УДК 517.54

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ON THE BOUNDARY BEHAVIOR OF CONJUGATE HARMONIC FUNCTIONS

It is proved that if a harmonic function u on the unit disk $\mathbb{D}$ in $\mathbb{C}$ has angular limits on a measurable set E of the unit circle, then its conjugate harmonic function v in $\mathbb{D}$ also has (finite !) angular limits a.e. on E and both boundary functions are measurable on E. The result is extended to arbitrary Jordan domains with rectifiable boundaries in terms of angular limits and of the natural parameter. This result is essentially based on the Fatou theorem on angular limits of bounded analytic functions and on the construction of Luzin and Priwalow to their uniqueness theorem for analytic and meromorphic functions. The result will have interesting applications to the study of the various Stieltjes integrals in the theory of harmonic and analytic functions and, in particular, of the Hilbert–Stieltjes inyegral.

Key words: *correlation, boundary behavior, conjugate harmonic functions, rectifiable Jordan curves, angular limits, boundary value problems.*

2010 Mathematics Subject Classification: Primary 30C62, 31A05, 31A20, 31A25, 31B25; Secondary 30E25, 31C05, 34M50, 35F45, 35Q15.

1. Introduction.

First of all, recall that a path in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ terminating at $\zeta = e^{i\vartheta} \in \partial\mathbb{D}$ is called **nontangential** at ζ if its part in a neighborhood of ζ lies inside of an angle in $\mathbb{D}$ with the vertex at ζ. Hence limits along all nontangential paths at ζ are also named **angular** at ζ. The latter is a traditional tool of the geometric function theory, see e.g. monographs [1]–[6]. Note that every closed rectifiable Jordan curve has a tangent a.e. with respect to the natural parameter and the angular limit has the same sense at its points with a tangent.

It is known the very delicate fact due to Lusin that harmonic functions in the unit circle with continuous (even absolutely continuous !) boundary data can have conjugate harmonic functions whose boundary data are not continuous functions, furthemore, they can even be even not essentially bounded in neighborhoods of each point of the unit circle, see e.g. Theorem VIII.13.1 in [7]. Thus, a correlation between boundary data of conjugate harmonic functions is not a simple matter, see also I.E in [3].

Denote by h^p, $p \in (0,\infty)$, the class of all harmonic functions u in $\mathbb{D}$ with

$$\sup_{r\in(0,1)} \left\{ \int\limits_0^{2\pi} |u(re^{i\vartheta})|^p \, d\vartheta \right\}^{\frac{1}{p}} < \infty \, .$$

It is clear that $h^p \subseteq h^{p'}$ for all $p > p'$ and, in particular, $h^p \subseteq h^1$ for all $p > 1$.

Remark 1. It is important that every function in the class h^1 has a.e. nontangential boundary limits, see e.g. Corollary IX.2.2 in [8].

It is also known that a harmonic function u in $\mathbb{D}$ can be represented as the Poisson integral

$$u(re^{i\vartheta}) \ = \ \frac{1}{2\pi} \int\limits_{0}^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2}\ \varphi(t)\ dt \tag{1.1}$$

with a function $\varphi \in L^p(0,2\pi)$, $p>1$, if and only if $u\in h^p$, see e.g. Theorem IX.2.3 in [8]. Thus, $u(z)\to\varphi(\vartheta)$ as $z\to e^{i\vartheta}$ along any nontangential path for a.e. ϑ, see e.g. Corollary IX.1.1 in [8]. Moreover, $u(z)\to\varphi(\vartheta_0)$ as $z\to e^{i\vartheta_0}$ at points ϑ_0 of continuity of the function φ, see e.g. Theorem IX.1.1 in [8].

Note also that $v\in h^p$ whenever $u\in h^p$ for all $p>1$ by the M. Riesz theorem, see [9], see also Theorem IX.2.4 in [8]. Generally speaking, this fact is not trivial but it follows immediately for $p=2$ from the Parseval equality, see e.g. the proof of Theorem IX.2.4 in [8]. The case $u\in h^1$ is more complicated.

The correlation of the boundary behavior of conjugate harmonic functions outside the classes h^p was not investigated at all. This is just the subject of the present article.

2. The case of the unit disk with respect to the arc length.

Here we apply in a certain part a construction of Luzin–Priwalow from the proof of their theorem on the boundary uniqueness for analytic functions, see [10], see also [3], Section III.D.1, and [6], Section IV.2.5.

Theorem 1. *Let $u:\mathbb{D}\to\mathbb{R}$ be a harmonic function that has angular limits on a measurable set E of the unit circle $\partial\mathbb{D}$. Then its conjugate harmonic functions v have (finite !) angular limits a.e. on E and both boundary functions are measurable on E.*

Remark 2. By the Luzin–Priwalow uniqueness theorem for meromorphic functions u as well as v cannot have infinite angular limits on a subset of $\mathbb{D}$ of a positive measure, see Section IV.2.5 in [6].

Proof. By Remark 2 we may consider that angular limits of u are finite everywhere on the set E. Moreover, the measurable set E admits a countable exhaustion by measure of the arc length with its closed subsets, see e.g. Theorem III(6.6) in [11], and hence with no loss of generality we may also consider that E is compact, see e.g. Proposition I.9.3 in [12].

Following [3], Section III.D.1, we set, for $\zeta\in\partial\mathbb{D}$,

$$S_\zeta \ = \ \left\{\ z\in\mathbb{D}:\ |z|\ >\ \frac{1}{\sqrt{2}}\ ,\ |\arg(\zeta-z)|\ <\ \frac{\pi}{4}\ \right\} \tag{2.1}$$

and

$$\mathfrak{D} \ = \ \bigcup_{\zeta\in E} S_\zeta\ \cup D_* \tag{2.2}$$

where

$$D_* = \left\{ z\in\mathbb{C}:\ |z|\ \le\ \frac{1}{\sqrt{2}}\ \right\}\ .$$

It is easy geometrically to see that $\partial\mathfrak{D}$ contains E and it is a rectifiable Jordan curve because $\partial\mathbb{D}\setminus E$ is open set and hence it consists of a countable collection of arcs of $\partial\mathbb{D}$, see the corresponding illustrations in [3], Section III.D.1.

By the construction, the radii of $\mathbb{D}$ to every $\zeta\in E$ belong to $\mathfrak{D}$ and the well defined real-valued function $\varphi(\zeta):=\lim\limits_{n\to\infty}\varphi_n(\zeta)$, $\varphi_n(\zeta):=u(r_n\zeta)$, $n=1,2,\ldots$, with arbitrary sequence $r_n\to 1-0$ as $n\to\infty$, is measurable, see e.g. Corollary 2.3.10 in [13]. Thus, by the known Egorov theorem, see e.g. Theorem 2.3.7 in [13], with no loss of generality we may assume that $\varphi_n\to\varphi$ uniformly on E and that φ is continuous on E, see e.g. Section 7.2 in [14].

Let us consider the sequence of the functions

$$\psi_n(\zeta)\ :=\ \sup_{z\in S_\zeta\cap D^n_\zeta}\ |u(z)-\varphi(\zeta)|\ ,\qquad \zeta\in E\ , \tag{2.3}$$

where $D^n_\zeta=\{z\in\mathbb{C}:|z-\zeta|<\varepsilon_n\}$ with $\varepsilon_n\searrow 0$ as $n\to\infty$. First of all, $\psi_n(\zeta)\to 0$ as $n\to\infty$ for every $\zeta\in E$. Moreover, the functions $\psi_n(\zeta)$ are measurable again by Corollary 2.3.10 in [13] because of $\psi_n(\zeta)=\lim\limits_{m\to\infty}\psi_{mn}(\zeta)$ as $m\to\infty$ where the functions

$$\psi_{mn}(\zeta)\ :=\ \max_{z\in\overline{S_\zeta}\cap R^{mn}_\zeta}\ |u(z)-\varphi(\zeta)|\ ,\qquad R^{mn}_\zeta:=\overline{D^n_\zeta}\setminus D^{n+m}_\zeta\ ,\qquad \zeta\in E\ , \tag{2.4}$$

are continuous. Indeed, $\psi_{mn}(\zeta)$ coincide with the Hausdorff distance between the compact sets $u(\,\overline{S_\zeta}\cap R^{mn}_\zeta)$ and $\{\varphi(\zeta)\}$, see e.g. Theorem 2.21.VII in [15], and any distance is continuous with respect to its variables, recall that both functions u and φ are continuous.

Again by the Egorov theorem with no loss of generality we may consider that $\psi_n\to 0$ uniformly on E. The latter implies that the restriction U of the harmonic function u to the domain $\mathfrak{D}$ is bounded. Indeed, let us assume that there exists a sequence of points $z_n\in\mathfrak{D}$ such that $|u(z_n)|\geq n$, $n=1,2,\ldots$. With no loss of generality we may consider that $z_n\to\zeta\in E$ because the function u is bounded on the compact subsets of $\mathbb{D}$ and by the construction $E=\partial\mathfrak{D}\cap\partial\mathbb{D}$ and E is compact. Moreover, by the construction of $\mathfrak{D}$, we also may consider that $z_n\in S_{\zeta_n}$, $\zeta_n\in E$, $n=1,2,\ldots$ and that $\zeta_n\to\zeta$ as $n\to\infty$. Consequently, it should be that $u(z_n)\to\varphi(\zeta)$ because $\psi_n(\zeta_n)\to 0$ as $\zeta_n\to\zeta$, see e.g. Theorem 7.1(2) and Proposition 7.1 in [14]. The latter conclusion contradicts the above assumption.

Further, by the construction the domain $\mathfrak{D}$ is simply connected and hence by the Riemann theorem there exists a conformal mapping $w=\omega(z)$ of $\mathfrak{D}$ onto $\mathbb{D}$, see e.g. Theorem II.2.1 in [8]. Note that the function $U_*:=U\circ\omega^{-1}$ is a bounded harmonic function in $\mathbb{D}$ and there exists its conjugate harmonic function V_* in $\mathbb{D}$, i.e. $F:=U_*+i\,V_*$ is an analytic function in $\mathbb{D}$. Let N be a positive number that is greater than $\sup\limits_{w\in\mathbb{D}}|U_*(w)|=\sup\limits_{z\in\mathfrak{D}}|U(z)|$. Then the analytic function $g(w):=F(w)/(N-F(w))$, $w\in\mathbb{D}$, is bounded. Thus, by the Fatou theorem, see e.g. Corollary III.A in [3], g has finite angular limits as $w\to W$ for a.e. $W\in\partial\mathbb{D}$. By Remark 2 these limits cannot be equal to 1 on a subset of $\partial\mathbb{D}$ of a positive measure. Consequently, the function $F(w)$ has also (finite!) angular limits as $w \to W$ for a.e. $W \in \partial\mathbb{D}$.

Let us consider the analytic function $f = F \circ \omega$ given in the domain $\mathfrak{D}$. By the construction $\mathrm{Re}\, f = U = u|_{\mathfrak{D}}$ and hence $V := \mathrm{Im}\, f$ is its conjugate harmonic function in $\mathfrak{D}$. By the standard uniqueness theorem for analytic functions, we have that $V = v|_{\mathfrak{D}}$ where v is a conjugate harmonic function for u in $\mathbb{D}$. Recall that the latter is unique up to an additive constant. Thus, it remains to prove that the function $f(z)$ has (finite !) angular limits as $z \to \zeta$ for a.e. $\zeta \in E$. For this goal, note that the rectifiable curve $\partial\mathfrak{D}$ has tangent a.e. with respect to its natural parameter. It is clear that tangents at points $\zeta \in E$ to $\partial\mathfrak{D}$ (where they exist !) coincide with the corresponding tangents at ζ to $\partial\mathbb{D}$.

By the Caratheodory theorem ω can be extended to a homeomorphism of $\overline{\mathfrak{D}}$ onto $\overline{\mathbb{D}}$ and, since $\partial\mathfrak{D}$ is rectifiable, by the theorem of F. and M. Riesz length $\omega^{-1}(\mathcal{E}) = 0$ whenever $\mathcal{E} \subset \partial\mathbb{D}$ with length $\mathcal{E} = 0$, see e.g. Theorems II.C.1 and II.D.2 in [3]. By the Lindelöf theorem, see e.g. Theorem II.C.2 in [3], if $\partial\mathfrak{D}$ has a tangent at a point ζ, then

$$\arg\,[\omega(\zeta) - \omega(z)] - \arg\,[\zeta - z] \to \mathrm{const} \qquad \text{as} \quad z \to \zeta\ .$$

In other words, the conformal images of sectors in $\mathfrak{D}$ with a vertex at $\zeta \in \partial\mathfrak{D}$ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $w = \omega(\zeta) \in \partial\mathbb{D}$ up to the corresponding shifts and rotations. Consequently, nontangential paths in $\mathbb{D}$ are transformed under ω^{-1} into nontangential paths in $\mathfrak{D}$ and inversely at the corresponding points of $\partial\mathbb{D}$ and $\partial\mathfrak{D}$.

Thus, in particular, $v(z)$ has finite angular limits $\varphi_*(\zeta)$ for a.e. $\zeta \in E$. Moreover, the function $\varphi_* : E \to \mathbb{R}$ is measurable because $\varphi_*(\zeta) = \lim\limits_{n\to\infty} v_n(\zeta)$ where $v_n(\zeta) := v(r_n\zeta)$, $n = 1, 2, \ldots$, with $r_n \to 1 - 0$ as $n \to \infty$, see e.g. Corollary 2.3.10 in [13]. □

In particular, we have the following consequence of Theorem 1.

Corollary 1. *Let $u : \mathbb{D} \to \mathbb{R}$ be a harmonic function that has angular limits a.e. on the unit circle $\partial\mathbb{D}$. Then its conjugate harmonic functions v in $\mathbb{D}$ also have angular limits a.e. on $\partial\mathbb{D}$ and both boundary functions are measurable.*

By Remark 1 we have also the next consequence of Theorem 1.

Corollary 2. *Let $u : \mathbb{D} \to \mathbb{R}$ be a harmonic function in the class h^1. Then its conjugate harmonic functions $v : \mathbb{D} \to \mathbb{R}$ have (finite !) angular limits $v(z) \to \varphi(\zeta)$ as $z \to \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$.*

3. The case of rectifiable Jordan domains.

Theorem 2. *Let D be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and $u : D \to \mathbb{R}$ be a harmonic function that has angular limits on a measurable set E of ∂D with respect to the natural parameter. Then its conjugate harmonic functions $v : D \to \mathbb{R}$ also have (finite !) angular limits a.e. on E with respect to the natural parameter and both boundary functions are measurable on E with respect to this parameter.*

Proof. Again by the Riemann theorem there exists a conformal mapping $w = \omega(z)$ of D onto $\mathbb{D}$ and by the Caratheodory theorem ω can be extended to a homeomorphism of $\overline{D}$ onto $\overline{\mathbb{D}}$. As known, a rectifiable curves have tangent a.e. with respect to the natural parameter. Hence ∂D has a tangent at every point ζ of the set E except its subset $\mathcal{E}$ with length $\mathcal{E} = 0$. By the Lindelöf theorem, for every $\zeta \in E \setminus \mathcal{E}$,

$$\arg\,[\omega(\zeta) - \omega(z)] - \arg\,[\zeta - z] \to const \qquad as \quad z \to \zeta\ .$$

Thus, the harmonic function $u_* := u \circ \omega^{-1}$ given in $\mathbb{D}$ has angular limits $\varphi_*(w)$ at all points w of the set $E_* := \omega(E \setminus \mathcal{E}) \subseteq \partial\mathbb{D}$. Consequently, by Theorem 1 its conjugate harmonic function $v_* : \mathbb{D} \to \mathbb{R}$ has (finite !) angular limits $\psi_*(w)$ at a.e. point $w \in E_*$ and the boundary functions $\varphi_* : E_* \to \mathbb{R}$ and $\psi_* : E_* \to \mathbb{R}$ are measurable. The harmonic function $v := v_* \circ \omega$ is conjugate for u because the function $f := f_* \circ \omega$, where $f_* := u_* + v_*$, is analytic. Finally, by theorems of Lindelöf and F. and M. Riesz v has (finite !) angular limits $\psi(\zeta) = \psi_*(\omega(\zeta))$ at a.e. point $\zeta \in E$.

The boundary functions $\varphi = \varphi_* \circ \omega$ and $\psi = \psi_* \circ \omega$ of u and v on E, correspondingly, are measurable functions on E because $\varphi(\zeta) = \lim\limits_{n\to\infty} \varphi_n(\zeta)$ for all $\zeta \in E$ and $\psi(\zeta) = \lim\limits_{n\to\infty} \psi_n(\zeta)$ for a.e. $\zeta \in E$, where the functions $\varphi_n(\zeta) := u_*(r_n\omega(\zeta))$ and $\psi_n(\zeta) := v_*(r_n\omega(\zeta))$ with $r_n \to 1 - 0$ as $n \to \infty$ are continuous, see e.g. Corollary 2.3.10 in [13]. □

Corollary 3. *Let D be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and $u : D \to \mathbb{R}$ be a harmonic function that has angular limits a.e. on ∂D with respect to the natural parameter. Then its conjugate harmonic functions $v : D \to \mathbb{R}$ also have (finite !) angular limits a.e. on ∂D and both boundary functions are measurable on E with respect to the natural parameter.*

Remark 3. These results can be extended to domains whose boundaries consist of a finite number of mutually disjoint rectifiable Jordan curves (through splitting into a finite collection of Jordan's domains !).

The established facts can be applied to various boundary value problems for harmonic and analytic functions in the plane, see e.g. [16]–[19].

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В. И. Рязанов

О граничном поведении сопряженных гармонических функций.

Доказывается, что если гармоническая функция u, заданная в единичном круге $\mathbb{D}$ комплексной плоскости $\mathbb{C}$, имеет угловые пределы на измеримом множестве E единичной окружности, то ее сопряженная гармоническая функция v в $\mathbb{D}$ также имеет угловые пределы п.в. на E и обе граничные функции п.в. конечны и измеримы на E. Затем этот результат распространяется на произвольные жордановы области со спрямляемыми границами в терминах угловых пределов относительно естественного параметра. Результат существенно основывается на теореме Фату об угловых пределах ограниченных аналитических функций и конструкции Лузина и Привалова к их теореме единственности для аналитических и мероморфных функций. Результат будет иметь интересные приложения к изучению различных интегралов Стилтьеса в теории гармонических и аналитических функций и, в частности, интеграла Гильберта–Стилтьеса.

Ключевые слова: *корреляция, граничное поведение, сопряженные гармонические функции, спрямляемые жордановы кривые, угловые пределы, краевые задачи.*

В. І. Рязанов

Про граничну поведінку пов'язаних гармонійних функцій.

Доводиться, що якщо гармонійна функція u, що задана в одиничному колі $\mathbb{D}$ комплексної площині $\mathbb{C}$, має кутові межі на вимірної множині E одиничного кола, то її сполучена гармонійна функція v в $\mathbb{D}$ також має кутові межі п.в. на E і обидві граничні функції п.в. кінцеві та вимірні на E. Потім цей результат поширюється на довільні жорданова області з границями, що спрямляються в термінах кутових меж щодо природного параметра. Результат істотно ґрунтується на теоремі Фату про кутові межи обмежених аналітичних функцій та конструкції Лузіна і Привалова до їх теоремі єдиності для аналітичних і мероморфних функцій. Результат буде мати цікаві додатки до вивчення різних інтегралів Стілтьєса в теорії гармонійних і аналітичних функцій і, зокрема, інтеграла Гільберта–Стілтьєса.

Ключові слова: *кореляція, гранична поведінка, пов'язані гармонійні функції, жорданові криві, що спрямляються, кутові границі, крайові задачі.*

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Получено 07.11.17