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Mathematical Analysis of a Generalized Chiral Quark Soliton Model

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Abstract. A generalized version of the so-called chiral quark soliton model (CQSM) in nuclear physics is introduced. The Hamiltonian of the generalized CQSM is given by a Dirac type operator with a mass term being an operator-valued function. Some mathematically rigorous results on the model are reported. The subjects included are: (i) supersymmetric structure; (ii) spectral properties; (iii) symmetry reduction; (iv) a unitarily equivalent model.

Key words: chiral quark soliton model; Dirac operator; supersymmetry; ground state; symmetry reduction

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1 Introduction

The chiral quark soliton model (CQSM) [5] is a model describing a low-energy effective theory of the quantum chromodynamics, which was developed in 1980's (for physical aspects of the model, see, e.g., [5] and references therein). The Hamiltonian of the CQSM is given by a Dirac type operator with iso-spin, which differs from the usual Dirac type operator in that the mass term is a matrix-valued function with an effect of an interaction between quarks and the pion field. It is an interesting object from the purely operator-theoretical point of view too. But there are few mathematically rigorous analyses for such Dirac type operators (e.g., [2], where the problem on essential self-adjointness of a Dirac operator with a variable mass term given by a scalar function is discussed).

In the previous paper [1] we studied some fundamental aspects of the CQSM in a mathematically rigorous way. In this paper we present a slightly general form of the CQSM, which we call a *generalized CQSM*, and report that results similar to those in [1] hold on this model too, at least, as far as some general aspects are concerned.

2 A Generalized CQSM

The Hilbert space of a Dirac particle with mass $M > 0$ and iso-spin $1/2$ is taken to be $L^2(\mathbb{R}^3;\mathbb{C}^4) \otimes \mathbb{C}^2$. For a generalization, we replace the iso-spin space $\mathbb{C}^2$ by an abitrary complex Hilbert space $\mathcal{K}$. Thus the Hilbert space $\mathcal{H}$ in which we work in the present paper is given by

$$\mathcal{H} := L^2(\mathbb{R}^3;\mathbb{C}^4) \otimes \mathcal{K}.$$

We denote by $\mathsf{B}(\mathcal{K})$ the Banach space of all bounded linear operators on $\mathcal{K}$ with domain $\mathcal{K}$. Let $T : \mathbb{R}^3 \to \mathsf{B}(\mathcal{K})$; $\mathbb{R}^3 \ni \boldsymbol{x} = (x_1, x_2, x_3) \mapsto T(\boldsymbol{x}) \in \mathsf{B}(\mathcal{K})$ be a Borel measurable mapping such that, for all $\boldsymbol{x} \in \mathbb{R}^3$, $T(\boldsymbol{x})$ is a non-zero bounded self-adjoint operator on $\mathcal{K}$ such that $\|T\|_\infty := \sup_{\boldsymbol{x}\in\mathbb{R}^3} \|T(\boldsymbol{x})\| < \infty$, where $\|T(\boldsymbol{x})\|$ denotes the operator norm of $T(\boldsymbol{x})$.

Example 1. In the original CQSM, $\mathcal{K} = \mathbb{C}^2$ and $T(\boldsymbol{x}) = \boldsymbol{\tau} \cdot \boldsymbol{n}(\boldsymbol{x})$, where $\boldsymbol{n} : \mathbb{R}^3 \to \mathbb{R}^3$ is a measurable vector field with $|\boldsymbol{n}(\boldsymbol{x})| = 1$, a.e. (almost everywhere) $\boldsymbol{x} \in \mathbb{R}^3$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is the set of the Pauli matrices.

We denote by $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ the Dirac matrices, i.e., 4×4-Hermitian matrices satisfying

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \qquad \{\alpha_j, \beta\} = 0, \qquad \beta^2 = 1, \qquad j, k = 1, 2, 3,$$

where $\{A, B\} := AB + BA$.

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be measurable, a.e., finite and

$$U_F := (\cos F) \otimes I + i(\sin F)\gamma_5 \otimes T,$$

where I denotes identity and $\gamma_5 := -i\alpha_1\alpha_2\alpha_3$. We set $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$ and $\nabla := (D_1, D_2, D_3)$ with D_j being the generalized partial differential operator in the variable x_j. Then the one particle Hamiltonian of a generalized CQSM is defined by

$$H := -i\boldsymbol{\alpha} \cdot \nabla \otimes I + M(\beta \otimes I)U_F$$

acting in the Hilbert space $\mathcal{H}$. For a linear operator L, we denote its domain by $D(L)$. It is well-known that $-i\boldsymbol{\alpha} \cdot \nabla$ is self-adjoint with $D(-i\boldsymbol{\alpha} \cdot \nabla) = \cap_{j=1}^3 D(D_j)$. Since the operator $M(\beta \otimes I)U_F$ is bounded and self-adjoint, it follows that H is self-adjoint with domain $D(H) = \cap_{j=1}^3 D(D_j \otimes I) = H^1(\mathbb{R}^3; \mathbb{C}^4 \otimes \mathcal{K})$, the Sobolev space of order 1 consisting of $\mathbb{C}^4 \otimes \mathcal{K}$-valued measurable functions on $\mathbb{R}^3$. In the context of the CQSM, the function F is called a profile function. In what follows we sometimes omit the symbol of tensor product $\otimes$ in writing equations down.

Example 2. Usually profile functions are assumed to be rotation invariant with boundary conditions

$$F(0) = -\pi, \qquad \lim_{|\boldsymbol{x}|\to\infty} F(\boldsymbol{x}) = 0.$$

The following are concrete examples [6]:

(I) $F(\boldsymbol{x}) = -\pi \exp(-|\boldsymbol{x}|/R), \qquad R = 0.55 \times 10^{-15}\,\mathrm{m};$

(II) $F(\boldsymbol{x}) = -\pi\{a_1 \exp(-|\boldsymbol{x}|/R_1) + a_2 \exp(-|\boldsymbol{x}|^2/R_2^2)\},$
$a_1 = 0.65, \qquad R_1 = 0.58 \times 10^{-15}\,\mathrm{m}, \qquad a_2 = 0.35, \qquad R_2 = \sqrt{0.3} \times 10^{-15}\,\mathrm{m};$

(III) $F(\boldsymbol{x}) = -\pi \left(1 - \dfrac{|\boldsymbol{x}|}{\sqrt{\lambda^2 + |\boldsymbol{x}|^2}}\right), \qquad \lambda = \sqrt{0.4} \times 10^{-15}\,\mathrm{m}.$

We say that a self-adjoint operator A on $\mathcal{H}$ has chiral symmetry if $\gamma_5 A \subset A\gamma_5$.

Proposition 1. *The Hamiltonian H has no chiral symmetry.*

Proof. It is easy to check that, for all $\psi \in D(H)$, $\gamma_5\psi \in D(H)$ and $[\gamma_5, H]\psi = 2M\gamma_5\beta U_F\psi$. Note that $U_F \neq 0$. Hence, $[\gamma_5, H] \neq 0$ on $D(H)$. ∎

We note that, if F and T are differentiable on $\mathbb{R}^3$ with $\sup_{\boldsymbol{x}\in\mathbb{R}^3}|\partial_j F(\boldsymbol{x})| < \infty$ and $\sup_{\boldsymbol{x}\in\mathbb{R}^3}\|\partial_j T(\boldsymbol{x})\| < \infty$ $(j = 1, 2, 3)$, then the square of H takes the form

$$H^2 = (-\Delta + M^2) \otimes I - iM\beta\boldsymbol{\alpha} \cdot (\nabla U_F) + M^2 \sin^2 F \otimes (T^2 - I).$$

This is a Schrödinger operator with an operator-valued potential.

3 Operator matrix representation

For more detailed analyses of the model, it is convenient to work with a suitable representation of the Dirac matrices. Here we take the following representation of α_j and β (the Weyl representation):

$$\alpha_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where σ_1, σ_2 and σ_3 are the Pauli matrices. Let $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ and

$$\Phi_F := (\cos F) \otimes I + i(\sin F) \otimes T.$$

Then we have the following operator matrix representation for H:

$$H = \begin{pmatrix} -i\boldsymbol{\sigma} \cdot \nabla & M\Phi_F^* \\ M\Phi_F & i\boldsymbol{\sigma} \cdot \nabla \end{pmatrix}.$$

4 Supersymmetric aspects

Let $\xi : \mathbb{R}^3 \to \mathsf{B}(\mathcal{K})$ be measurable such that, for all $\boldsymbol{x} \in \mathbb{R}^3$, $\xi(\boldsymbol{x})$ is a bounded self-adjoint operator on $\mathcal{K}$ and $\xi(\boldsymbol{x})^2 = I$, $\forall\, \boldsymbol{x} \in \mathbb{R}^3$. Let

$$\Gamma(\boldsymbol{x}) := i\gamma_5\beta \otimes \xi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3.$$

We define an operator $\hat{\Gamma}$ on $\mathcal{H}$ by

$$(\hat{\Gamma}\psi)(\boldsymbol{x}) := \Gamma(\boldsymbol{x})\psi(\boldsymbol{x}), \qquad \psi \in \mathcal{H}, \qquad \text{a.e. } \boldsymbol{x} \in \mathbb{R}^3.$$

The following fact is easily proven:

Lemma 1. *The operator $\hat{\Gamma}$ is self-adjoint and unitary, i.e., it is a grading operator on $\mathcal{H}$: $\hat{\Gamma}^* = \hat{\Gamma}$, $\hat{\Gamma}^2 = I$.*

Theorem 1. *Suppose that ξ is strongly differentiable with $\sup_{\boldsymbol{x}\in\mathbb{R}^3} \|\partial_j \xi(\boldsymbol{x})\| < \infty$ $(j = 1, 2, 3)$ and*

$$\sum_{j=1}^{3} \alpha_j \otimes D_j \xi(\boldsymbol{x}) = M\gamma_5\beta\{\xi(\boldsymbol{x}), T(\boldsymbol{x})\} \sin F(\boldsymbol{x}). \tag{1}$$

Then $\hat{\Gamma}D(H) \subset D(H)$ and $\{\hat{\Gamma}, H\}\psi = 0$, $\forall\, \psi \in D(H)$.

Proof. For all $\psi \in D_0 := C_0^\infty(\mathbb{R}^3) \otimes_{\rm alg} (\mathbb{C}^4 \otimes \mathcal{K})$ ($\otimes_{\rm alg}$ denotes algebraic tensor product), we have

$$D_j\hat{\Gamma}\psi = i\gamma_5\beta \otimes (D_j\xi)\psi + i\gamma_5\beta \otimes \xi(D_j\psi). \tag{2}$$

By a limiting argument using the fact that D_0 is a core of $D_j \otimes I$, we can show that, for all $\psi \in D(D_j)$, $\hat{\Gamma}\psi$ is in $D(D_j)$ and (2) holds. Hence, for all $\psi \in D(H)$, $\hat{\Gamma}\psi \in D(H)$ and (2) holds. Thus we have for all $\psi \in D(H)$ $\{\hat{\Gamma}, H\}\psi = C_1\psi + C_2\psi$ with $C_1 := \sum_{j=1}^{3} \{\gamma_5\beta \otimes \xi, \alpha_j D_j\}$ and $C_2 := iM\{\gamma_5\beta \otimes \xi, \beta U_F\}$. Using the fact that $\{\gamma_5, \beta\} = 0$ and $[\gamma_5, \alpha_j] = 0$ $(j = 1, 2, 3)$, we obtain $C_1\psi = -\gamma_5\beta(\sum_{j=1}^{3} \alpha_j D_j \xi)\psi$. Similarly direct computations yield $(C_2\psi)(\boldsymbol{x}) = -M \sin F(\boldsymbol{x}) \otimes \{\xi(\boldsymbol{x}), T(\boldsymbol{x})\}\psi(x)$. Thus (1) implies $\{\hat{\Gamma}, H\}\psi = 0$. ■

Theorem 1 means that, under its assumption, H may be interpreted as a generator of a supersymmetry with respect to $\hat{\Gamma}$.

Example 3. Consider the case $\mathcal{K} = \mathbb{C}^2$. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable function such that

$$\left(1 + C^2\right) f(\boldsymbol{x})^2 + g(\boldsymbol{x})^2 = 1.$$

with a real constant $C \neq 0$ and $\boldsymbol{n}(\boldsymbol{x}) := (f(\boldsymbol{x}), Cf(\boldsymbol{x}), g(\boldsymbol{x}))$. Then $|\boldsymbol{n}(\boldsymbol{x})| = 1$, $\forall\, \boldsymbol{x} \in \mathbb{R}^3$. Let

$$\xi := \frac{C}{\sqrt{1+C^2}}\tau_1 - \frac{1}{\sqrt{1+C^2}}\tau_2, \qquad T(\boldsymbol{x}) := \boldsymbol{\tau} \cdot \boldsymbol{n}(\boldsymbol{x}).$$

Then $\xi^2 = I$ and (ξ, T) satisfies (1).

To state spectral properties of H, we recall some definitions. For a self-adjoint operator S, we denote by $\sigma(S)$ the spectrum of S. The point spectrum of S, i.e., the set of all the eigenvalues of S is denoted $\sigma_{\mathrm{p}}(S)$. An isolated eigenvalue of S with finite multiplicity is called a discrete eigenvalue of S. We denote by $\sigma_{\mathrm{d}}(S)$ the set of all the discrete eigenvalues of S. The set $\sigma_{\mathrm{ess}}(S) := \sigma(S) \setminus \sigma_{\mathrm{d}}(S)$ is called the essential spectrum of S.

Theorem 2. *Under the same assumption as in Theorem 1, the following holds:*

(i) $\sigma(H)$ *is symmetric with respect to the origin of* $\mathbb{R}$, *i.e., if* $\lambda \in \sigma(H)$, *then* $-\lambda \in \sigma(H)$.

(ii) $\sigma_{\#}(H)$ $(\# = \mathrm{p}, \mathrm{d})$ *is symmetric with respect to the origin of* $\mathbb{R}$ *with*

$$\dim \ker(H - \lambda) = \dim \ker(H - (-\lambda))$$

for all $\lambda \in \sigma_{\#}(H)$.

(iii) $\sigma_{\mathrm{ess}}(H)$ *is symmetric with respect to the origin of* $\mathbb{R}$.

Proof. Theorem 1 implies a unitary equivalence of H and $-H$ ($\hat{\Gamma} H \hat{\Gamma}^{-1} = -H$). Thus the desired results follow. ∎

Remark 1. Suppose that the assumption of Theorem 1 holds. In view of supersymmetry breaking, it is interesting to compute $\dim\ker H$. This is related to the index problem: Let

$$\mathcal{H}_+ := \ker(\hat{\Gamma} - 1), \quad \mathcal{H}_- := \ker(\hat{\Gamma} + 1)$$

and

$$H_\pm := H|\mathcal{H}_\pm.$$

Then H_+ (resp. H_-) is a densely defined closed linear operator from $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) to $\mathcal{H}_-$ (resp. $\mathcal{H}_+$) with $D(H_+) = D(H) \cap \mathcal{H}_+$ (resp. $D(H_-) = D(H) \cap D(H_-)$). Obviously

$$\ker H = \ker H_+ \oplus \ker H_-.$$

The analytical index of H_+ is defined by

$$\mathrm{index}(H_+) := \dim\ker H_+ - \dim\ker H_+^*,$$

provided that at least one of $\dim\ker H_+$ and $\dim\ker H_+^*$ is finite. We conjecture that, for a class of F and T, $\mathrm{index}(H_+) = 0$.

5 The essential spectrum and finiteness of the discrete spectrum of $\boldsymbol{H}$

5.1 Structure of the spectrum of $\boldsymbol{H}$

Theorem 3. *Suppose that* $\dim \mathcal{K} < \infty$ *and*

$$\lim_{|\boldsymbol{x}|\to\infty} F(\boldsymbol{x}) = 0. \tag{3}$$

Then

$$\sigma_{\mathrm{ess}}(H) = (-\infty, -M] \cup [M, \infty), \tag{4}$$

$$\sigma_{\mathrm{d}}(H) \subset (-M, M). \tag{5}$$

Proof. We can rewrite H as $H = H_0 \otimes I + V$ with $H_0 := -i\boldsymbol{\alpha}\cdot\nabla + M\beta$ and $V := M(\beta \otimes I)(U_F - I)$. We denote by χ_R $(R > 0)$ the characteristic function of the set $\{\boldsymbol{x} \in \mathbb{R}^3 | \, |\boldsymbol{x}| < R\}$. It is well-known that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H_0 - z)^{-1}\chi_R$ is compact [7, Lemma 4.6]. Since $\mathcal{K}$ is finite dimensional, it follows that $(H_0 \otimes I - z)^{-1}\chi_R \otimes I$ is compact. We have

$$\|V(\boldsymbol{x})\| \le M(|\cos F(\boldsymbol{x}) - 1| + |\sin F(\boldsymbol{x})|\|T\|_\infty) \le M\left(\frac{|F(\boldsymbol{x})|^2}{2} + |F(\boldsymbol{x})|\|T\|_\infty\right).$$

Hence, by (3), we have $\lim_{R\to\infty} \sup_{|\boldsymbol{x}|>R} \|V(\boldsymbol{x})\| = 0$. Then, in the same way as in the method described on [7, pp. 115–117], we can show that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $(H - z)^{-1} - (H_0 \otimes I - z)^{-1}$ is compact. Hence, by a general theorem (e.g., [7, Theorem 4.5]), $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0 \otimes I)$. Since $\sigma_{\mathrm{ess}}(H_0) = (-\infty, -M] \cup [M, \infty)$ ([7, Theorem 1.1]), we obtain (4). Relation (5) follows from (4) and $\sigma_{\mathrm{d}}(H) = \sigma(H) \setminus \sigma_{\mathrm{ess}}(H)$. ■

5.2 Bound for the number of discrete eigenvalues of $\boldsymbol{H}$

Suppose that $\dim \mathcal{K} < \infty$ and (3) holds. Then, by Theorem 3, we can define the number of discrete eigenvalues of H counting multiplicities:

$$N_H := \dim \operatorname{Ran} E_H((-M, M)), \tag{6}$$

where E_H is the spectral measure of H.

To estimate an upper bound for N_H, we introduce a hypothesis for F and T:

Hypothesis (A).

(i) $T(\boldsymbol{x})^2 = I$, $\forall\, \boldsymbol{x} \in \mathbb{R}^3$ and T is strongly differentiable with $\sum_{j=1}^{3}(D_jT(\boldsymbol{x}))^2$ being a multiplication operator by a scalar function on $\mathbb{R}^3$.

(ii) $F \in C^1(\mathbb{R}^3)$.

(iii) $\sup_{\boldsymbol{x}\in\mathbb{R}^3} |D_jF(\boldsymbol{x})| < \infty$, $\sup_{\boldsymbol{x}\in\mathbb{R}^3} \|D_jT(\boldsymbol{x})\| < \infty$ $(j = 1, 2, 3)$.

Under this assumption, we can define

$$V_F(\boldsymbol{x}) := \sqrt{|\nabla F(\boldsymbol{x})|^2 + \sum_{j=1}^{3}(D_jT(\boldsymbol{x}))^2 \sin^2 F(\boldsymbol{x})}.$$

Theorem 4. *Let* $\dim \mathcal{K} < \infty$*. Assume* (3) *and Hypothesis* (A)*. Suppose that*

$$C_F := \int_{\mathbb{R}^6} \frac{V_F(\boldsymbol{x})V_F(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2}\, d\boldsymbol{x}d\boldsymbol{y} < \infty.$$

Then N_H *is finite with*

$$N_H \le \frac{(\dim \mathcal{K})M^2 C_F}{4\pi^2}.$$

A basic idea for the proof of Theorem 4 is as follows. Let

$$L(F) := H^2 - M^2.$$

Then we have

$$L(F) = -\Delta + M \begin{pmatrix} 0 & W_F^* \\ W_F & 0 \end{pmatrix}$$

with $W_F := i\boldsymbol{\sigma} \cdot \nabla \Phi_F$. Note that

$$W_F^* W_F = W_F W_F^* = V_F^2.$$

Let

$$L_0(F) := -\Delta - M V_F.$$

For a self-adjoint operator S, we introduce a set

$$N_-(S) := \text{the number of negative eigenvalues of } S \text{ counting multiplicities.}$$

The following is a key lemma:

Lemma 2.

$$N_H \le N_-(L(F)) \le N_-(L_0(F)). \tag{7}$$

Proof. For each $\lambda \in \sigma_{\mathrm{d}}(H) \cap (-M, M)$, we have $\ker(H - \lambda) \subset \ker(L(F) - E_\lambda)$ with $E_\lambda = \lambda^2 - M^2 < 0$. Hence the first inequality of (7) follows. The second inequality of (7) can be proven in the same manner as in the proof of [1, Lemma 3.3], which uses the min-max principle. ■

On the other hand, one has

$$N_-(L_0(F)) \le \frac{(\dim \mathcal{K})M^2 C_F}{4\pi^2}$$

(the Birman–Schwinger bound [4, Theorem XIII.10]). In this way we can prove Theorem 4.

As a direct consequence of Theorem 4, we have the following fact on the absence of discrete eigenvalues of H:

Corollary 1. *Assume* (3) *and Hypothesis* (A)*. Let* $(\dim \mathcal{K})M^2 C_F < 4\pi^2$*. Then* $\sigma_{\mathrm{d}}(H) = \varnothing$*, i.e.,* H *has no discrete eigenvalues.*

6 Existence of discrete ground states

Let A be a self-adjoint operator on a Hilbert space and bounded from below. Then

$$E_0(A) := \inf \sigma(A)$$

is finite. We say that A has a *ground state* if $E_0(A) \in \sigma_{\rm p}(A)$. In this case, a non-zero vector in $\ker(A - E_0(A))$ is called a *ground state of* A. Also we say that A has a discrete ground state if $E_0(A) \in \sigma_{\rm d}(A)$.

Definition 1. Let

$$E_0^+(H) := \inf\left[\sigma(H) \cap [0,\infty)\right], \qquad E_0^-(H) := \sup\left[\sigma(H) \cap (-\infty,0]\right].$$

(i) If $E_0^+(H)$ is an eigenvalue of H, then we say that H *has a positive energy ground state* and we call a non-zero vector in $\ker(H - E_0^+(H))$ a *positive energy ground state* of H.

(ii) If $E_0^-(H)$ is an eigenvalue of H, then we say that H *has a negative energy ground state* and we call a non-zero vector in $\ker(H - E_0^-(H))$ a *negative energy ground state* of H.

(iii) If $E_0^+(H)$ (resp. $E_0^-(H)$) is a discrete eigenvalue of H, then we say that H has a *discrete positive* (resp. *negative*) *energy ground state.*

Remark 2. If the spectrum of H is symmetric with respect to the origin of $\mathbb{R}$ as in Theorem 2, then $E_0^+(H) = -E_0^-(H)$, and H has a positive energy ground state if and only if it has a negative energy ground state.

Assume Hypothesis (A). Then the operators

$$S_\pm(F) := -\Delta \pm M(D_3 \cos F)$$

are self-adjoint with $D(S_\pm(F)) = D(\Delta)$ and bounded from below.

As for existence of discrete ground states of the Dirac operator H, we have the following theorem:

Theorem 5. *Let* $\dim \mathcal{K} < \infty$. *Assume Hypothesis* (A) *and* (3). *Suppose that* $E_0(S_+(F)) < 0$ *or* $E_0(S_-(F)) < 0$. *Then* H *has a discrete positive energy ground state or a discrete negative ground state.*

Proof. We describe only an outline of proof. We have

$$\sigma_{\rm ess}(L(F)) = [0,\infty), \qquad \sigma_{\rm d}(L(F)) \subset [-M^2, 0).$$

Hence, if $L(F)$ has a discrete eigenvalue, then H has a discrete eigenvalue in $(-M, M)$. By the min-max principle, we need to find a unit vector Ψ such that $\langle \Psi, L(F)\Psi\rangle < 0$. Indeed, for each $f \in D(\Delta)$, we can find vectors $\Psi_f^\pm \in D(L(F))$, such that $\langle \Psi_f^\pm, L(F)\Psi_f^\pm\rangle = \langle f, S_\pm f\rangle$. By the present assumption, there exists a non-zero vector $f_0 \in D(\Delta)$ such that $\langle f_0, S_+(F) f_0\rangle < 0$ or $\langle f_0, S_-(F) f_0\rangle < 0$. Thus the desired results follow. ■

To find a class of F such that $E_0(S_+(F)) < 0$ or $E_0(S_-(F)) < 0$, we proceed as follows. For a constant $\varepsilon > 0$ and a function f on $\mathbb{R}^d$, we define a function f_ε on $\mathbb{R}^d$ by

$$f_\varepsilon(x) := f(\varepsilon x), \qquad x \in \mathbb{R}^d.$$

The following are key Lemmas.

Lemma 3. *Let* $V : \mathbb{R}^d \to \mathbb{R}$ *be in* $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ *and*

$$S_\varepsilon := -\Delta + V_\varepsilon.$$

Suppose that:

(i) *For all* $\varepsilon > 0$, S_ε *is self-adjoint, bounded below and* $\sigma_{\mathrm{ess}}(S_\varepsilon) \subset [0, \infty)$.

(ii) *There exists a nonempty open set* $\Omega \subset \{x \in \mathbb{R}^d | V(x) < 0\}$.

Then then there exists a constant $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, S_ε *has a discrete ground state.*

Proof. A basic idea for the proof of this lemma is to use the min-max principle (see [1, Lemma 4.3]). ∎

Lemma 4. $V : \mathbb{R}^d \to \mathbb{R}$ *be continuous with* $V(x) \to 0 (|x| \to \infty)$. *Suppose that* $\{x \in \mathbb{R}^d | V(x) < 0\} \neq \varnothing$. *Then:*

(i) $-\Delta + V$ *is self-adjoint and bounded below.*

(ii) $\sigma_{\mathrm{ess}}(-\Delta + V) = [0, \infty)$.

(iii) S_ε *has a discrete ground state for all* $\varepsilon \in (0, \varepsilon_0)$ *with some* $\varepsilon_0 > 0$.

Proof. The facts (i) and (ii) follow from the standard theory of Schrödinger operators. Part (iii) follow from a simple application of Lemma 3 (for more details, see the proof of [1, Lemma 4.4]). ∎

We now consider a one-parameter family of Dirac operators:

$$H_\varepsilon := (-i)\boldsymbol{\alpha} \cdot \nabla + \frac{1}{\varepsilon} M(\beta \otimes I) U_{F_\varepsilon}.$$

Theorem 6. *Let* $\dim \mathcal{K} < \infty$. *Assume Hypothesis* (A) *and* (3). *Suppose that* $D_3 \cos F$ *is not identically zero. Then there exists a constant* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, H_ε *has a discrete positive energy ground state or a discrete negative ground state.*

Proof. This follows from Theorem 5 and Lemma 4 (for more details, see the proof of [1, Theorem 4.5]). ∎

7 Symmetry reduction of $\boldsymbol{H}$

Let T_1, T_2 and T_3 be bounded self-adjoint operators on $\mathcal{K}$ satisfying

$$T_j^2 = I, \qquad j = 1, 2, 3,$$
$$T_1 T_2 = i T_3, \qquad T_2 T_3 = i T_1, \qquad T_3 T_1 = i T_2.$$

Then it is easy to see that the anticommutation relations

$$\{T_j, T_k\} = 2\delta_{jk} I, \qquad j, k = 1, 2, 3$$

hold. Since each T_j is a unitary self-adjoint operator with $T_j \neq \pm I$, it follows that

$$\sigma(T_j) = \sigma_{\mathrm{p}}(T_j) = \{\pm 1\}.$$

We set $\boldsymbol{T} = (T_1, T_2, T_3)$.

In this section we consider the case where $T(\boldsymbol{x})$ is of the following form:

$$T(\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{T},$$

where $\boldsymbol{n}(\boldsymbol{x})$ is the vector field in Example 1. We use the cylindrical coordinates for points $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z,$$

where $\theta \in [0, 2\pi)$, $r > 0$. We assume the following:

Hypothesis (B). There exists a continuously differentiable function $G : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that

(i) $F(\boldsymbol{x}) = G(r, z)$, $\boldsymbol{x} \in \mathbb{R}^3 \setminus \{0\}$;

(ii) $\lim_{r+|z|\to\infty} G(r,z) = 0$;

(iii) $\sup_{r>0, z\in\mathbb{R}} (|\partial G(r,z)/\partial r| + |\partial G(r,z)/\partial z|) < \infty$.

We take the vector field $\boldsymbol{n} : \mathbb{R}^3 \to \mathbb{R}^3$ to be of the form

$$\boldsymbol{n}(\boldsymbol{x}) := \big(\sin\Theta(r,z)\cos(m\theta), \sin\Theta(r,z)\sin(m\theta), \cos\Theta(r,z)\big),$$

where $\Theta : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and m is a natural number.

Let L_3 be the third component of the angular momentum acting in $L^2(\mathbb{R}^3)$ and

$$K_3 := L_3 \otimes I + \frac{1}{2}\Sigma_3 \otimes I + \frac{m}{2} I \otimes T_3 \tag{8}$$

with $\Sigma_3 := \sigma_3 \oplus \sigma_3$. It is easy to see that K_3 is a self-adjoint operator acting in $\mathcal{H}$.

Lemma 5. *Assume that*

$$\Theta(\varepsilon r, \varepsilon z) = \Theta(r, z), \qquad (r,z) \in (0,\infty) \times \mathbb{R}, \qquad \varepsilon > 0. \tag{9}$$

Then, for all $t \in \mathbb{R}$ and $\varepsilon > 0$, the operator equality

$$e^{itK_3} H_\varepsilon e^{-itK_3} = H_\varepsilon \tag{10}$$

holds.

Proof. Similar to the proof of [1, Lemma 5.2]. We remark that, in the calculation of

$$e^{itK_3} T(\boldsymbol{x}) e^{-itK_3} = \sum_{j=1}^{3} e^{itL_3} n_j(\boldsymbol{x}) e^{-itL_3} e^{itmT_3} T_j e^{-itmT_3},$$

the following formulas are used:

$$(T_1 \cos mt - T_2 \sin mt) e^{itmT_3} = T_1, \qquad (T_1 \sin mt + T_2 \cos mt) e^{itmT_3} = T_2. \qquad \blacksquare$$

Definition 2. We say that two self-adjoint operators on a Hilbert space strongly commute if their spectral measures commute.

Lemma 6. *Assume (9). Then, for all $\varepsilon > 0$, H_ε and K_3 strongly commute.*

Proof. By (10) and the functional calculus, we have for all $s, t \in \mathbb{R}$ $e^{itK_3} e^{isH_\varepsilon} e^{-itK_3} = e^{isH_\varepsilon}$, which is equivalent to $e^{itK_3} e^{isH_\varepsilon} = e^{isH_\varepsilon} e^{itK_3}$, $s, t \in \mathbb{R}$. By a general theorem (e.g., [3, Theorem VIII.13]), this implies the strong commutativity of K_3 and H_ε. $\blacksquare$

Lemma 6 implies that H_ε is reduced by eigenspaces of K_3. Note that

$$\sigma(K_3) = \sigma_{\rm p}(K_3) = \left\{ \ell + \frac{s}{2} + \frac{mt}{2} \,\middle|\, \ell \in \mathbb{Z}, s = \pm 1, t = \pm 1 \right\}.$$

The eigenspace of K_3 with eigenvalue $\ell + (s/2) + (mt/2)$ is given by

$$\mathcal{M}_{\ell,s,t} := \mathcal{M}_\ell \otimes \mathcal{C}_s \otimes \mathcal{T}_t$$

with $\mathcal{C}_s := \ker(\Sigma_3 - s)$ and $\mathcal{T}_t := \ker(T_3 - t)$. Then $\mathcal{H}$ has the orthogonal decomposition

$$\mathcal{H} = \oplus_{\ell \in \mathbb{Z}, s,t \in \{\pm 1\}} \mathcal{M}_{\ell,s,t}.$$

Thus we have:

Lemma 7. *Assume (9). Then, for all $\varepsilon > 0$, H_ε is reduced by each $\mathcal{M}_{\ell,s,t}$.*

We denote by $H_\varepsilon(\ell, s, t)$ by the reduced part of H_ε to $\mathcal{M}_{\ell,s,t}$ and set

$$H(\ell, s, t) := H_1(\ell, s, t).$$

For $s = \pm 1$ and $\ell \in \mathbb{Z}$, we define

$$L_s(G, \ell) := -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\ell^2}{r^2} + \frac{\partial^2}{\partial z^2} + sMD_z \cos G$$

acting in $L^2((0,\infty) \times \mathbb{R}, rdrdz)$ with domain

$$D(L_s(G, \ell)) := C_0^\infty((0,\infty) \times \mathbb{R})$$

and set

$$\mathcal{E}_0(L_s(G, \ell)) := \inf_{f \in C_0^\infty((0,\infty)\times\mathbb{R}), \|f\|_{L^2((0,\infty)\times\mathbb{R}, rdrdz)} = 1} \langle f, L_s(G, \ell) f \rangle.$$

The following theorem is concerned with the existence of discrete ground states of $H(\ell, s, t)$.

Theorem 7. *Assume Hypothesis* (B) *and* (9)*. Fix an $\ell \in \mathbb{Z}$ arbitrarily, $s = \pm 1$ and $t = \pm 1$. Suppose that* $\dim \mathcal{T}_t < \infty$ *and*

$$\mathcal{E}_0(L_s(G, \ell)) < 0.$$

Then $H(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.

Proof. Similar to the proof of Theorem 5 (for more details, see the proof of [1, Theorem 5.5]). ■

Theorem 8. *Assume Hypothesis* (B) *and* (9)*. Suppose that* $\dim \mathcal{T}_t < \infty$ *and that* $D_z \cos G$ *is not identically zero. Then, for each $\ell \in \mathbb{Z}$, there exists a constant $\varepsilon_\ell > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\ell)$, each $H_\varepsilon(\ell, s, t)$ has a discrete positive energy ground state or a discrete negative ground state.*

Proof. Similar to the proof of Theorem 6 (for more details, see the proof of [1, Theorem 5.6]). ■

Theorem 8 immediately yields the following result:

Corollary 2. *Assume Hypothesis* (B) *and* (9)*. Suppose that* $\dim \mathcal{T}_t < \infty$ *and that* $D_z \cos G$ *is not identically zero. Let ε_ℓ be as in Theorem 8 and, for each $n \in \mathbb{N}$ and $k > n$ ($k, n \in \mathbb{Z}$), $\nu_{k,n} := \min_{n+1 \le \ell \le k} \varepsilon_\ell$. Then, for each $\varepsilon \in (0, \nu_{k,n})$, H_ε has at least $(k - n)$ discrete eigenvalues counting multiplicities.*

Proof. Note that $\sigma_{\rm p}(H_\varepsilon) = \cup_{\ell \in \mathbb{Z}, s,t = \pm 1} \sigma_{\rm p}(H_\varepsilon(\ell, s, t))$. ■

8 A unitary transformation

We go back again to the generalized CQSM defined in Section 2. It is easy to see that the operator

$$X_F := \frac{1+\gamma_5}{2}\exp\left(iF\otimes\frac{T}{2}\right)+\frac{1-\gamma_5}{2}\exp\left(-iF\otimes\frac{T}{2}\right)$$

is unitary. Under Hypothesis (A), we can define the following operator-valued functions:

$$B_j(\boldsymbol{x}) := \frac{1}{2}D_j[F(\boldsymbol{x})T(\boldsymbol{x})], \qquad \boldsymbol{x}\in\mathbb{R}^3, \qquad j=1,2,3.$$

We set

$$\boldsymbol{B} := (B_1, B_2, B_3)$$

and introduce

$$H(\boldsymbol{B}) := (-i)\boldsymbol{\alpha}\cdot\nabla + M\beta - \boldsymbol{\sigma}\cdot\boldsymbol{B}$$

acting in $\mathcal{H}$. Since $\boldsymbol{\sigma}\cdot\boldsymbol{B}$ is a bounded self-adjoint operator, $H(\boldsymbol{B})$ is self-adjoint with $D(H(\boldsymbol{B})) = \cap_{j=1}^3 D(D_j\otimes I)$.

Proposition 2. *Assume Hypothesis* (A) *and that* $T(\boldsymbol{x})$ *is independent of* $\boldsymbol{x}$*. Then*

$$X_F H X_F^{-1} = H(\boldsymbol{B}).$$

Proof. Similar to the proof of [1, Proposition 6.1]. ■

Using this proposition, we can prove the following theorem:

Theorem 9. *Let* $\dim\mathcal{K} < \infty$*. Assume Hypothesis* (A) *and that* $T(\boldsymbol{x})$ *is independent of* $\boldsymbol{x}$*. Suppose that*

$$\lim_{|\boldsymbol{x}|\to\infty} |\nabla F(\boldsymbol{x})| = 0.$$

Then

$$\sigma_{\text{ess}}(H) = (-\infty, -M]\cup[M,\infty). \tag{11}$$

Proof. By Proposition 2, we have $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H(\boldsymbol{B}))$. By the present assumption, $B_j(\boldsymbol{x}) = D_jF(\boldsymbol{x})T(\mathbf{0})/2$. Hence

$$\sup_{|\boldsymbol{x}|>R}\|\boldsymbol{\sigma}\cdot\boldsymbol{B}(\boldsymbol{x})\| \le \sum_{j=1}^3(\|T(\mathbf{0})\|/2)\sup_{|\boldsymbol{x}|>R}|D_jF(\boldsymbol{x})| \to 0 \qquad (R\to\infty).$$

Therefore, as in the proof of Theorem 3, we conclude that $\sigma_{\text{ess}}(H(\boldsymbol{B})) = (-\infty,-M]\cup[M,\infty]$. Thus (11) follows. ■

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[1] Arai A., Hayashi K., Sasaki I., Spectral properties of a Dirac operator in the chiral quark soliton model, *J. Math. Phys.*, 2005, V.46, N 5, 052360, 12 pages.

[2] Kalf H., Yamada O., Essential self-adjointness of n-dimensional Dirac operators with a variable mass term, *J. Math. Phys.*, 2001, V.42, 2667–2676.

[3] Reed M., Simon B., Methods of modern mathematical physics I: Functional analysis, New York, Academic Press, 1972.

[4] Reed M., Simon B., Methods of modern mathematical physics IV: Analysis of operators, New York, Academic Press, 1978.

[5] Sawado N., The $SU(3)$ dibaryons in the chiral quark soliton model, *Phys. Lett. B*, 2002, V.524, 289–296.

[6] Sawado N., Private communication.

[7] Thaller B., The Dirac equation, Springer-Verlag, 1992.