ : \tilde{\mathcal{M}}(\theta) \to \tilde{\mathcal{M}}(1-\theta)$ is the obvious lift of s_2 with $\tilde{t} \mapsto \tilde{t} + \pi i$. The formula for s_2^+ is not obvious and its computation is given below.

Put $B := (s_1^+)^2 = (s_2^+)^2$. Then B is the automorphism of $\tilde{\mathcal{M}(\theta)} = M(\theta) \times \tilde{T}$, which is the identity on $M(\theta)$ and $B : \tilde{t} \mapsto \tilde{t} + 2\pi i$.

The group $\langle s_1^+, s_2^+ \rangle$ generated by s_1^+, s_2^+ (for their action on θ, $\tilde{t}$) has $\langle B \rangle \cong \mathbb{Z}$ as normal subgroup and $\langle s_1^+, s_2^+ \rangle / \langle B \rangle$ is the affine Weyl group of type A_1.

Computation of the Bäcklund transformation s_2^+. A point $\xi \in \mathcal{M}(\theta)$, lying in the affine open subset defined by $c_0 \neq 0$ and $c_1 \neq 0$, is represented by the operator in standard form $z\frac{d}{dz} + \begin{pmatrix} az^{-1} & b \\ 1 - qz^{-1} & -az^{-1} \end{pmatrix}$, where $b = z - \frac{t\theta}{2q} + \frac{a^2 - \frac{t^2}{4}}{q^2} + \frac{a^2 - \frac{t^2}{4}}{q} z^{-1}$. The map s_2^+ changes this operator into $z\frac{d}{dz} + A$, where A is obtained from the above matrix by $t \mapsto -t$ and adding $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. The point $s_2^+(\xi) \in \mathcal{M}(1-\theta)$ is supposed to be represented by the operator $z\frac{d}{dz} + \tilde{A}$, where $\tilde{A} = \begin{pmatrix} \tilde{a}z^{-1} & \tilde{b} \\ 1 - \tilde{q}z^{-1} & -\tilde{a}z^{-1} \end{pmatrix}$ with $\tilde{b} = z - \frac{t(1-\theta)}{2\tilde{q}} + \frac{\tilde{a}^2 - \frac{t^2}{4}}{\tilde{q}^2} + \frac{\tilde{a}^2 - \frac{t^2}{4}}{\tilde{q}} z^{-1}$. Since the two matrix differential operators represent the same irreducible differential module over $\mathbb{C}(z)$, there is a $T \in \mathrm{GL}(2, \mathbb{C}(z)) \neq 0$, unique up to multiplication by a constant, such that $(z\frac{d}{dz} + A)T = T(z\frac{d}{dz} + \tilde{A})$. A local computation shows that T has the form $T_{-2}z^{-2} + T_{-1}z^{-1} + T_0 \neq 0$ with constant matrices T_*. The $\tilde{a}$, $\tilde{q}$ and the entries of the T_* are the unknows in the identity $(z\frac{d}{dz} + A)(T_{-2}z^{-2} + T_{-1}z^{-1} + T_0) = (T_{-2}z^{-2} + T_{-1}z^{-1} + T_0)(z\frac{d}{dz} + \tilde{A})$. A Maple computation yields

$$\tilde{q} = -\frac{t(\theta q + 2a - t)}{2q^2}$$

and

$$\tilde{a} = \frac{t(4a^2 - 4at + 2aq + 2\theta a\ q + q^2\theta + t^2 - tq\theta - qt - 2q^3)}{4q^3}.$$

The isomorphism s_2^+ respects the foliations. For a leaf one has $q' = \frac{q+2a}{t}$ and substitution in the first formula produces $\tilde{q} = -\frac{t(\theta t + tq' - q - t)}{2q^2}$ for this Bäcklund transformation on solutions of PIII(D$_7$).

The Bäcklund transformation $s_2^+ s_1^+$ maps a solution q for the parameter θ to the solution $\frac{t(\theta q - 2a + t)}{2q^2}$, with $a = \frac{tq' - q}{2}$, with parameter $1 + \theta$.

2.5 Remarks

1. One considers for $(M, t) \in \mathbf{S}$ the connection $(\mathcal{V}_0, \nabla)$ with generic fibre M and the local data $z\frac{d}{dz} + \begin{pmatrix} \frac{1}{4} & z \\ 1 & -\frac{1}{4} \end{pmatrix}$ at $z = \infty$ and $z\frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ with $\omega = \frac{tz^{-1} + \theta}{2}$ at $z = 0$. The second exterior power of $(\mathcal{V}_0, \nabla)$ is (O, d) and thus $\mathcal{V}_0$ has degree 0. Since $(\mathcal{V}_0, \nabla)$ is irreducible, there are two possibilities for $\mathcal{V}_0$ namely $O \oplus O$ and $O(1) \oplus O(-1)$. Suppose that $\mathcal{V}_0 \cong O(1) \oplus O(-1)$. Then one can identify $\mathcal{V}_0$ with $O([\infty])e_1 + O(-[\infty])e_2$ and for a good choice of e_1, e_2 one obtains the operator $\nabla_{z\frac{d}{dz}} = z\frac{d}{dz} + \begin{pmatrix} 0 & b \\ z^{-1} & 0 \end{pmatrix}$ with $b = z^2 + \cdots$. One concludes that the locus of the modules M with $\mathcal{V}_0 = O(1) \oplus O(-1)$ is the set of the closed points of $q^{-1}(\infty)$ for the map $q := -\frac{c_0}{c_1} : \mathcal{M}(\theta) \to \mathbb{P}^1$ (see also Section 3.3.4).

2. *Algebraic solutions of* PIII(D_7). One easily finds the algebraic solution(s) q with $q^3 = \frac{t^2}{2}$ for PIII(D_7) with $\theta = 0$. Using the Bäcklund transformations one finds an algebraic solution for PIII(D_7) for every $\theta \in \mathbb{Z}$. According to [5, 6], these are all the algebraic solutions of PIII(D_7).

More precisely, $q_j(\tilde{t}) = e^{2\pi ij/3}\frac{e^{2\tilde{t}/3}}{\sqrt[3]{2}}$, $j = 0, 1, 2$ are algebraic solution for $\theta = 0$. We note that $q_1(\tilde{t}) = q_0(\tilde{t} + 4\pi i)$ and $q_2(\tilde{t}) = q_0(\tilde{t} + 2\pi i)$. Since $\alpha = 1$ and $\mathrm{top}^3 = 1$, these solutions are mapped to a single point of $\mathcal{R}(1)$ corresponding to $c_1c_2 = -3$, $e = -i$ and certain values for the invariants ℓ_{12}, ℓ_{14}, ℓ_{23}, ℓ_{34} (which we cannot make explicit). The isomonodromic family for this solution q is

$$z\frac{d}{dz} + \begin{pmatrix} -\frac{q}{6}z^{-1} & b \\ -qz^{-1} + 1 & \frac{q}{6}z^{-1} \end{pmatrix} \qquad \text{with} \quad b = z + \left(\frac{1}{36} - \frac{q}{2}\right) + q\left(\frac{1}{36} - \frac{q}{2}\right)z^{-1}$$

and

$$q^3 = \frac{t^2}{2}.$$

It is not clear what makes this family and the corresponding point of $\mathcal{R}(1)$ so special.

3. *Special solutions of* PIII(D_7). Consider an isomonodromy family for which the Stokes matrices are trivial, i.e., $c_1 = c_2 = e = 0$. Then $\alpha = i$ or $\alpha = -i$. In the first case one computes that $\ell_{12} = \ell_{14} = \frac{1}{2}$, $\ell_{23} = \ell_{34} = -\frac{1}{2}$ and one finds a unique point of $\mathcal{R}(i)$ and a special solution $q(\tilde{t})$ of PIII(D_7) for $\theta = \frac{1}{2}$. Using Bäcklund transformations one obtains a similar special solution for any $\theta \in \frac{1}{2} + \mathbb{Z}$. Y. Ohyama informed us that the condition $c_1 = c_2 = 0$ implies that the corresponding solution q of PIII(D_7) is a univalent function of t and is meromorphic at $t = 0$. Further $\theta \in \frac{1}{2} + \mathbb{Z}$ is equivalent to $e = 0$. See [2] for details.

3 The family $(1, -, 1)$

3.1 Definition of the family

The set $\mathbf{S}$ consists of the equivalence classes of pairs (M, t), where M is a differential module M over $\mathbb{C}(z)$ and $t \in \mathbb{C}^*$ such that: $\dim M = 2$, $\Lambda^2 M$ is the trivial module, M has two singularities 0 and ∞, both singularities have Katz invariant 1, the (generalized) eigenvalues are normalized to $\pm\frac{t}{2}z^{-1}$ at 0 and $\pm\frac{t}{2}z$ at ∞. Further, two pairs (M_1, t_1) and (M_2, t_2) are called equivalent if there exists an isomorphism $M_1 \to M_2$ and $t_1 = t_2$.

As in Section 2, we will have to replace T by its universal covering $\tilde{T} = \mathbb{C} \to T$, $\tilde{t} \mapsto e^{\tilde{t}}$. Write $\tilde{\mathbf{S}} = \mathbf{S} \times_{\mathbf{T}} \tilde{\mathbf{T}}$. Define for $\alpha, \beta \in \mathbb{C}^*$ the subset $\mathbf{S}(\alpha, \beta)$ of $\mathbf{S}$ consisting of the pairs (M, t) such that $\mathbb{C}((z)) \otimes M$ is represented by $z\frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ with $\omega = \frac{tz^{-1}+\theta_0}{2}$, $\alpha = e^{\pi i\theta_0}$ and $\mathbb{C}((z^{-1})) \otimes M$ is represented by $z\frac{d}{dz} + \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix}$ with $\tau = \frac{tz+\theta_\infty}{2}$, $\beta = e^{\pi i\theta_\infty}$. Further $\tilde{\mathbf{S}}(\alpha, \beta) = \mathbf{S}(\alpha, \beta) \times_{\mathbf{T}} \tilde{\mathbf{T}}$.

3.2 The monodromy space

For $(M, t) \in \mathbf{S}$, the *monodromy data* are given by (compare [10]): the symbolic solutions spaces $V(0)$ and $V(\infty)$ at $z = 0$ and $z = \infty$ (including formal monodromies and Stokes matrices) and the link $L : V(0) \to V(\infty)$. We make this more explicit.

The module $\mathbb{C}((z)) \otimes M$ has a basis E_1, E_2 with $\delta(E_1 \wedge E_2) = 0$ and $\delta E_1 = -\frac{tz^{-1}+\theta_0}{2}E_1$, $\delta E_2 = \frac{tz^{-1}+\theta_0}{2}E_2$. We note that t is used to distinguish between E_1 and E_2. This basis is unique up to a transformation $E_1 \mapsto c_1z^mE_1$, $E_2 \mapsto c_2z^{-m}E_2$ with $c_1, c_2 \in \mathbb{C}^*$, $m \in \mathbb{Z}$. After fixing θ_0, the E_1, E_2 are unique up to multiplication by constants. The symbolic solution space $V(0)$ at $z = 0$ is $\mathbb{C}e_1 + \mathbb{C}e_2$, with $e_1 = e^{-\frac{t}{2}z^{-1}+\frac{\theta_0}{2}\log z}E_1$ and $e_2 = e^{+\frac{t}{2}z^{-1}-\frac{\theta_0}{2}\log z}E_2$.

Now $\alpha = e^{\pi i\theta_0}$ is well defined and does not depend on the choices for E_1, E_2.

Similarly, $\mathbb{C}((z^{-1}))\otimes M = \mathbb{C}((z^{-1})F_1 \oplus \mathbb{C}((z^{-1}))F_2$ with $\delta(F_1\wedge F_2) = 0$, $\delta F_1 = -\frac{tz+\theta_\infty}{2}F_1$ and $\delta F_2 = \frac{tz+\theta_\infty}{2}F_2$. The space $V(\infty)$ has basis $f_1 = e^{\frac{tz}{2}+\frac{\theta_\infty}{2}\log z}F_1$ and $f_2 = e^{-\frac{tz}{2}-\frac{\theta_\infty}{2}\log z}F_2$ over $\mathbb{C}$. Moreover, $\beta = e^{\pi i\theta_\infty}$.

For the basis e_1, e_2 of $V(0)$, the formal monodromy and the Stokes matrices are:

$$\begin{pmatrix}\alpha & 0\\ 0 & \frac{1}{\alpha}\end{pmatrix},\qquad \begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix},\qquad \begin{pmatrix}1 & a_2\\ 0 & 1\end{pmatrix}\qquad \text{with product}\qquad \begin{pmatrix}\alpha & \alpha a_2\\ \frac{a_1}{\alpha} & \frac{1+a_1a_2}{\alpha}\end{pmatrix}.$$

This product is the topological monodromy top_0 at $z=0$.

For the basis f_1, f_2 of $V(\infty)$, the formal monodromy and the Stokes matrices are:

$$\begin{pmatrix}\beta & 0\\ 0 & \frac{1}{\beta}\end{pmatrix},\qquad \begin{pmatrix}1 & 0\\ b_1 & 1\end{pmatrix},\qquad \begin{pmatrix}1 & b_2\\ 0 & 1\end{pmatrix}\qquad \text{with product}\qquad \begin{pmatrix}\beta & \beta b_2\\ \frac{b_1}{\beta} & \frac{1+b_1b_2}{\beta}\end{pmatrix}.$$

This is the topological monodromy top_∞ at $z=\infty$.

The link $L: V(0)\to V(\infty)$ with matrix $\begin{pmatrix}\ell_1 & \ell_2\\ \ell_3 & \ell_4\end{pmatrix}$ has determinant 1.

The relations are given by the matrix equality

$$\begin{pmatrix}\beta & \beta b_2\\ \frac{b_1}{\beta} & \frac{1+b_1b_2}{\beta}\end{pmatrix} = L\circ\begin{pmatrix}\alpha & \alpha a_2\\ \frac{a_1}{\alpha} & \frac{1+a_1a_2}{\alpha}\end{pmatrix}\circ L^{-1}.$$

In particular, $\beta = \ell_1\ell_4\alpha + \ell_2\ell_4\frac{a_1}{\alpha} - \ell_1\ell_3\alpha a_2 - \ell_2\ell_3\frac{1+a_1a_2}{\alpha}$. This defines a variety $\mathcal{T}$, given by the variables $\alpha, a_1, a_2, \ell_1, \dots, \ell_4$ with the only restrictions $\ell_1\ell_4 - \ell_2\ell_3 = 1$, $\alpha\neq 0$ and $\ell_1\ell_4\alpha + \ell_2\ell_4\frac{a_1}{\alpha} - \ell_1\ell_3\alpha a_2 - \ell_2\ell_3\frac{1+a_1a_2}{\alpha}\neq 0$.

For fixed values of $\alpha,\beta\in\mathbb{C}^*$ we obtains a variety $\mathcal{T}(\alpha,\beta)$ defined by the variables $a_1, a_2, \ell_1, \dots, \ell_4$ and the relations:

$$\ell_1\ell_4 - \ell_2\ell_3 = 1 \qquad \text{and} \qquad \beta = \ell_1\ell_4\alpha + \ell_2\ell_4\frac{a_1}{\alpha} - \ell_1\ell_3\alpha a_2 - \ell_2\ell_3\frac{1+a_1a_2}{\alpha}.$$

The group $\mathbb{G}_m\times\mathbb{G}_m$ acts on $\mathcal{T}$ and $\mathcal{T}(\alpha,\beta)$, by base change $(e_1,e_2,f_1,f_2)\mapsto(\gamma e_1, \gamma^{-1}e_2, \delta f_1, \delta^{-1}f_2)$. The categorical quotient of $\mathcal{T}$ by $\mathbb{G}_m\times\mathbb{G}_m$ is $\mathcal{R}\to\mathcal{P}$ with parameter space $\mathcal{P} = \mathbb{C}^*\times\mathbb{C}^*$ given by (α,β). This is a family of affine cubic surfaces $\mathcal{R}(\alpha,\beta)$ (this is the categorical quotient of $\mathcal{T}(\alpha,\beta)$) given by the equation

$$x_1x_2x_3 + x_1^2 + x_2^2 + (1+\alpha\beta)x_1 + (\alpha+\beta)x_2 + \alpha\beta = 0,$$

where

$$x_1 = \ell_1\ell_4 - 1,\qquad x_2 = \alpha a_2\ell_1\ell_3 - \alpha\ell_1\ell_4,\qquad x_3 = \frac{1+a_1a_2}{\alpha} + \alpha.$$

Observation: $\mathcal{R}(\alpha,\beta)$ is simply connected if $\alpha\neq\beta^{\pm 1}$.

Define U by removing the two lines $\{(0,-\beta,*)\}$, $\{(-1,0,*)\}$ from $\mathcal{R}(\alpha,\beta)$. The image of the projection $(x_1,x_2,x_3)\in U\mapsto(x_1,x_2)\in\mathbb{C}^2$ is $\mathbb{C}^*\times\mathbb{C}^*\cup\{(0,-\alpha),(-\alpha\beta,0)\}$. This image is simply connected. For $x_1x_2\neq 0$, the fiber is one point. For $x_1x_2 = 0$, the fiber is an affine line. It follows that U is simply connected and thus $\mathcal{R}(\alpha,\beta)$ is simply connected, too.

Definition of the link and the topological monodromies. A construction similar to the one in Section 2 is needed for the definition of the link. For $\tilde{t} = |t|e^{i\phi}\in\tilde{T}$, $\phi\in\mathbb{R}$ the singular directions at $z=0$ and $z=\infty$ are ϕ, $\phi-\pi$ and $-\phi$, $\pi-\phi$. On the universal covering of $\mathbb{P}^1\setminus\{0,\infty\}$ one considers the path $\tilde{z} = re^{id(r)}$, $0<r<\infty$ with $d(r) = \frac{1}{1+r}(\phi-\frac{\pi}{2}) + \frac{r}{1+r}(\frac{\pi}{2}-\phi)$. The link $L: V(0)\to V(\infty)$ is defined by (multi)summation at zero in the direction $\phi-\frac{\pi}{2}$, followed by analytic continuation along the above path and finally the inverse of (multi)summation in the direction $\frac{\pi}{2} - \phi$ at infinity. Now the map $\tilde{\mathbf{S}}(\alpha, \beta) \to \mathcal{R}(\alpha, \beta) \times \tilde{T}$ is well defined.

In the general case, i.e., $\alpha \neq \beta^{\pm 1}$, the space $\mathcal{R}(\alpha, \beta)$ is the geometric quotient of $\mathcal{T}(\alpha, \beta)$ and this space is nonsingular. Therefore the natural map $\tilde{\mathbf{S}}(\alpha, \beta) \to \mathcal{R}(\alpha, \beta) \times \tilde{T}$ is a bijection.

Let $(M, \tilde{t}) \in \tilde{\mathbf{S}}(\alpha, \beta)$. Then M is reducible if and only its monodromy data (in $\mathcal{R}(\alpha, \beta)$) is reducible. Further $\mathcal{R}(\alpha, \beta)$ contains reducible monodromy data if and only if $\alpha = \beta^{\pm 1}$. Thus $\mathbf{S}(\alpha, \beta)$ contains reducible modules if and only if $\alpha = \beta^{\pm 1}$. We will first investigate the general case $\alpha \neq \beta^{\pm 1}$. The special case $\alpha = \beta^{\pm 1}$ presents many difficulties and will be handled later on.

3.3 The general case $\boldsymbol{\alpha \neq \beta^{\pm 1}}$

3.3.1 The moduli space $\boldsymbol{\mathcal{M}(\theta_0, \theta_\infty)}$

Fix θ_0, θ_∞ with $\alpha = e^{i\pi\theta_0}$, $\beta = e^{i\pi\theta_\infty}$. We will construct a moduli space $\mathcal{M}(\theta_0, \theta_\infty)$ of connections on the bundle $Oe_1 \oplus O(-[0])e_2$ on $\mathbb{P}^1$ such that the map, which associates to a connection in this space its generic fiber, is a bijection $\mathcal{M}(\theta_0, \theta_\infty) \to \mathbf{S}(\alpha, \beta)$.

The elements of the *set* $\mathcal{M}(\theta_0, \theta_\infty)$ are the connections $\nabla : \mathcal{V} \to \Omega(2[0] + 2[\infty]) \otimes \mathcal{V}$ defined by: the generic fiber M satisfies $(M, t) \in \mathbf{S}(\alpha, \beta)$; the invariant lattices at $z = 0$ and $z = \infty$ are given by the local matrix differential operators $z\frac{d}{dz} + \begin{pmatrix} \frac{tz^{-1}+\theta_0}{2} & 0 \\ 0 & -\frac{tz^{-1}+\theta_0}{2} + 1 \end{pmatrix}$ and $z\frac{d}{dz} + \begin{pmatrix} \frac{tz+\theta_\infty}{2} & 0 \\ 0 & -\frac{tz+\theta_\infty}{2} \end{pmatrix}$. Since the second exterior power of M is trivial, the degree of $\mathcal{V}$ is -1. By assumption M is irreducible and therefore the type of $\mathcal{V}$ is $O \oplus O(-1)$ and one can identify $\mathcal{V}$ with $Oe_1 \oplus O(-[0])e_2$. By construction the map $\mathcal{M}(\theta_0, \theta_\infty) \to \mathbf{S}(\alpha, \beta)$ is bijective.

The operator $\nabla_{z\frac{d}{dz}}$ has, with respect to the basis $\{e_1, e_2\}$, the form $z\frac{d}{dz} + \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a = a_{-1}z^{-1} + a_0 + a_1 z$, $b = b_{-2}z^{-2} + \cdots + b_1 z$, $c = c_0 + c_1 z$. The lattice condition at $z = 0$ is equivalent to $a(a-1) + bc \in (\frac{tz^{-1}+\theta_0}{2})^2 - (\frac{tz^{-1}+\theta_0}{2}) + \mathbb{C}[[z]]$. This leads to the equations

$$a_{-1}^2 + b_{-2}c_0 = \frac{t^2}{4}, \qquad 2a_{-1}a_0 - a_{-1} + b_{-2}c_1 + b_{-1}c_0 = t\left(\frac{\theta_0}{2} - \frac{1}{2}\right).$$

The lattice condition at $z = \infty$ is equivalent to $a^2 + bc \in (\frac{tz+\theta_\infty}{2})^2 + \mathbb{C}[[z^{-1}]]$ and one finds the equations

$$a_1^2 + b_1 c_1 = \frac{t^2}{4}, \qquad 2a_0 a_1 + b_0 c_1 + b_1 c_0 = \frac{t\theta_\infty}{2}.$$

Define the (quasi-)affine space $\mathcal{A}(\theta_0, \theta_\infty)$ by the variables a_*, b_*, c_*, t, the four equations and the open condition $(c_0, c_1) \neq (0, 0)$. For the general case $\alpha \neq \beta^{\pm 1}$, the module M and the corresponding connection are irreducible and thus $(c_0, c_1) \neq (0, 0)$ holds (see Remarks 3.2.3).

The space $\mathcal{A}(\theta_0, \theta_\infty)$ is divided out by the group of the automorphisms of the bundle $Oe_1 \oplus O(-[0])e_2$. This action amounts to dividing $\mathcal{A}(\theta_0, \theta_\infty)$ by the group G of transformations $e_1 \mapsto e_1$, $e_2 \mapsto \lambda e_2 + (\gamma z^{-1} + \delta)e_1$.

Proposition 3.1. *The quotient of* $\mathcal{A}(\theta_0, \theta_\infty)$ *by* G *is geometric and has no singularities.*

Proof. The open subset $\mathcal{A}(\theta_0, \theta_\infty)_1$, defined by $c_1 \neq 0$, contains the closed 'standard subset' ST_1, given by the connections

$$z\frac{d}{dz} + \begin{pmatrix} a_{-1}z^{-1} & b \\ z + c_0 & -a_{-1}z^{-1} \end{pmatrix} \qquad \text{with} \qquad b = b_{-2}z^{-2} + \cdots + b_1 z,$$

$$b_1 = \frac{t^2}{4}, \qquad b_0 = -\frac{t^2}{4}c_0 + \frac{t\theta_\infty}{2}, \qquad b_{-2} = a_{-1} - b_{-1}c_0 + t\left(\frac{\theta_0}{2} - \frac{1}{2}\right).$$

and the equation

$$a_{-1}^2 + c_0\left(a_{-1} - b_{-1}c_0 + t\left(\frac{\theta_0}{2} - \frac{1}{2}\right)\right) - \frac{t^2}{4} = 0.$$

The natural morphism $G \times ST_1 \to \mathcal{M}(\theta_0, \theta_\infty)_1$ is an isomorphism.

Similarly, let ST_0 denote the closed subset of the open subset $\mathcal{A}(\theta_0, \theta_\infty)_0$, defined by $c_0 \neq 0$, be given by the connections

$$z\frac{d}{dz} + \begin{pmatrix} a_1 z & b \\ c_1 z + 1 & -a_1 z \end{pmatrix} \qquad \text{with} \qquad b = b_{-2}z^{-2} + \cdots + b_1 z,$$

$$b_{-2} = \frac{t^2}{4}, \qquad b_{-1} = t\left(\frac{\theta_0}{2} - \frac{1}{2}\right) - \frac{t^2}{4}c_1, \qquad b_1 = \frac{t\theta_\infty}{2} - b_0 c_1$$

and the equation

$$a_1^2 + \left(\frac{t\theta_\infty}{2} - b_0 c_1\right)c_1 - \frac{t^2}{4} = 0.$$

Again $G \times ST_0 \to \mathcal{A}(\theta_0, \theta_\infty)_0$ is an isomorphism. The quotient of $\mathcal{A}(\theta_0, \theta_\infty)$ by G is obtained by gluing the two non singular spaces ST_1 and ST_0. ■

Now $\mathcal{M}(\theta_0, \theta_\infty)$ is, as algebraic variety, defined as the quotient of $\mathcal{A}(\theta_0, \theta_\infty)$ by G. Since this is a geometric quotient, the map $\mathcal{M}(\theta_0, \theta_\infty)(\mathbb{C}) \to \mathbf{S}(\alpha, \beta)$ is bijective.

Remarks 3.2. 1. As in Section 2, the Observation, one sees that, after scaling some of the variables, the two charts ST_1, ST_0 of $\mathcal{M}(\theta_0, \theta_\infty)$, have the form $U_1 \times T$, $U_2 \times T$ with simply connected spaces U_1, U_2.

Define $M(\theta_0, \theta_\infty) = f^{-1}(1)$, *where* $f : \mathcal{M}(\theta_0, \theta_\infty) \to T$ *is the canonical morphism.* Then $M(\theta_0, \theta_\infty)$ is simply connected.

Further $\tilde{\mathcal{M}}(\theta_0, \theta_\infty) := \mathcal{M}(\theta_0, \theta_\infty) \times_T \tilde{T}$ is the universal covering of $\mathcal{M}(\theta_0, \theta_\infty)$.

2. Let $q : \mathcal{M}(\theta_0, \theta_\infty) \to \mathbb{P}^1$ denote the morphism given by $q = -\frac{c_0}{c_1}$.

3. In the cases $\alpha = \beta^{\pm 1}$, the space $\mathcal{A}(\theta_0, \theta_\infty)$ is defined as before, and including the assumption $(c_0, c_1) \neq (0, 0)$. The space $\mathcal{M}(\theta_0, \theta_\infty)$ is again the geometric and non singular quotient of $\mathcal{A}(\theta_0, \theta_\infty)$ by G. The canonical map $\mathcal{M}(\theta_0, \theta_\infty)(\mathbb{C}) \to \mathbf{S}(\alpha, \beta)$ *is injective and, in general, not surjective.*

Indeed, the open condition $(c_0, c_1) \neq (0, 0)$ is valid for $(M, t) \in \mathbf{S}(\alpha, \beta)$ such that M is irreducible but may exclude certain reducible modules in $\mathbf{S}(\alpha, \beta)$. We note that $c_0 = c_1 = 0$ implies $\pm\frac{\theta_\infty}{2} = \frac{\theta_0}{2} - \epsilon$ with $\epsilon \in \{0, 1\}$.

3.3.2 The Okamoto–Painlevé space $\tilde{\mathcal{M}}(\theta_0, \theta_\infty)$ for $\frac{\theta_0}{2} \pm \frac{\theta_\infty}{2} \notin \mathbb{Z}$

The moduli space $\mathcal{M}(\theta_0, \theta_\infty)$ is replaced by $\tilde{\mathcal{M}}(\theta_0, \theta_\infty) := \mathcal{M}(\theta_0, \theta_\infty) \times_T \tilde{T}$. The bijections $\tilde{\mathbf{S}}(\alpha, \beta) \to \mathcal{R}(\alpha, \beta) \times \tilde{T}$ and $\mathcal{M}(\theta_0, \theta_\infty) \to \mathbf{S}(\alpha, \beta)$ imply that the analytic morphism $\tilde{\mathcal{M}}(\theta_0, \theta_\infty) \to \mathcal{R}(\alpha, \beta) \times \tilde{T}$ is bijective and hence an analytic isomorphism. As in Theorem 2.3, using arguments presented in [8, 9, 12] this implies the following result.

Theorem 3.3. *Suppose that* $\frac{\theta_0}{2} \pm \frac{\theta_\infty}{2} \notin \mathbb{Z}$ *(equivalently* $\alpha \neq \beta^{\pm 1}$*). Then* $\tilde{\mathcal{M}}(\theta_0, \theta_\infty) \to \tilde{T}$*, provided with the foliation given by the fibers of* $\tilde{\mathcal{M}}(\theta_0, \theta_\infty) \to \mathcal{R}(\alpha, \beta)$ *(i.e., the isomonodromy families), is the Okamoto–Painlevé space corresponding to the equation* PIII(D$_6$)*, namely*

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} - \frac{4(\theta_0 - 1)}{t} + \frac{4\theta_\infty q^2}{t} + 4q^3 - \frac{4}{q}.$$

Moreover, this equation satisfies the Painlevé property.

Observations 3.4. 1. The above formula differs slightly from the one given in [10, Section 4.5]. This is due to different choices of the standard matrix differential operator.

2. The transformation $t \mapsto -t$, $\theta_\infty \mapsto -\theta_\infty$ and $\theta_0 \mapsto -\theta_0 + 2$ leaves the family of matrix differential operators invariant. This has the consequence that a solution $q(t)$ of PIII(D_6) with parameters θ_0 and θ_∞, yields the solution $q(-t)$ of PIII(D_6) with parameters $-\theta_0+2$ and $-\theta_\infty$. This can also be seen directly from the differential equation.

3. The solutions $q(t)$ of PIII(D_6) are in fact meromorphic functions in $\tilde{t} \in \tilde{T} = \mathbb{C}$. Thus $Q(\tilde{t}) := q(e^{\tilde{t}})$ is well defined and satisfies the equation

$$Q'' = \frac{(Q')^2}{Q} - 4(\theta_0 - 1)e^{\tilde{t}} + 4\theta_\infty Q^2 e^{\tilde{t}} + 4Q^3 e^{2\tilde{t}} - \frac{4e^{2\tilde{t}}}{Q}, \qquad \text{where} \qquad {}' = \frac{d}{d\tilde{t}}.$$

4. The space of initial conditions is analytically isomorphic to $\mathcal{R}(\alpha,\beta)$ and can also be identified with $M(\theta_0,\theta_\infty)$. Indeed, the extended Riemann–Hilbert isomorphism $\tilde{\mathcal{M}}(\theta_0,\theta_\infty) \to \mathcal{R}(\alpha,\beta) \times \tilde{T}$ induces an analytic isomorphism $M(\theta_0,\theta_\infty) \to \mathcal{R}(\alpha,\beta)$.

3.3.3 Verification of the formula in Theorem 2.3

On the chart ST_1 of $\mathcal{M}(\theta_0,\theta_\infty)$ the matrix differential operator has the form $z\frac{d}{dz} + A = z\frac{d}{dz} + a_{-1}z^{-1}H + bE_1 + (z-q)E_2$, where $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We will use $[H,E_1] = 2E_2$, $[H,E_2] = -2E_2$, $[E_1,E_2] = H$. Further

$$q := -c_0, b = b_{-2}z^{-2} + b_{-1}z^{-1} + b_0 + b_1 z,$$

where $b_{-2} \neq 0$ (by assumption),

$$b_1 = \frac{t^2}{4}, \qquad b_0 = q\frac{t^2}{4} + \frac{t\theta_\infty}{2}, \qquad b_{-2} = a_{-1} + qb_{-1} + t\left(\frac{\theta_0}{2} - \frac{1}{2}\right),$$
$$a_{-1}^2 - q\left(a_{-1} + qb_{-1} + t\left(\frac{\theta_0}{2} - \frac{1}{2}\right)\right) - \frac{t^2}{4} = 0.$$

Now q, a_{-1} are considered as (meromorphic) functions of t and A is a matrix depending on z and t. The family $z\frac{d}{dz} + A$ is isomonodromic if and only if there is an operator of the form $\frac{d}{dt} + B = \frac{d}{dt} + B_H H + B_1 E_1 + B_2 E_2$ which commutes with $z\frac{d}{dz} + A$. This is equivalent to $\frac{d}{dt}(A) = z\frac{d}{dz}(B) + [B,A]$.

Further $B_* = B_H, B_1, B_2$ are functions of t, z and are supposed to have the form $B_{*,-2}(t)z^{-2} + B_{*,-1}(t)z^{-1} + B_{*,0}(t) + B_{*,1}(t)z$. The two operators commute if and only if $a'_{-1}z^{-1}H + b'E_1 - q'E_2$ is equal to

$$\overset{\circ}{B}_H H + \overset{\circ}{B}_1 E_1 + \overset{\circ}{B}_2 E_2 - \left[B_H H + B_1 E_1 + B_2 E_2, a_{-1}z^{-1}H + bE_1 + (z-q)E_2\right],$$

where $\overset{\circ}{X}$ stands for $z\frac{d}{dz}(X)$ and $X' := \frac{d}{dt}(X)$. On obtains the equations

$$(H) \quad a'_{-1}z^{-1} = \overset{\circ}{B}_H - B_1(z-q) + B_2 b,$$
$$(E_1) \quad b' = \overset{\circ}{B}_1 - 2B_H b + 2B_1 a_{-1}z^{-1},$$
$$(E_2) \quad -q' = \overset{\circ}{B}_2 + 2B_H(z-q) - 2B_2 a_{-1}z^{-1}.$$

A Maple computation shows that this system of differential equations for q, a_{-1} is equivalent to

$$q' = \frac{4a_{-1} - q}{t}, \qquad a'_{-1} = \frac{4a_{-1}^2 - t^2 + q(t - a_{-1} - t\theta_0) + q^3 t\theta_\infty + q^4 t^2}{tq}.$$

The equation

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} - \frac{4(\theta_0 - 1)}{t} + \frac{4\theta_\infty q^2}{t} + 4q^3 - \frac{4}{q}$$

follows by substitution. Using the transformation $z \mapsto z^{-1}$ one finds that $Q = \frac{1}{q}$ satisfies the PIII(D$_6$) equation with $\theta_0 - 1$ and θ_∞ interchanged:

$$Q'' = \frac{(Q')^2}{Q} - \frac{Q'}{t} - \frac{4\theta_\infty}{t} + \frac{4(\theta_0 - 1)Q^2}{t} + 4Q^3 - \frac{4}{Q}.$$

3.3.4 After a remark by Yousuke Ohyama

Let $(M, t) \in \mathbf{S}(\alpha, \beta)$ with $\alpha = e^{\pi i \theta_0}$, $\beta = e^{\pi i \theta_\infty}$ have the property that M is irreducible. Consider the connection $(\mathcal{W}, \nabla)$ with generic fiber M and locally represented by

$$z\frac{d}{dz} + \begin{pmatrix} \frac{tz^{-1}+\theta_0}{2} & 0 \\ 0 & -\frac{tz^{-1}+\theta_0}{2} \end{pmatrix} \quad \text{at } z = 0 \qquad \text{and} \qquad z\frac{d}{dz} + \begin{pmatrix} \frac{tz+\theta_\infty}{2} & 0 \\ 0 & -\frac{tz+\theta_\infty}{2} \end{pmatrix} \quad \text{at } z = \infty.$$

The second exterior product of $(\mathcal{W}, \nabla)$ is trivial and thus $\Lambda^2\mathcal{W}$ has degree 0. Since M is irreducible one has $\mathcal{W} \cong O(k) \oplus O(-k)$ with $k \in \{0, 1\}$.

Suppose that the connection $\mathcal{W}$ has type $O(1) \oplus O(-1)$. Then we can identify $\mathcal{W}$ with $O([0])B_1 \oplus O(-[0])B_2$. Put $D := \nabla_{z\frac{d}{dz}}$. Now DB_1 is not a multiple of B_1 since M is irreducible. After multiplying B_1 with a scalar, the matrix of D with respect to the basis B_1, B_2 has the form $\begin{pmatrix} \alpha & \beta \\ z & -\alpha \end{pmatrix}$ with $\alpha = \alpha_{-1}z^{-1} + \alpha_0 + \alpha_1 z$, $\beta = \beta_{-3}z^{-3} + \cdots + \beta_1 z$. The base vector B_2 can be replaced by $B_2 + hB_1$ with $h = h_0 + h_{-1}z^{-1} + h_{-2}z^{-2}$. The result is a new representation of D, namely

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{-1} \left\{ z\frac{d}{dz} + \begin{pmatrix} \alpha & \beta \\ z & -\alpha \end{pmatrix} \right\} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} = z\frac{d}{dz} + \begin{pmatrix} \alpha - hz & 2\alpha h - h^2 z + z\frac{dh}{dz} + \beta \\ z & -\alpha + hz \end{pmatrix}.$$

For unique h_0, h_{-1} and at most two values of h_{-2}, the last operator is

$$z\frac{d}{dz} + \begin{pmatrix} a_{-1}z^{-1} b \\ z - a_{-1}z^{-1} \end{pmatrix}, \qquad \text{where} \qquad b = b_{-2}z^{-2} + b_{-1}z^{-1} + b_0 + b_1 z.$$

Let e_1, e_2 denote the new basis. Then $\mathcal{V} = Oe_1 \oplus O(-[0])e_2$ and the corresponding point $\xi \in \mathcal{M}(\theta_0, \theta_\infty)$ satisfies $q(\xi) = 0$. The converse holds, too. One observes that $q^{-1}(0) \subset \mathcal{M}(\theta_0, \theta_\infty)$ has two connected components, each one isomorphic to $\mathbb{A}^1 \times T$. We note that the map $Q := \frac{1}{q}$ can also be used in this context, since for a monodromic family Q satisfies a PIII(D$_6$) equation (see the end of Section 3.3.3).

According to Malgrange, the locus where the bundle $\mathcal{W}$ is not free is the tau-divisor. Thus we find that the tau-divisor coincides with the locus $Q^{-1}(\infty) \subset \mathcal{M}(\theta_0, \theta_\infty)$. The statement: 'the tau-divisor coincides with $q^{-1}(\infty)$' holds for PI, PII, PIII(D$_7$), PIII(D$_8$), PIV, too (see [8, 9, 12]).

3.4 The cases $\alpha = \beta^{\pm 1}$

3.4.1 Geometric quotients of the monodromy data

We use here the notation of Section 3.2.

1. If $\alpha = \beta \neq \pm 1$, then $\mathcal{R}(\alpha, \alpha)$ has a singular point, namely $(x_1, x_2, x_3) = (0, -\alpha, \alpha + \alpha^{-1})$. The preimage in $\mathcal{T}(\alpha, \alpha)$ of this singular point consists of the tuples $(a_1, a_2, \ell_1, \ldots, \ell_4)$ such that

the matrices L, top_0 have the form $\begin{pmatrix}\ell_1 0\\ \ell_3\ell_4\end{pmatrix}$, $\begin{pmatrix}\alpha & 0\\ \frac{a_1}{\alpha} & \frac{1}{\alpha}\end{pmatrix}$ or $\begin{pmatrix}\ell_1 & \ell_2\\ 0 & \ell_4\end{pmatrix}$, $\begin{pmatrix}\alpha & \alpha a_2\\ 0 & \frac{1}{\alpha}\end{pmatrix}$. In particular, $\mathcal{R}(\alpha,\alpha)$ is *not a geometric quotient*.

The remedy consists of replacing $\mathcal{T}(\alpha,\alpha)$ by $\mathcal{T}(\alpha,\alpha)^*$ which is the complement of the closed subset of $\mathcal{T}(\alpha,\alpha)$ given by the equations $\ell_2=\ell_3=a_1=a_2=0$. We claim that $\mathcal{T}(\alpha,\alpha)^*$ has a nonsingular geometric quotient by the action of $\mathbb{G}_m\times\mathbb{G}_m$. A proof is obtained by writing $\mathcal{T}(\alpha,\alpha)^*$ as the union of the four affine open subsets $\ell_2\neq 0$, $\ell_3\neq 0$, $a_1\neq 0$ and $a_2\neq 0$. On each of these subsets one explicitly computes the quotient by $\mathbb{G}_m\times\mathbb{G}_m$, which turns out to be nonsingular and geometric. Gluing these four quotients produces the required geometric quotient which will be denoted by $\mathcal{R}(\alpha,\alpha)^*$.

Let $\mathbf{S}(\alpha,\alpha)^*$ the complement in $\mathbf{S}(\alpha,\alpha)$ of the set of the modules which are direct sums and $\tilde{\mathbf{S}}(\alpha,\alpha)^*=\mathbf{S}(\alpha,\alpha)^*\times_T\tilde{T}$. Then the canonical map $\tilde{\mathbf{S}}(\alpha,\alpha)^*\to\mathcal{R}(\alpha,\alpha)^*\times\tilde{T}$ is bijective.

Define the closed space $\mathcal{T}(\alpha,\alpha)^*_{\mathrm{red}}$ of $\mathcal{T}(\alpha,\alpha)^*$ by the condition that the data is reducible. This space has two irreducible components, given in terms of the matrices L, top_0 by:

(a) $\begin{pmatrix}\ell_1 & 0\\ \ell_3 & \ell_4\end{pmatrix}$, $\begin{pmatrix}\alpha & 0\\ \frac{a_1}{\alpha} & \frac{1}{\alpha}\end{pmatrix}$ with $\ell_1\ell_4=1$ and $(\ell_3,a_1)\neq 0$. One easily verifies that the map which sends (L,top_0) to $(\ell_3:a_1)\in\mathbb{P}^1$ is the geometric quotient.

(b) $\begin{pmatrix}\ell_1 & \ell_2\\ 0 & \ell_4\end{pmatrix}$, $\begin{pmatrix}\alpha & \alpha a_2\\ 0 & \frac{1}{\alpha}\end{pmatrix}$ with $\ell_1\ell_4=1$ and $(\ell_2,a_2)\neq 0$. One easily verifies that the map which sends the (L,top_0) to $(\ell_2:a_2)\in\mathbb{P}^1$ is the geometric quotient.

Therefore the 'reducible locus' $\mathcal{R}(\alpha,\alpha)^*_{\mathrm{red}}$ (i.e., corresponding to reducible monodromy data) is the union of two, not intersecting, projective lines.

2. The case $\alpha=\beta^{-1}\neq\pm 1$ can be handled as in (1). One finds (with a similar notation) a geometric quotient $\mathcal{R}(\alpha,\alpha^{-1})^*$ of $\mathcal{T}(\alpha,\alpha^{-1})^*$ and a bijection $\tilde{\mathbf{S}}(\alpha,\alpha^{-1})^*\to\mathcal{R}(\alpha,\alpha^{-1})^*\times\tilde{T}$. Further $\mathcal{R}(\alpha,\alpha^{-1})^*_{\mathrm{red}}$ is the union of two, non intersecting, projective lines.

3. $\alpha=\beta=1$. The categorical quotient $\mathcal{R}(1,1)$ of $\mathcal{T}(1,1)$ has two singular points, namely $(x_1,x_2,x_3)=(0,-1,2)$ and $(x_1,x_2,x_3)=(-1,0,2)$. The preimage of the first singular point consists of the pairs (L,top_0) equal to $\left(\begin{pmatrix}\ell_1 & \ell_2\\ 0 & \ell_4\end{pmatrix},\begin{pmatrix}1 & a_2\\ 0 & 1\end{pmatrix}\right)$ or to $\left(\begin{pmatrix}\ell_1 & 0\\ \ell_3 & \ell_4\end{pmatrix},\begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix}\right)$. The preimage of the second singular point consists of the pairs (L,top_0) equal to $\left(\begin{pmatrix}0 & \ell_2\\ \ell_3 & \ell_4\end{pmatrix},\begin{pmatrix}1 & a_2\\ 0 & 1\end{pmatrix}\right)$ or to $\left(\begin{pmatrix}\ell_1 & \ell_2\\ 0 & \ell_4\end{pmatrix},\begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix}\right)$. Clearly, $\mathcal{R}(1,1)$ is not a geometric quotient.

The locus of the points in $\mathcal{T}(1,1)$ which describe the monodromy data for modules in $\mathbf{S}(1,1)$ which are direct sums is the union of the two closed sets $a_1=a_2=\ell_2=\ell_3=0$ and $a_1=a_2=\ell_1=\ell_4=0$. Let $\mathcal{T}(1,1)^*\subset\mathcal{T}(1,1)$ denote the complement of this locus. This set is the union of the six open subsets given by the inequalities $a_1\neq 0$, $a_2\neq 0$, $\ell_{12}\neq 0$, $\ell_{13}\neq 0$, $\ell_{24}\neq 0$ and $\ell_{34}\neq 0$. The group $\mathbb{G}_m\times\mathbb{G}_m$ acts on each of these open affine sets and the categorical quotient is a geometric quotient and is nonsingular. Therefore the quotient $\mathcal{R}(1,1)^*$ of $\mathcal{T}(1,1)^*$, obtained by gluing the six quotients, is a geometric quotient and nonsingular.

Example. The open subset $\ell_{12}\neq 0$ is defined by the variables $\ell_1,\ldots,\ell_4$, a_1, a_2 and relations $0=-\ell_{23}a_{12}+\ell_{24}a_1-\ell_{13}a_2$, $\ell_{14}-\ell_{13}-1=0$, $\ell_{12}\neq 0$. Division by $\mathbb{G}_m\times\mathbb{G}_m$ is equivalent to the normalisation $\ell_1=\ell_2=1$. Elimination of ℓ_4 by $\ell_4=\ell_3+1$ yields the equation $\ell_3(a_1-a_2+a_1a_2)+a_1=0$. This is a nonsingular surface. Using the projection $(\ell_3,a_1,a_2)\mapsto(a_1,a_2)$ one finds that this surface is simply connected.

Similar computations lead to the statements: $\mathcal{R}(1,1)^*$ *is a nonsingular geometric quotient and is simply connected. The natural map* $\tilde{\mathbf{S}}(1,1)^*\to\mathcal{R}(1,1)^*\times\tilde{T}$ *is a well defined bijection. The reducible locus* $\mathcal{R}(1,1)^*_{\mathrm{red}}$ *is the union of four, non intersecting, projective lines.*

4. $\alpha=\beta=-1$. The categorical quotient $\mathcal{R}(-1,-1)$ has two singular points, namely $(x_1,x_2,x_3)=(0,1,-2)$ and $(x_1,x_2,x_3)=(-1,0,-2)$. As in (3), one defines $\mathcal{T}(-1,-1)^*$ and its

geometric non singular quotient $\mathcal{R}(-1,-1)^*$. The space $\mathcal{R}(-1,-1)^*$ contains four, non intersecting, projective lines. These lines correspond to the reducible locus of $\mathcal{R}(-1,-1)^*$. As in (3) one defines $\tilde{\mathbf{S}}(-1,-1)^*$ and concludes:

$\mathcal{R}(-1,-1)^*$ is a nonsingular geometric quotient and is simply connected.

The natural map $\tilde{\mathbf{S}}(-1,-1)^* \to \mathcal{R}(-1,-1)^* \times \tilde{T}$ is a bijection.

3.4.2 Reducible modules in S

We use here the notation of Sections 3.2 and 3.4.1.

Observations 3.5. Let $N \subset M$ be a 1-dimensional submodule, then $\mathbb{C}((z)) \otimes N = \mathbb{C}((z))E_i$ and $\mathbb{C}((z^{-1})) \otimes N = \mathbb{C}((z^{-1}))F_j$ with $i,j \in \{1,2\}$. Since N has no other singularities than 0, ∞ one has $N = \mathbb{C}(z)n$ with $\delta(n) = (\frac{\pm tz^{-1} \pm tz}{2} + d)n$, where $d \in \mathbb{C}$ is unique modulo $\mathbb{Z}$. Indeed, $\delta(n) = \big(\frac{\pm tz^{-1} \pm tz}{2} + f\big)n$, where $f \in \mathbb{C}(z)$ has no poles at 0 and ∞. Using that N has only singularities at 0 and ∞, one can change the generator n of N such that f is a constant d. Any other base vector of N with this property has the form $z^k n$ with $k \in \mathbb{Z}$. Further $d \in \pm\theta_0/2 + \mathbb{Z}$ and $d \in \pm\theta_\infty/2 + \mathbb{Z}$ and hence $\alpha = \beta^{\pm 1}$.

Proposition 3.6 (Reducible modules)**.** 1. *A module $(M,t) \in \mathbf{S}$ is reducible if and only if there are $i,j \in \{1,2\}$ such that $a_i = 0$, $b_j = 0$ and $L(\mathbb{C}e_i) = \mathbb{C}f_j$.*

2. *Let $(M,t) \in \mathbf{S}$ be reducible, but not a direct sum of two submodules of dimension one. Then there are unique elements $\epsilon_1, \epsilon_2 \in \{-1,1\}$, a complex number d, unique modulo $\mathbb{Z}$, and a polynomial $c = c_1 z + c_0 \neq 0$, unique up to multiplication by a scalar, such that M is represented by the matrix differential operator*

$$z\frac{d}{dz} + \begin{pmatrix} -\frac{\epsilon_1 tz^{-1} + \epsilon_2 tz}{2} - d & 0 \\ c_1 z + c_0 & \frac{\epsilon_1 tz^{-1} + \epsilon_2 tz}{2} + d \end{pmatrix}.$$

3. *For $\alpha \neq \pm 1$, the reducible locus $\mathbf{S}(\alpha,\alpha)^*_{\mathrm{red}}$ of $\mathbf{S}(\alpha,\alpha)^*$ is represented by the union of the two families in* (2) *given by $e^{2\pi i d} = \alpha$, $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_1 = \epsilon_2 = -1$. Each of the two families is isomorphic to $\mathbb{P}^1 \times T$, by sending the matrix differential operator to $((c_1 : c_0), t) \in \mathbb{P}^1 \times T$.*

The isomorphism $\tilde{\mathbf{S}}(\alpha,\alpha)^*_{\mathrm{red}} \to \mathcal{R}(\alpha,\alpha)^*_{\mathrm{red}} \times \tilde{T}$ yields two isomorphism $\mathbb{P}^1 \times \tilde{T} \to \mathbb{P}^1 \times \tilde{T}$. These have the form $(p,\tilde{t}) \mapsto (A(\tilde{t})p, \tilde{t})$ where $A(\tilde{t})$ is an automorphism of $\mathbb{P}^1$, depending on $\tilde{t}$.

4. *A similar result holds for $\alpha \neq \pm 1$ and $\alpha = \beta^{-1}$.*

5. *$\mathbf{S}(1,1)^*_{\mathrm{red}}$ is represented by the families $d \in \mathbb{Z}$ and the four possibilities of ϵ_1, ϵ_2. Then $\tilde{\mathbf{S}}(1,1)^*_{\mathrm{red}}$ identifies with the disjoint union of four copies of $\mathbb{P}^1 \times \tilde{T}$. The same holds for $\mathcal{R}(1,1)^*_{\mathrm{red}} \times \tilde{T}$. The isomorphism $\tilde{\mathbf{S}}(1,1)^*_{\mathrm{red}} \to \mathcal{R}(1,1)^*_{\mathrm{red}} \times \tilde{T}$ yields four isomorphism $\mathbb{P}^1 \times \tilde{T} \to \mathbb{P}^1 \times \tilde{T}$. These have the form described in* (3)*.*

6. *The case $\alpha = \beta = -1$ is similar to case* (5)*.*

Proof. 1. This follows from Observations 3.5 and the statement that a differential module over $\mathbb{C}(z)$ is determined by its monodromy data (i.e., ordinary monodromy, Stokes matrices and links) and the formal classification of the singular points (see [10, Theorem 1.7]).

2. For convenience we consider the case $i = j = 1$ of (1). Using the above Observation, one finds that M has a basis m_1, m_2 such that $z\partial(m_2) = am_2$ and $z\partial(m_1) = -am_1 + fm_2$ with $a := \frac{tz^{-1} + tz}{2} + d$ and $f \in \mathbb{C}(z)$. If we fix d, then m_2 is unique up to multiplication by a scalar. Further, m_1 is unique up to a transformation $m_1 \mapsto \lambda m_1 + hm_2$ with $\lambda \in \mathbb{C}^*$, $h \in \mathbb{C}(z)$. This transformation changes f into $\lambda f + 2ah + zh'$.

We start considering the subgroup of transformations with $\lambda = 1$ and $h \in \mathbb{C}(z)$. For a suitable h the term $f_1 := f + 2ah + zh'$ has in $\mathbb{C}^*$ at most poles of order one. A pole of order one of f_1 in $\mathbb{C}^*$ cannot disappear by a transformation of the form under consideration. Since M has only singularities at 0 and ∞ we conclude that $f_1 \in \mathbb{C}[z,z^{-1}]$. For suitable $h \in \mathbb{C}[z,z^{-1}]$ the term $c := f_1 + 2ah + zh'$ is a polynomial of degree ≤ 1 and is $\neq 0$, by assumption. For any $h \in \mathbb{C}(z)$, $h \neq 0$ the term $c + 2ah + zh'$ is not a polynomial of degree ≤ 1. This yields a unique c for this subgroup of transformations. Finally, the transformation $m_1 \mapsto \lambda m_1$ shows that c is unique up to multiplication by a scalar.

Cases (3)–(6) are consequences of the above computation of the $\mathcal{R}(\alpha,\beta)^*_{\mathrm{red}}$. ■

3.4.3 The reducible connections in $\mathcal{M}(\theta_0,\theta_\infty)$

The image of the injective map $\mathcal{M}(\theta_0,\theta_\infty) \to \mathbf{S}(\alpha,\beta)^*$, where $\alpha = e^{\pi i\theta_0}$, $\beta = e^{\pi i\theta_\infty}$, will be denoted by $\mathbf{S}(\theta_0,\theta_\infty)$. By Remarks 3.2 part (3), this image contains the irreducible modules (M,t). For $\alpha \neq \beta^{\pm 1}$, $\mathbf{S}(\theta_0,\theta_\infty)$ is equal to $\mathbf{S}(\alpha,\beta) = \mathbf{S}(\alpha,\beta)^*$. Now we consider the other cases.

Proposition 3.7. *Suppose $\alpha = \beta \neq \pm 1$. Then $\mathbf{S}(\theta_0,\theta_\infty)$ consists of:*

(a) *the irreducible elements (M,t);*

(b) *the $(\epsilon_1,\epsilon_2) = (1,1)$ family of reducible modules $\Leftrightarrow \theta_0 - \theta_\infty \geq 2$;*

(c) *the $(\epsilon_1,\epsilon_2) = (-1,-1)$ family of reducible modules $\Leftrightarrow \theta_0 - \theta_\infty \leq 0$.*

Proof. Part (a) is known. Consider an element (M,t) in the $(\epsilon_1,\epsilon_2) = (1,1)$ family of reducible modules. Let the connection $(\mathcal{V},\nabla)$, corresponding to (M,t), be defined as in Section 3.3.1. Then $\mathcal{V}$ can be identified with the vector bundle $O(k[0])e_1 + O((-k-1)[0])e_2$ for some integer $k \geq 0$. Then (M,t) lies in $\mathbf{S}(\theta_0,\theta_\infty) \Leftrightarrow k = 0$.

Consider the case $k > 0$. Then $\nabla_{z\frac{d}{dz}} e_1 = (a_{-1}z^{-1} + a_0 + a_1 z)e_1$ and $\nabla_{z\frac{d}{dz}}(z^{-k}e_1) = (a_{-1}z^{-1} + a_0 + a_1 z - k)(z^{-k}e_1)$.

Comparing with the prescribed local operator at $z = 0$ yields the possibilities: $a_{-1}z^{-1} + a_0$ equals (i) $\frac{t}{2}z^{-1} + \frac{\theta_0}{2}$ or (ii) $-\frac{t}{2}z^{-1} - \frac{\theta_0}{2} + 1$.

Comparing with the prescribed local operator at $z = \infty$ yields the possibilities: $a_0 + a_1 z$ equals (A) $\frac{t}{2}z + \frac{\theta_\infty}{2}$ or (B) $-\frac{t}{2}z - \frac{\theta_\infty}{2}$. Combining one obtains

$$\begin{array}{llllll} (i), & (A) & a_{-1} = \dfrac{t}{2}, & a_1 = \dfrac{t}{2}, & \theta_0 = \theta_\infty - 2k, & (\epsilon_1,\epsilon_2) = (1,1), \\ (i), & (B) & a_{-1} = \dfrac{t}{2}, & a_1 = -\dfrac{t}{2}, & \theta_0 = -\theta_\infty - 2k, & (\epsilon_1,\epsilon_2) = (1,-1), \\ (ii), & (A) & a_{-1} = -\dfrac{t}{2}, & a_1 = \dfrac{t}{2}, & \theta_0 = -\theta_\infty + 2k + 2, & (\epsilon_1,\epsilon_2) = (-1,1), \\ (ii), & (B) & a_{-1} = -\dfrac{t}{2}, & a_1 = -\dfrac{t}{2}, & \theta_0 = \theta_\infty + 2k + 2, & (\epsilon_1,\epsilon_2) = (-1,-1). \end{array}$$

Since $\alpha = \beta \neq \pm 1$, reducible modules of types $(1,-1)$ and $(-1,1)$ are not present in $\mathbf{S}(\alpha,\alpha)^*$. Now we consider the presence of reducible modules of type $(1,1)$ in $\mathbf{S}(\theta_0,\theta_\infty)$.The condition $\theta_0 - \theta_\infty \geq 0$ is necessary because of (i), A.

Consider the case $\theta_0 = \theta_\infty = 2d$. A reducible module of type $(1,1)$ yields a connection on $\mathcal{W} = Of_1 \oplus Of_2$ with the local data $z\frac{d}{dz} + \begin{pmatrix} -(\frac{t}{2}z^{-1} + d) & 0 \\ * & \frac{t}{2}z^{-1} + d \end{pmatrix}$ at $z = 0$ and $z\frac{d}{dz} + \begin{pmatrix} -(\frac{t}{2}z + d) & 0 \\ * & \frac{t}{2}z + d \end{pmatrix}$ at $z = \infty$. Now $\mathcal{V} = O(-[0])f_1 \oplus Of_2 = Oe_1 \oplus O(-[0])e_2$, with $e_1 = f_2$, $e_2 = f_1$, has the required local data and the matrix of $\nabla_{z\frac{d}{dz}}$ with respect to the basis e_1, e_2 is $\begin{pmatrix} \omega & * \\ 0 & -\omega \end{pmatrix}$. Thus $c_0 = c_1 = 0$ and the reducible modules of type $(1,1)$ are not present according to the construction of $\mathcal{M}(\theta_0,\theta_\infty)$.

Consider the case $\theta_0 = \theta_\infty + 2$. The standard form of Proposition 3.6.2 for type $(1,1)$ belongs to $\mathcal{M}(2d+2, 2d)$. Further, see Remarks 3.2.3, $(c_0, c_1) \neq (0,0)$ holds for $\theta_0 - \theta_\infty > 2$. This proves (b). The proof of (c) is similar. ■

Similarly one shows: $\alpha = \beta^{-1} \neq \pm 1$. Then $\mathbf{S}(\theta_0, \theta_\infty)$ consists of:

(a) the irreducible elements (M, t);

(b) the $(\epsilon_1, \epsilon_2) = (1, -1)$ family of reducible modules $\Leftrightarrow$ $\theta_0 + \theta_\infty \geq 2$;

(c) the $(\epsilon_1, \epsilon_2) = (-1, 1)$ family of reducible modules $\Leftrightarrow$ $\theta_0 + \theta_\infty \leq 0$.

$\alpha = \beta = \pm 1$. Then $\mathbf{S}(\theta_0, \theta_\infty)$ consists of:

(a) the irreducible elements (M, t);

(b) the $(\epsilon_1, \epsilon_2) = (1, 1)$ family of reducible modules $\Leftrightarrow$ $\theta_0 - \theta_\infty \geq 2$;

(c) the $(\epsilon_1, \epsilon_2) = (1, -1)$ family of reducible modules $\Leftrightarrow$ $\theta_0 + \theta_\infty \geq 2$;

(d) the $(\epsilon_1, \epsilon_2) = (-1, 1)$ family of reducible modules $\Leftrightarrow$ $\theta_0 + \theta_\infty \leq 0$;

(e) the $(\epsilon_1, \epsilon_2) = (-1, -1)$ family of reducible modules $\Leftrightarrow$ $\theta_0 - \theta_\infty \leq 0$.

3.4.4 $\mathcal{M}(\theta_0, \theta_\infty)$ for $\alpha = \beta^{\pm 1}$

Let $\mathcal{R}(\theta_0 . \theta_\infty)$ denote the open subspace of $\mathcal{R}(\alpha, \beta)^*$ which corresponds to the subset $\mathbf{S}(\theta_0, \theta_\infty)$ of $\mathbf{S}^*(\alpha, \beta)$, defined in Section 3.4.3. By Sections 3.4.1–3.4.3, the extended Riemann–Hilbert morphism $\tilde{\mathcal{M}}(\theta_0, \theta_\infty) \to \mathcal{R}(\theta_0, \theta_\infty) \times \tilde{T}$ is a well defined analytic isomorphism. This has as consequence:

Theorem 3.3 holds for the cases $\frac{\theta_0}{2} \pm \frac{\theta_\infty}{2} \in \mathbb{Z}$ with $\mathcal{R}(\alpha, \beta)$ replaced by $\mathcal{R}(\theta_0, \theta_\infty)$.

3.4.5 Isomonodromy for reducible connections

The fibers of the locally defined map $\mathbf{S}(\alpha, \alpha^{\pm 1})^*_{\rm red} \to \mathcal{R}(\alpha, \alpha^{\pm 1})^*_{\rm red}$ are the isomonodromy families of reducible modules. As a start, we consider the reducible familly of type $(\epsilon_1, \epsilon_2) = (1, 1)$ lying in $\mathcal{M}(2d+2, 2d)$. This family is represented by $z\frac{d}{dz} + \begin{pmatrix} -\frac{tz^{-1}+tz}{2} - d & 0 \\ z - q & \frac{tz^{-1}+tz}{2} + d \end{pmatrix}$. For an isomonodromy subfamily of this, q is a function of t and the Stokes data at 0 and ∞ and the link are fixed. Isomonodromy is equivalent to the statement that the above matrix differential operator commutes with an operator of the form $\frac{d}{dt} + B_{-1}z + B_0 + B_1 z$, where the tracefree 2×2 matrices B_{-1}, B_0, B_1 depend on t only. This leads to the equation

$$\begin{pmatrix} -\frac{z^{-1}+z}{2} & 0 \\ -q' & \frac{z^{-1}+z}{2} \end{pmatrix} = -B_1 z^{-1} + B_1 z - \left[B_{-1}z^{-1} + B_0 + B_1 z, \begin{pmatrix} -\frac{tz^{-1}+tz}{2} - d & 0 \\ z - q & \frac{z^{-1}+z}{2} + d \end{pmatrix} \right].$$

A computation yields the equation $q' = -2q^2 - \frac{4d-1}{t} q - 2$. The solutions of this equation have the form $\frac{1}{2}\frac{y'}{y}$, where y is a non zero solution of the Bessel equation $y'' + \frac{4d-1}{t} y' + 4y = 0$. One obtains in a similar way for an isomonodromic family of reducible modules of type (ϵ_1, ϵ_2) the equation

$$q' = -2\epsilon_2 q^2 - \frac{4d-1}{t} q - 2\epsilon_1.$$

The solutions are $q = \frac{\epsilon_2}{2}\frac{y'}{y}$ where y is a solution of the Bessel equation $y'' + \frac{4d+1}{t}y' + 4\epsilon_1\epsilon_2 y = 0$. These equations are consistent with the formula of Theorem 3.3

$$q'' = \frac{(q')^2}{q} - \frac{q'}{t} - \frac{4(\theta_0 - 1)}{t} + \frac{4\theta_\infty q^2}{t} + 4q^3 - \frac{4}{q}$$

for isomonodromic families in $\mathcal{M}(\theta_0, \theta_\infty)$. According to [5] we found in this way all Riccati solutions for PIII(D_6), up to the action of the Bäcklund transformations.

Remark 3.8. The assumption that a function q satisfies two distinct PIII(D_6) equations leads to $q^4 = 1$. Thus we found the algebraic solutions $q = \pm 1$ for $\theta_\infty = \theta_0 - 1$ and $q = \pm i$ for $-\theta_\infty = \theta_0 - 1$. According to [5] these are all the algebraic solutions of PIII(D_6), up to the action of the Bäcklund transformations.

4 Bäcklund transformations for PIII(D_6)

4.1 Automorphisms of S

We start with a table of generators for the group Aut($\mathbf{S}$) of 'natural' automorphism of $\mathbf{S}$, in terms of their action on the parameters α, β and on t, z.

	α	β	t	z
σ_1	α^{-1}	β^{-1}	$-t$	z
σ_2	$-\alpha$	$-\beta$	t	z
σ_3	α	β^{-1}	it	iz
σ_4	β	α	t	z^{-1}

These generators are defined as follows.

1. $\sigma_1 : (M, t) \mapsto (M, -t)$. This induces bijections $\mathbf{S}(\alpha, \beta) \to \mathbf{S}(\alpha^{-1}, \beta^{-1})$. Indeed, the basis vectors e_1, e_2 of $V(0)$ are interchanged and the same holds for the basis f_1, f_2 of $V(\infty)$.

2. Define the differential module $N = \mathbb{C}(z)b$ by $\delta b = \frac{1}{2}b$. Then $\sigma_2 : (M, t) \mapsto (M \otimes N, t)$. Since $\Lambda^2(M \otimes N) = N^{\otimes 2}$ is the trivial module, $(M \otimes N, t)$ belongs to $\mathbf{S}$. Let E_1, E_2 be a basis of $\mathbb{C}((z)) \otimes M$ such that $\delta E_1 = -\frac{tz^{-1}+\theta_0}{2}E_1$ and $\delta E_2 = \frac{tz^{-1}+\theta_0}{2}E_2$. Then the formal module $\mathbb{C}((z)) \otimes (M \otimes N)$ has basis $E_1 \otimes b$, $E_2 \otimes b$ and $\delta(E_1 \otimes b) = (-\frac{tz^{-1}+\theta_0}{2} + \frac{1}{2})(E_1 \otimes b)$ and similarly $\delta(E_2 \otimes b) = (\frac{tz^{-1}+\theta_0}{2} + \frac{1}{2})(E_2 \otimes b)$. Thus $e^{\pi i(\theta_0+1)} = -\alpha$ is the eigenvalue of the formal monodromy at $z = 0$. The same argument shows that $-\beta$ is the eigenvalue of the formal monodromy at $z = \infty$.

3. Let ϕ be a $\mathbb{C}$-linear automorphism of the field $\mathbb{C}(z)$, such that $z\frac{d}{dz} \circ \phi = \mu \cdot \phi \circ z\frac{d}{dz}$ for some $\mu \in \mathbb{C}^*$. There are two possibilities:

(a) $\phi(z) = cz$ with $c \in \mathbb{C}^*$, $\mu = 1$, (b) $\phi(z) = cz^{-1}$ with $c \in \mathbb{C}^*$, $\mu = -1$.

For $(M, t) \in \mathbf{S}$ one considers $(\mathbb{C}(z) \otimes_\phi M, \dots)$. As additive group $\mathbb{C}(z) \otimes_\phi M$ is identified with M. Its structure as vector space is given by the new scalar multiplication $f * m := \phi(f)m$. In case (a), the differential structure is given by the original δ and in case (b) the differential structure is given by $-\delta$.

(a) $\phi(z) = cz$. The formal local module $\mathbb{C}((z)) \otimes M$ with basis E_1, E_2 and $\delta E_1 = -\frac{tz^{-1}+\theta_0}{2}E_1$, $\delta E_2 = \frac{tz^{-1}+\theta_0}{2}E_2$ is transformed into $\mathbb{C}((z)) \otimes_\phi M$. Now $\delta E_1 = -\frac{tcz^{-1}+\theta_0}{2} * E_1$ and $\delta E_2 = \frac{tcz^{-1}+\theta_0}{2} * E_2$. The basis F_1, F_2 of $\mathbb{C}((z^{-1})) \otimes M$ with $\delta F_1 = -\frac{tz+\theta_\infty}{2}F_1$ and $\delta F_2 = \frac{tz+\theta_\infty}{2}F_2$, yields for the new structure the formulas $\delta F_1 = -\frac{tc^{-1}z+\theta_\infty}{2} * F_1$ and $\delta F_2 = \frac{tc^{-1}z+\theta_\infty}{2} * F_2$.

The condition $tc = \pm tc^{-1}$ implies that $c^4 = 1$. We define σ_3 by $\phi(z) = iz$. This yields the new module $(\mathbb{C}(z) \otimes_\phi M, it)$ with new α equal to $e^{\pi i\theta_0}$ and new β equal to $e^{-\pi i\theta_\infty}$.

(b) We define σ_4 by $\phi(z) = z^{-1}$. The new module $(\mathbb{C}(z) \otimes_\phi M, t)$ has new $\alpha = e^{\pi i\theta_\infty}$ and new $\beta = e^{\pi i\theta_0}$.

Comments. Using the first three columns of the table we will consider $\mathrm{Aut}(\mathbf{S})$ as an automorphism group of the algebraic variety $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$. A straightforward computation shows that $\mathrm{Aut}(\mathbf{S})$ is the product $\langle\sigma_1\rangle \times \langle\sigma_2\rangle \times \langle\sigma_3,\sigma_4\rangle$, where $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ have order two and $\langle\sigma_3,\sigma_4\rangle$ is the dihedral group D_4 of order eight.

The action of $\mathrm{Aut}(\mathbf{S})$ *on the monodromy data.*

1. σ_1. The map $t \mapsto -t$ has as consequence that the basis vectors e_1, e_2 of $V(0)$ are permuted and the same holds for $V(\infty)$. The singular directions and the Stokes maps do not change. The new topological monodromies are $\begin{pmatrix} \frac{1+a_1a_2}{\alpha} & \frac{a_1}{\alpha} \\ \alpha a_2 & \alpha \end{pmatrix}$ at $z = 0$ and $\begin{pmatrix} \frac{1+b_1b_2}{\beta} & \frac{b_1}{\beta} \\ \beta b_2 & \beta \end{pmatrix}$ at $z = \infty$. The matrix of the link L is now $\begin{pmatrix} \ell_4 & \ell_3 \\ \ell_2 & \ell_1 \end{pmatrix}$. The matrix relation remains the same. This amounts to the change $\alpha \mapsto \alpha^{-1}$, $\beta \mapsto \beta^{-1}$, $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $\ell_1 \leftrightarrow \ell_4$, $\ell_2 \leftrightarrow \ell_3$. This induces an automorphism of $\mathcal{R}$ given by the formula $(\alpha,\beta,x_1,x_2,x_3) \mapsto (\alpha^{-1},\beta^{-1},x_1\alpha^{-1}\beta^{-1},x_2\alpha^{-1}\beta^{-1},x_3)$.

2. σ_2. The map $(M,t) \mapsto (M \otimes N, t)$ has as consequence that the formal monodromies at $z = 0$ and $z = \infty$ are multiplied by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (and thus $\alpha \mapsto -\alpha$, $\beta \mapsto -\beta$). The Stokes matrices do not change. The same holds for the link L. The induced automorphism of $\mathcal{R}$ is given by the formula $(\alpha,\beta,x_1,x_2,x_3) \mapsto (-\alpha,-\beta,x_1,-x_2,-x_3)$.

3. σ_3. The effect of this transformation is: the singular directions change over $\frac{\pi}{2}$; the basis vectors of $V(0)$ are permuted; the basis of $V(\infty)$ is unchanged; the Stokes map and the link remain the same, however the corresponding matrices change. The induced automorphism of $\mathcal{R}$ is given by $(\alpha,\beta,x_1,x_2,x_3) \mapsto (\alpha^{-1},\beta,\alpha^{-1}x_1,\alpha^{-1}x_2,x_3)$.

4. σ_4. The spaces $V(0)$ and $V(\infty)$ are interchanged; the other data are unchanged. The induced automorphism of $\mathcal{R}$ is given by $(\alpha,\beta,x_1,x_2,x_3) \mapsto (\beta,\alpha,x_1,x_2,x_3)$.

4.2 Bäcklund transformations

Define the map $exp : \mathbb{C}^3 \to (\mathbb{C}^*)^3$ by $(\theta_0,\theta_\infty,\tilde{t}) \mapsto (\alpha,\beta,t) := (e^{\pi i\theta_0}, e^{\pi i\theta_\infty}, e^{\tilde{t}})$. We will define the group $B(\mathbf{S})$ of the *Bäcklund transformations* as the affine automorphisms of $\mathbb{C}^3$ which respect the equivalence relation defined by exp and which map to elements of $\mathrm{Aut}(\mathbf{S})$. By definition there is an exact sequence of groups $0 \to B(\mathbf{S})_0 \to B(\mathbf{S}) \to \mathrm{Aut}(\mathbf{S}) \to 1$, where $B(\mathbf{S})_0$ is the group of affine transformations of $\mathbb{C}^3$, generated by: $B_1 : (\theta_0,\theta_\infty,\tilde{t}) \mapsto (2+\theta_0,\theta_\infty,\tilde{t})$, $B_2 : (\theta_0,\theta_\infty,\tilde{t}) \mapsto (\theta_0,2+\theta_\infty,\tilde{t})$ and $B_3 : (\theta_0,\theta_\infty,\tilde{t}) \mapsto (\theta_0,\theta_\infty,2\pi i+\tilde{t})$.

The aim is to give each Bäcklund transformation the interpretation of a morphism between the various moduli spaces $\tilde{\mathcal{M}}(*,*)$, preserving the foliations by isomonodromic families, and to compute the effect on solutions of PIII(D_6). First we investigate the group $B(\mathbf{S})$.

The affine map B_3 is not considered in the literature. Its action on $\tilde{\mathcal{M}}(\theta_0,\theta_\infty)$ is obvious since $\tilde{\mathcal{M}}(\theta_0,\theta_\infty) = \mathcal{M}(\theta_0,\theta_\infty) \times_T \tilde{T}$ and $\tilde{T} = \mathbb{C} \to T = \mathbb{C}^*$ is the map $\tilde{t} \mapsto e^{\tilde{t}}$. The effect of B_3 on solutions of PIII(D_6) is far from obvious. A solution is a function $q(\tilde{t}) =$ "$q(e^{\tilde{t}})$". Since the equation depends only on t, the function $q(2\pi i+\tilde{t})$ satisfies the same equation. It seems that no formula for $q(2\pi i+\tilde{t})$ in terms of $q(\tilde{t})$ and its derivative is present in the literature.

Generators for $B(\mathbf{S})$ are given by their action on $\mathbb{C}^3$ and the variables t, z.

	θ_0	θ_∞	$\tilde{t}$	t	z
s_1	$2-\theta_0$	$-\theta_\infty$	$\pi i+\tilde{t}$	$-t$	z
s_2	$1+\theta_0$	$1+\theta_\infty$	$\tilde{t}$	t	z
s_3	θ_0	$-\theta_\infty$	$\frac{\pi i}{2}+\tilde{t}$	it	iz
s_4	θ_∞	θ_0	$\tilde{t}$	t	z^{-1}

These elements generate $B(\mathbf{S})$ because: s_* is mapped to σ_* for $* = 1,2,3,4$; $B_3 = s_1^2 = s_3^4$; $B_1 = s_3 s_1^{-1} s_4 s_3 s_4$ and $B_2 = B_1^{-1} s_2^2$.

The group $\langle B_3 \rangle$, generated by B_3, is isomorphic to $\mathbf{Z}$ and lies in the center of $B(\mathbf{S})$. Put $\overline{B(\mathbf{S})} = B(\mathbf{S})/\langle B_3 \rangle$ and let $\overline{s}_*$ denote the image of s_* in this quotient. Then $\overline{s}_3^2$ has order two and lies in the center of $\overline{B(\mathbf{S})}$.

We compare this with Okamoto's paper [7]. The group of the Bäcklund transformations B of equation PIII$'$(D$_6$)

$$\frac{d^2Q}{dx^2} = \frac{1}{Q}\left(\frac{dQ}{dx}\right)^2 - \frac{1}{x}\frac{dQ}{dx} + \frac{Q^2(\gamma Q + \alpha)}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4Q}$$

is computed to be the affine Weyl group of type B_2. The substitution $x = t^2$, $Q = tq$ transforms the equation into

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{\alpha q^2 + \beta}{t} + \gamma q^3 + \frac{\delta}{q},$$

and thus in our notation $\alpha = 4\theta_\infty$, $\beta = -4(\theta_0 - 1)$, $\gamma = 4$, $\delta = -4$. In a sense, PIII(D$_6$) is a degree two covering of PIII$'$(D$_6$). It can be seen that there is a surjective homomorphism $\overline{B(\mathbf{S})} \to B$ with kernel $\langle \overline{s}_3^2 \rangle$.

Our approach using moduli spaces explains the Bäcklund transformations presented in [4]. In contrast to this, the new transformations of [13] do not seem to have a simple modular interpretation.

4.3 Bäcklund transformations of the moduli spaces

Let $s \in B(\mathbf{S})$ have image $\sigma \in \mathrm{Aut}(\mathbf{S})$. Choose α, β and write $\alpha' = \sigma(\alpha)$, $\beta' = \sigma(\beta)$. Choose θ_0, θ_∞, θ_0', θ_∞' such that $e^{\pi i \theta_0} = \alpha, \dots, e^{\pi i \theta_\infty'} = \beta'$ and $s(\theta_0) = \theta_0'$, $s(\theta_\infty) = \theta_\infty'$. Now σ induces a bijection $\mathbf{S}(\alpha,\beta) \xrightarrow{\sigma} \mathbf{S}(\alpha',\beta')$ and consider

$$\mathcal{M}(\theta_0,\theta_\infty) \to \mathbf{S}(\alpha,\beta) \xrightarrow{\sigma} \mathbf{S}(\alpha',\beta') \leftarrow \mathcal{M}(\theta_0',\theta_\infty').$$

The first and the last arrow are injective. Their images contain the locus of the irreducible modules and the loci of the (ϵ_1,ϵ_2)-reducible modules depending on the $\theta_0,\dots,\theta_\infty'$ (see Section 3.4.3). Thus we obtain a, maybe partially defined, map, again denoted by $s : \mathcal{M}(\theta_0,\theta_\infty) \to \mathcal{M}(\theta_0',\theta_\infty')$. It can be seen from the definitions of the elements of $\mathrm{Aut}(\mathbf{S})$ that the Bäcklund transformation s is a birational algebraic map. Using the expression for $s(\tilde{t})$ one obtains a birational morphism, again denoted by $s : \tilde{\mathcal{M}}(\theta_0,\theta_\infty) \to \tilde{\mathcal{M}}(\theta_0',\theta_\infty')$ which respects the foliations (i.e., the isomonodromic families). In particular, s maps 'generic' solutions for the parameters θ_0, θ_∞ and the variable $\tilde{t}$ to 'generic' solutions for the parameters θ_0', θ_∞' and the variable $s(\tilde{t})$. The term 'generic' means here the solutions corresponding to irreducible modules and the ones for (ϵ_1,ϵ_2)-reducible modules (i.e., Riccati solutions) which are present in both $\mathcal{M}(\theta_0,\theta_\infty)$ and $\mathcal{M}(\theta_0',\theta_\infty')$.

4.4 Formulas for the Bäcklund transformations

4.4.1 $s_1 : \tilde{\mathcal{M}}(\theta_0,\theta_\infty) \to \tilde{\mathcal{M}}(2-\theta_0,-\theta_\infty)$

Using the first charts of the two spaces and their variables, s_1 has the form $(a_{-1}, b_{-2}, \dots, b_1, c_0, \tilde{t}) \mapsto (a_{-1}, b_{-2}, \dots, b_1, c_0, \tilde{t} + i\pi)$. The formula for s_1 on the second charts is similar. The induced map for the PIII(D$_6$) equations is $t \mapsto -t$, $\frac{d}{dt} \mapsto -\frac{d}{dt}$, $q(\tilde{t}) \mapsto q(\tilde{t} + i\pi)$.

4.4.2 $s_2 : \boldsymbol{\mathcal{M}}(\theta_0,\theta_\infty) \to \boldsymbol{\mathcal{M}}(1+\theta_0, 1+\theta_\infty)$

A point on the first chart of the first space is represented by the differential operator $z\frac{d}{dz} + \begin{pmatrix} & az^{-1}b \\ z-q-az^{-1} & \end{pmatrix}$, where $a := a_{-1}$, $q := -c_0$, $b = b_1 z + b_0 + b_{-1}z^{-1} + b_{-2}z^{-2}$ and the $b_1, \dots, b_{-2}$ are polynomials in t, q, q^{-1}, using the notation of Section 3.3.1. This is transformed by s_2 into the operator $z\frac{d}{dz} + A$ with $A = \begin{pmatrix} az^{-1} + \frac{1}{2} & b \\ z-q-az^{-1} & +\frac{1}{2}\end{pmatrix}$. We want to compute an operator $z\frac{d}{dz} + \tilde{A}$ with $\tilde{A} = \begin{pmatrix} \tilde{a}z^{-1} & \tilde{b} \\ z-\tilde{q} & -\tilde{a}z^{-1}\end{pmatrix}$, $\tilde{b} = \tilde{b}_1 z + \tilde{b}_0 + \tilde{b}_{-1}z^{-1} + \tilde{b}_{-2}z^{-2}$ and $\tilde{b}_1, \dots, \tilde{b}_{-2}$ polynomials in t, $\tilde{q}$, $\tilde{q}^{-1}$, representing a point on the first chart of $\mathcal{M}(\frac{1}{2} + \frac{\theta_0}{2}, \frac{1}{2} + \frac{\theta_\infty}{2})$, which is equivalent to $z\frac{d}{dz} + A$. Thus we have to solve an equation of the type $\{z\frac{d}{dz} + A\}T = T\{z\frac{d}{dz} + \tilde{A}\}$ with $T \in \mathrm{GL}(2, \mathbb{C}(z))$. A local computation shows that T has the form $T_0 + T_{-1}z^{-1} + T_{-2}z^{-2} \neq 0$ with 'constant' matrices T_0, T_{-1}, T_{-2}. A Maple computation yields the solution

$$\tilde{q} = -\frac{tq^2 - q\theta_0 - t + 2a}{q(tq^2 + q\theta_\infty - t + 2a)}, \qquad \tilde{a} = \frac{\text{long}}{2q^2(tq^2 + q\theta_\infty - t + 2a)^2},$$

$$\begin{aligned}\text{long} = {} & 8a^3 - 4aq^2t^2 + 8a^2q^2t - qt^2 + 2aq^4t^2 - 8a^2t + 2at^2 - 4a^2q + 4aqt - q^5t + qt^2\theta_0 \\ & - q^5t^2\theta_0 + q^2t\theta_0^2 - 4a^2q\theta_0 + 2aq^2\theta_0 + q^4t\theta_0 - q^2t\theta_0 - q^5t^2\theta_\infty - 4q^4t\theta_\infty^2 + 4a^2q\theta_\infty \\ & + qt^2\theta_\infty - 2aq^2\theta_\infty - q^4t\theta_\infty + q^2t^2\theta_\infty + q^3\theta_0\theta_\infty - 4aq^3t\theta_0 - 4aqt\theta_\infty + q^2t\theta_0\theta_\infty \\ & - q^4t\theta_0\theta_\infty - 2aq^2\theta_0\theta_\infty - 4aq^3t + 2q^3t.\end{aligned}$$

The induced map for solutions of PIII(D$_6$) is obtained from the formula for $\tilde{q}$ and the equality $q' = \frac{4a-q}{t}$.

Comments on the formulas. The term q in the denominator of the formulas is due to our choice of working on the first charts of the spaces $\mathcal{M}(\theta_0,\theta_\infty)$ and $\mathcal{M}(\theta_0+1,\theta_\infty+1)$. This term does not produce singularities for s_2.

The denominator $(tq^2 + q\theta_\infty - t + 2a)$ is due to a reducible locus of type $(\epsilon_1,\epsilon_2) = (-1,1)$. More precisely, $tq^2 + q\theta_\infty - t + 2a = 0$ describes the reducible locus of $\mathcal{M}(\theta_0,\theta_\infty)$ if and only if $\theta_0 + \theta_\infty \in 2\mathbb{Z}$ and $\theta_0 + \theta_\infty \leq 0$.

A priori, s_2 is not defined on this locus if moreover $\theta_0 + \theta_\infty = 0$. One computes that for $\theta_\infty + \theta_0 = 0$ the formulas reduce to the rational map $\tilde{q} = -q^{-1}$, $\tilde{a} = \frac{-q+2a}{2q^2}$ and thus s_2 is well defined on this locus of type $(\epsilon_1,\epsilon_2) = (-1,1)$. In fact, s_2 maps this locus to the reducible locus of type $(\epsilon_1,\epsilon_2) = (1,-1)$, which is present in $\mathcal{M}(\theta_0+1,\theta_\infty+1)$.

4.4.3 $s_3 : \tilde{\boldsymbol{\mathcal{M}}}(\theta_0,\theta_\infty) \to \tilde{\boldsymbol{\mathcal{M}}}(\theta_0,-\theta_\infty)$

On the first charts the map is given by: $(a_{-1}, b_{-2}, \dots, b_1, c_0, \tilde{t}) \mapsto (-ia_{-1}, -ib_{-2}, b_{-1}, ib_0, -b_1, -ic_0, \tilde{t} + i\frac{\pi}{2})$ and similarly on the second charts. For the PIII(D$_6$) equations, the map is $t \mapsto it$, $\frac{d}{dt} \mapsto -i\frac{d}{dt}$, $q(\tilde{t}) \mapsto -iq(\tilde{t} + i\frac{\pi}{2})$.

4.4.4 $s_4 : \boldsymbol{\mathcal{M}}(\theta_0,\theta_\infty) \to \boldsymbol{\mathcal{M}}(\theta_\infty,\theta_0)$

Consider a point on the chart $c_1 \neq 0$ of $\mathcal{M}(\theta_0,\theta_\infty)$ (see Section 3.3.1) represented by $z\frac{d}{dz} + A(z)$, where the entries of $A(z)$ are polynomial in z, z^{-1}, t, $a := a_{-1}$, $q := -c_0$, q^{-1}. Now s_4 transforms this operator into $z\frac{d}{dz} - A(z^{-1})$. We suppose that a transformation $T := T_{-1}z^{-1} + T_0 + T_1 z \neq 0$ brings this operator into a point of the chart $c_1 \neq 0$ of $\mathcal{M}(\theta_\infty,\theta_0)$ represented by an operator $z\frac{d}{dz} + \tilde{A}(z)$, where the entries of $\tilde{A}(z)$ are polynomials in z, z^{-1}, t, $\tilde{a} = \tilde{a}_{-1}$, $\tilde{q} = -\tilde{c}_0$, $\tilde{q}^{-1}$. A Maple computation shows that there is a unique solution in terms of $\tilde{q}$ and $\tilde{a}$ of the equation

$\{z\frac{d}{dz} - A(z^{-1})\}T = T\{z\frac{d}{dz} + \tilde{A}(z)\}$, namely

$$\tilde{q} = \frac{q(-q^2t - \theta_\infty q - t + 2a)}{(-q^2t - \theta_0 q - t + 2a)}, \qquad \tilde{a} = \frac{\text{long}}{2(-q^2t - \theta_0 q - t + 2a)^2},$$

$$\begin{aligned}\text{long} = {} & 4q^2at^2 + qt^2\theta_0 + q^5t^2\theta_\infty + 4q^3at\theta_0 - q^5t^2\theta_0 - q^4t\theta_\infty\theta_0 - q^2t\theta_0\theta_\infty + 4qt\theta_\infty a \\ & + q^2t\theta_0^2 + q^4t\theta_\infty^2 - t^2q\theta_\infty - 4q\theta_0 a^2 - 4a^2\theta_\infty q + 2a\theta_0 q^2\theta_\infty - 8a^2q^2t + 2aq^4t^2 \\ & + 2at^2 - 8a^2t + 8a^3.\end{aligned}$$

The induced map for solutions of PIII(D_6) is obtained from the formula for $\tilde{q}$ and the equality $q' = \frac{4a-q}{t}$.

Comments on the formulas. The denominator in these formulas is due to a possible reducible locus of type $(\epsilon_1, \epsilon_2) = (-1, -1)$. This locus is present in $\mathcal{M}(\theta_0, \theta_\infty)$ if and only if $\theta_0 - \theta_\infty \in 2\mathbb{Z}$ and $\theta_0 - \theta_\infty \leq 0$. This locus is not present in $\mathcal{M}(\theta_\infty, \theta_0)$ if moreover $\theta_0 - \theta_\infty < 0$. However, in the critical case $\theta_0 = \theta_\infty$, s_4 turns out to be the identity.

If $\theta_0 - \theta_\infty \in 2\mathbb{Z}$ and $\theta_0 - \theta_\infty \geq 2$, then the reducible modules of type $(1, 1)$ are present in $\mathcal{M}(\theta_0, \theta_\infty)$ and not in $\mathcal{M}(\theta_\infty, \theta_0)$. The corresponding term $2a + tq^2 + (2d - 1)q + t$ in the denominator of the map s_4 does not occur, because s_4 maps this reducible locus to the reducible locus of type $(-1, -1)$ of $\mathcal{M}(\theta_\infty, \theta_0)$.

Using the formulas for $s_1, \ldots, s_4$ one can deduce formulas for B_1 and B_2. We will however derive these by the direct method used for s_2 and s_4.

4.4.5 The transformation $B_1 : \theta_0 \mapsto 2 + \theta_0$, $\theta_\infty \mapsto \theta_\infty$, $\tilde{t} \mapsto \tilde{t}$

We compute the birational map B_1 on the open part of the chart $c_1 \neq 0$ of $\mathcal{M}(\theta_0, \theta_\infty)$ where $q = -c_0 \neq 0$ (see Section 3.3.1). A point is represented by an operator $z\frac{d}{dz} + A$. The entries of A are polynomials in $q = -c_0$, q^{-1} and $a = a_{-1}$. Further we suppose that the image under B_1 lies in the open part of $\mathcal{M}(2 + \theta_0, \theta_\infty)$ given by $c_1 \neq 0$ and has the form $z\frac{d}{dz} + \tilde{A}$, where the entries of $\tilde{A}$ are polynomials in $\tilde{q}$, $\tilde{q}^{-1}$ and $\tilde{a} = \tilde{a}_{-1}$. Since the two differential operators represent the same differential module, there exists a $T \in \mathrm{GL}(2, \mathbb{C}(z))$ such that $z\frac{d}{dz} + \tilde{A} = T^{-1}(z\frac{d}{dz} + A)T$.

Local calculations at $z = 0$ and $z = \infty$ predict that T has the form $T = T_{-2}z^{-2} + T_{-1}z^{-1} + T_0$, $T_{-2} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ and $\det T \in \mathbb{C}^*$.

Further it is assumed that $z\frac{d}{dz} + A$ is not (ϵ_1, ϵ_2)-reducible for the critical cases $(\epsilon_1, \epsilon_2) = (-1, -1)$, $\theta_\infty = \theta_0$ and $(\epsilon_1, \epsilon_2) = (-1, 1)$, $\theta_\infty = -\theta_0$, where this reducible locus is present in $\mathcal{M}(\theta_0, \theta_\infty)$ and not in $\mathcal{M}(2 + \theta_0, \theta_\infty)$. Maple produces the formulas

$$\tilde{q} = \frac{q(-4a^2 + 4at - t^2 + \theta_0^2q^2 + 2t\theta_\infty q^3 + t^2q^4)}{(2a - t - \theta_0 q + tq^2)(2a - t - \theta_0 q - tq^2)},$$

$$\tilde{a} = \frac{\text{long}}{(2a - t - \theta_0 q + tq^2)^2(2a - t - \theta_0 q - tq^2)^2},$$

where "long" means too long for copying. Substitution of $a = \frac{tq'+q}{4}$ in the formula for $\tilde{q}$ yields the formula for B_1 with respect to solutions.

In the first critical case $(\epsilon_1, \epsilon_2) = (-1, -1)$, $\theta_\infty = \theta_0$ one obtains

$$\tilde{q} = -\frac{q(2a - t + \theta_0 q + tq^2)}{2a - t - \theta_0 q + tq^2}, \qquad \tilde{a} = \frac{\text{long}}{(2a - t - \theta_0 q + tq^2)^2}.$$

The reducible locus is given by $a = \frac{t}{2} + \frac{\theta_0}{2}q + \frac{t}{2}q^2$. Thus the above map extends to the reducible locus and produces there the formulas

$$\tilde{q} = -\frac{q(2tq + 2\theta_0 + 1)}{2tq + 1}, \qquad \tilde{a} = \frac{\text{long}}{(2tq + 1)^2}.$$

For the second critical case $(\epsilon_1,\epsilon_2)=(-1,1)$, $\theta_\infty=-\theta_0$ one finds

$$\tilde{q}=\frac{q(-2a+t-\theta_0 q+tq^2)}{2a-t-\theta_0 q-tq^2},\qquad \tilde{a}=\frac{\text{long}}{(2a-t-\theta_0 q+tq^2)^2}.$$

The reducible locus is given by $a=\frac{t}{2}+\frac{\theta_0}{2}q-\frac{t}{2}q^2$. On this locus one has $\tilde{q}=-q+\frac{\theta_0}{t}$.

Comment. As in Sections 4.4.2 and 4.4.4, the map B_1 is well defined on the reducible loci because B_1 changes the type (ϵ_1,ϵ_2) of the reducible loci.

4.4.6 The transformation $B_2:\theta_0\mapsto\theta_0,\ \theta_\infty\mapsto 2+\theta_\infty,\ \tilde{t}\mapsto\tilde{t}$

Let an object of $\mathcal{M}(\theta_0,\theta_\infty)$ be represented by a standard operator $z\frac{d}{dz}+A$. We expect that the transformation B_2 yields an object of $\mathcal{M}(\theta_0,2+\theta_\infty)$, represented by a standard operator $z\frac{d}{dz}+\tilde{A}$. Then $z\frac{d}{dz}+\tilde{A}=T^{-1}(z\frac{d}{dz}+A)T$ for a certain $T\in\mathrm{GL}(2,\mathbb{C}(z))$.

Local calculations at $z=0$ and $z=\infty$ show that T has the form $T_{-1}z^{-1}+T_0+T_1z$ with $T_{-1}=\begin{pmatrix}0&*\\0&0\end{pmatrix}$, $\det T_1=0$, $\det T\in\mathbb{C}^*$.

The matrix A depends on q and $a:=a_{-1}$ and the matrix $\tilde{A}$ depends on $\tilde{q}$ and $\tilde{a}:=\tilde{a}_{-1}$. We have to solve the equation $T(z\frac{d}{dz}+\tilde{A})=(z\frac{d}{dz}+A)T$.

Maple produced the formula

$$\tilde{q}=-\frac{(2a+t+\theta_\infty q+tq^2)(2a-t+\theta_\infty q+tq^2)q}{4aq^2t+2qt-t^2-2q^3t-q^2\theta_\infty^2+4a^2-4aq-2qt\theta_0-2q^2\theta_\infty+t^2q^4}.$$

The denominator of $\tilde{a}$ is the square of the denominator of $\tilde{q}$ and the numerator of $\tilde{a}$ is too large to copy here. The substitution of $a=\frac{tq'+q}{4}$ in the formula for $\tilde{q}$ yields the B_2 map for the solutions of PIII(D_6).

The two cases where B_2 is, *a priori*, not defined on the reducible locus are:

1. $(\epsilon_1,\epsilon_2)=(1,1)$ and $\frac{\theta_0}{2}=\frac{\theta_\infty}{2}+1$. The reducible locus is given by $2a+t+\theta_\infty q+tq^2=0$. After substitution of $\frac{\theta_0}{2}=\frac{\theta_\infty}{2}+1$, the denominator of $\tilde{q}$ factors as $(2a+t+\theta_\infty q+tq^2)(2a-t-\theta_\infty q-2q+tq^2)$. Further $\tilde{q}=-\frac{(2a-t+\theta_\infty q+tq^2)q}{2a-t-\theta_\infty q-2q+tq^2}$ On the reducible locus one has $\tilde{q}=\frac{tq}{t+\theta_\infty q+q}$ and B_2 is well defined.

2. $(\epsilon_1,\epsilon_2)=(-1,1)$ and $\frac{\theta_0}{2}=-\frac{\theta_\infty}{2}$. The reducible locus is given by $2a-t+\theta_\infty q+tq^2=0$. After substitution of $\frac{\theta_0}{2}=-\frac{\theta_\infty}{2}$, the denominator of $\tilde{q}$ factors as $(2a-t+\theta_\infty q+tq^2)(2a+t-\theta_\infty q-2q+tq^2)$. Further $\tilde{q}=-\frac{(2a+t+\theta_\infty q+tq^2)q}{2a+t-\theta_\infty q-2q+tq^2}$. On the reducible locus one has $\tilde{q}=\frac{tq}{t-\theta_\infty q-q}$ and B_2 us well defined.

Comment. As in Sections 4.4.2, 4.4.4 and 4.4.5, the map B_2 is well defined because B_2 changes the types (ϵ_1,ϵ_2) of the reducible loci.

Acknowledgments

The authors thank Yousuke Ohyama for his helpful answers to our questions and his remarks concerning the tau-divisor (see Section 3.3.4).

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