Український математичний вісник
Том 13 **(2016)**, № 1, 28 – 67

1–D Schrödinger operators with local interactions on a discrete set with unbounded potentia

Aleksandra Yu. Ananieva

(*Presented by M.M. Malamud*)

Abstract. We study spectral properties of the One-dimensional Schrödinger operators $H_{X,\alpha,q} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + \sum_{x_n \in X} \alpha_n \delta(x - x_n)$ with local interactions, $d_* = 0$ and an unbounded potential q being a piecewise constant function by using the technique of boundary triplets and the corresponding Weyl functions. Using various sufficient conditions for the self-adjointness, discreteness of Jacobi matrices, we obtain a self-adjointness, discreteness condition for the operator $H_{X,\alpha,q}$.

2000 MSC. 34L10.

Key words and phrases. Schrödinger operator, unbounded potential, self-adjoint, local point interaction, discreteness.

1. Introduction

Let $\mathbb{R}_+ = [0, +\infty)$, and let $X = \{x_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$ be a strictly increasing sequence ($x_{n+1} > x_n$ for all $n \in \mathbb{N}$) such that $x_n \to +\infty$. We set $x_0 = 0$, $d_n := x_n - x_{n-1}$ for all $n \in \mathbb{N}$ and

$$d_* := \inf_{n\in\mathbb{N}} d_n = \inf_{n\in\mathbb{N}} (x_n - x_{n-1}), \qquad d^* := \sup_{n\in\mathbb{N}} d_n = \sup_{n\in\mathbb{N}} (x_n - x_{n-1}).$$

Let $H_{X,\alpha,q}$ be the minimal symmetric operator associated in $L^2(\mathbb{R}_+)$ with the differential expression

$$\ell_{X,\alpha,q} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + \sum_{x_n \in X} \alpha_n \delta(x - x_n), \qquad x \geq 0. \tag{1.1}$$

Received 29.11.2014

Namely, assuming that $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $q : \mathbb{R}_+ \to \mathbb{R}$ is locally square integrable function on $\mathbb{R}_+$, $q \in L^2_{\text{loc}}(\mathbb{R}_+)$, define the pre-minimal operator $H^0_{X,\alpha,q}$ in $L^2(\mathbb{R}_+)$ by the differential expression

$$\tau_q := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) \tag{1.2}$$

on the domain

$$\operatorname{dom}(H^0_{X,\alpha,q}) = \Big\{ f \in W^{2,2}_{\text{comp}}(\mathbb{R}_+ \setminus X) : f'(0) = 0, \qquad (1.3)$$
$$\left. \begin{array}{c} f(x_n+) = f(x_n-) \\ f'(x_n+) - f'(x_n-) = \alpha_n f(x_n) \end{array}, n \in \mathbb{N} \right\}.$$

Clearly, $H^0_{X,\alpha,q}$ is symmetric and we denote its closure by $H_{X,\alpha,q}$. Note that if all the $\alpha_n = 0$, the operator $H_{X,0,q} =: H^N_q$ is the Neumann realization of the expression (1.2).

The operator $H_{X,\alpha,q}$ describes δ-interactions on a discrete set $X = \{x_n\}_{n\in\mathbb{N}}$, and the coefficient α_n is called the strength of the interaction at $x = x_n$. Let us stress that the operator $H_{X,\alpha,q}$ is symmetric but not automatically self-adjoint even in the case $q \equiv 0$ (see [21, 22, 34]).

Schrödinger operators with point interaction on a finite or a discrete set arise in various physical applications (see [3]). In recent years spectral properties of the operator $H_{X,\alpha,q}$ have been studied in numerous papers (see, e.g., [4, 7, 8, 15, 16, 18, 20–25, 27, 30, 31, 33, 34], and also [22] for a comprehensive overview).

Here we study spectral properties of the Hamiltonian $H_{X,\alpha,q}$ with $d_* = 0$ and an unbounded potential q being a piecewise constant function. Namely, later on in this paper we make the following assumption:

Hypothesis 1. *Assume that*

$$q(x) \equiv q_n > 0, \quad x \in (x_{n-1}, x_n), \tag{1.4}$$

for all $n \in \mathbb{N}$, and the sequence $\{q_n\}_{n\in\mathbb{N}}$ satisfies the following condition:

$$\sup_{n\in\mathbb{N}} d_n \sqrt{q_n} =: c < \infty. \tag{1.5}$$

Let us mention that (1.5) covers the very important case in our considerations:

$$d_n \sqrt{q_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.6}$$

Our main tool is a powerful approach developed recently in [21]. Namely, applying the technique of boundary triplets and the corresponding Weyl function (see [12, 13, 17]), it was shown in [21] that spectral properties of the operator $H_{X,\alpha,q}$ with a bounded potential $q \in L^\infty(\mathbb{R}_+)$ closely correlate with the corresponding properties of a certain class of Jacobi matrices. Similar results were obtained later for Schrödinger operators with a matrix-valued potential [24] as well as for Dirac operators [9]. Our main aim is to extend the results of [21] to the case of unbounded potentials satisfying Hypothesis 1. Namely, consider the following Jacobi (three-diagonal) matrix

$$B_{X,\alpha,q} = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{1.7}$$

where

$$a_n = -\frac{\sqrt{q_{n+1}}}{r_n r_{n+1} \sinh(d_{n+1}\sqrt{q_{n+1}})}, \qquad r_n := \sqrt{d_n + d_{n+1}}, \tag{1.8}$$

$$b_n = \frac{\alpha_n}{d_n + d_{n+1}} + \frac{\sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})}{d_n + d_{n+1}}, \quad n \in \mathbb{N} \tag{1.9}$$

Our main result reads as follows.

Theorem 1.1. *Assume that Hypothesis 1 holds, $H_{X,\alpha,q}$ is the minimal symmetric operator associated with* (1.1)*. Let also $B_{X,\alpha,q}$ be the minimal operator associated with the Jacobi matrix* (1.7)*. Then:*

(i) The deficiency indices of $H_{X,\alpha,q}$ and $B_{X,\alpha,q}$ are equal and

$$\mathrm{n}_\pm(H_{X,\alpha,q}) = \mathrm{n}_\pm(B_{X,\alpha,q}) \le 1.$$

In particular, $H_{X,\alpha,q}$ is self-adjoint if and only if $B_{X,\alpha,q}$ is self-adjoint.

(ii) The operator $H_{X,\alpha,q}$ is lower semibounded if and only if so is the operator $B_{X,\alpha,q}$.

In addition, assume that $H_{X,\alpha,q}$ (and hence $B_{X,\alpha,q}$) is self-adjoint. Then:

(iii) The operator $H_{X,\alpha,q}$ is nonnegative if and only if so is $B_{X,\alpha,q}$.

(iv) The total multiplicities of the negative spectra of $H_{X,\alpha,q}$ and $B_{X,\alpha,q}$ coincide:

$$\kappa_-(H_{X,\alpha,q}) = \kappa_-(B_{X,\alpha,q}). \tag{1.10}$$

(v) *For any* $p \in (0, \infty]$, *the following equivalence holds:*

$$E_{H_{X,\alpha,q}}(\mathbb{R}_-)H_{X,\alpha,q} \in \mathfrak{S}_p \iff E_{B_{X,\alpha,q}}(\mathbb{R}_-)B_{X,\alpha,q} \in \mathfrak{S}_p.$$

In particular, the negative part of the spectrum $H_{X,\alpha,q}$ *is discrete if and only if the same holds for the negative spectrum of* $B_{X,\alpha,q}$.

(vi) $\sigma_c(H_{X,\alpha,q}) \subseteq [0, \infty)$ *if and only if* $\sigma_c(B_{X,\alpha,q}) \subseteq [0, \infty)$.

(vii) $\sigma_c(H_{X,\alpha,q}) \subset (0, \infty)$ *if and only if* $\sigma_c(B_{X,\alpha,q}) \subset (0, \infty)$.

(viii) *The operator* $H_{X,\alpha,q}$ *has purely discrete spectrum if and only if* $\lim\limits_{n\to\infty} d_n = 0$ *and* $B_{X,\alpha,q}$ *has purely discrete spectrum.*

(ix) *Let* $\widetilde{\alpha} = \{\widetilde{\alpha}_k\}_{k=1}^{\infty} \subset \mathbb{R}$, *and let* $B_{X,\widetilde{\alpha},q}$ *be the minimal operator associated with the matrix* (1.7) *and constructed by the sequence* $\widetilde{\alpha}$ *instead of* α. *If* $H_{X,\widetilde{\alpha},q} = H^*_{X,\widetilde{\alpha},q}$ *then* $B_{X,\widetilde{\alpha},q} = B^*_{X,\widetilde{\alpha},q}$, *and for any* $p \in (0, +\infty]$ *the following equivalence holds:*

$$(H_{X,\alpha,q} - \mathrm{i})^{-1} - (H_{X,\widetilde{\alpha},q} - \mathrm{i})^{-1} \in \mathfrak{S}_p \iff (B_{X,\alpha,q} - \mathrm{i})^{-1} - (B_{X,\widetilde{\alpha},q} - \mathrm{i})^{-1} \in \mathfrak{S}_p.$$

Combining Theorem 1.1(i) with the Carleman test (see, e.g., [1, Chapter II]), we obtain the following result.

Proposition 1.2. *Assume that Hypothesis 1 holds. Then the Hamiltonian* $H_{X,\alpha,q}$ *is self-adjoint for any* $\alpha = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *provided that*

$$\sum_{n=1}^{\infty} d_n^2 = \infty. \tag{1.11}$$

Note that this result is sharp. Namely, if $\{d_n\}_{n=1}^{\infty} \in l^2$ and the coefficients $X = \{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\alpha_n \in \mathbb{R}$ satisfy certain concavity assumptions, then the operator $H_{X,\alpha,q}$ is symmetric with $\mathrm{n}_\pm(H_{X,\alpha,q}) = 1$ (see Proposition 6.8). Note that in the case $q \in L^\infty(\mathbb{R}_+)$ Proposition 1.2 was first proved in [21]. More general result was proved later in [30].

Investigating discreteness and absolute continuity of spectra of the operator $B_{X,\alpha,q}$ we arrive at the following sufficient condition (see Propositions 6.17 and 6.23).

Proposition 1.3. *Assume that Hypothesis 1 holds and* $\lim_{n\to\infty} d_n\sqrt{q_n} = 0$. *Assume also that* $\lim_{n\to\infty} d_n = 0$ *and the operator* $B_{X,\alpha,q}$ *is self-adjoint. If*

$$\lim_{n\to\infty} \left|\frac{\alpha_{n-1}}{d_n} + q_n\right| = \infty, \quad \lim_{k\to\infty} \frac{1}{d_k(\alpha_k + q_{k+1}d_{k+1})} > -\frac{1}{4}$$

$$\textit{and} \qquad \lim_{n\to\infty} \frac{1}{d_n \alpha_{n-1}} > -\frac{1}{4}, \tag{1.12}$$

then the operator $H_{X,\alpha,q}$ *has discrete spectrum.*

This result is of interest only in the case when the operator $H_{X,\alpha,q}$ is not semi-bounded since the lower-bounded below Hamiltonian $H_{X,\alpha,q}$ is always self-adjoint (see [4]).

Proposition 1.4. *Assume that Hypothesis 1 holds, and assume that*

$$\sum_{n=1}^{\infty} \frac{|\alpha_n|}{d_{n+1}} < \infty. \tag{1.13}$$

Then absolutely continuous part $H^{\mathrm{ac}}_{X,\alpha,q}$ *of the Hamiltonian* $H_{X,\alpha,q}$ *is unitarily equivalent to the operator* $H^N_q := H_{X,0,q}$ *that is the Neumann realization of* (1.2) *in* $L^2(\mathbb{R}_+)$*. In particular,*

$$\sigma_{ac}(H_{X,\alpha,q}) = \sigma_{ac}(H^N_q), \tag{1.14}$$

where $\mathrm{dom}(H^N_q) = \mathrm{dom}(H_{X,0,q}) \subset \{W^{2,2}(\mathbb{R}_+) : f'(0) = 0\}$.

If, in addition $q \in L^1(\mathbb{R}_+)$*, then* $\sigma_{ac}(H_{X,\alpha,q}) = \mathbb{R}_+$.

The main results are announced in [5].

Notation. Let $\mathfrak{H}$, $\mathcal{H}$ stand for the separable Hilbert spaces. Further, $[\mathfrak{H},\mathcal{H}]$ denotes the set of bounded operators from $\mathfrak{H}$ to $\mathcal{H}$; $[\mathfrak{H}] := [\mathfrak{H},\mathfrak{H}]$; $\mathfrak{S}_p(\mathcal{H})$, $(p \in (0,\infty))$, denotes the Neumann-Schatten ideal in $\mathcal{H}$. In particular, $\mathfrak{S}_\infty(\mathcal{H})$ is the set of compact operators in $\mathcal{H}$, $\mathfrak{S}_1(\mathcal{H})$ is the trace class of the operators in $\mathcal{H}$, $\mathcal{C}(\mathfrak{H})$ and $\widetilde{\mathcal{C}}(\mathfrak{H})$ are the sets of closed operators and linear relations in $\mathfrak{H}$, respectively. Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows $\mathrm{dom}(T)$, $\ker(T)$, and $\mathrm{ran}(T)$ denote the domain, the kernel and the range of T, respectively; $\sigma(T)$, $\rho(T)$ and $\widehat{\rho}(T)$ denote the spectrum, the resolvent set and the set of regular type points of T, respectively; $R_T(\lambda) := (T - \lambda I)^{-1}$, $\lambda \in \rho(T)$, is the resolvent of T.

By $W^{2,2}(\mathbb{R}_+ \setminus X)$, $W^{2,2}_0(\mathbb{R}_+ \setminus X)$, and $W^{2,2}_{\mathrm{loc}}(\mathbb{R}_+ \setminus X)$ we denote the Sobolev spaces

$$\begin{aligned}
W^{2,2}(\mathbb{R}_+ \setminus X) &:= \{f \in L^2(\mathbb{R}_+) : f, f' \in AC_{loc}(\mathbb{R}_+ \setminus X), f'' \in L^2(\mathbb{R}_+)\},\\
W^{2,2}_0(\mathbb{R}_+ \setminus X) &:= \{f \in W^{2,2}(\mathbb{R}_+) : f(x_k) = f'(x_k) = 0 \text{ for all } x_k \in X\},\\
W^{2,2}_{\mathrm{comp}}(\mathbb{R}_+ \setminus X) &:= \{f \in W^{2,2}(\mathbb{R}_+ \setminus X) : \mathrm{supp}\, f \text{ is compact in } \mathbb{R}_+\}.
\end{aligned}$$

Let I be a subset of $\mathbb{Z}$, $I \subseteq \mathbb{Z}$. We denote by $l^2(I,\mathcal{H})$ the Hilbert space of $\mathcal{H}$-valued sequences such that $\|f\|^2 = \sum_{n\in I} \|f_n\|^2_{\mathcal{H}} < \infty$; $l^2_0(I,\mathcal{H})$ is a set of sequences with a finite number of nonzero components; we also abbreviate $l^2 := l^2(\mathbb{N},\mathbb{C})$, $l^2_0 := l^2_0(\mathbb{N},\mathbb{C})$.

2. Preliminaries

2.1. Boundary triplets and Weyl functions

In this section we briefly recall the notion of abstract boundary triplets and associated Weyl functions in the extension theory of symmetric operators (for a detailed study of boundary triplets we refer the reader to [12, 13, 17]).

Linear relations, boundary triplets, and self-adjoint extensions

1. The set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$ is the set of closed linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. Recall that $\mathrm{dom}(\Theta) = \big\{f : \{f, f'\} \in \Theta\big\}$, $\mathrm{ran}(\Theta) = \big\{f' : \{f, f'\} \in \Theta\big\}$ and $\mathrm{mul}\,(\Theta) = \big\{f' : \{0, f'\} \in \Theta\big\}$ are the domain, the range and the multivalued part of Θ , respectively. A closed linear operator A in $\mathcal{H}$ is identified with its graph $\mathrm{gr}(A)$, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in $\mathcal{H}$ is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. In particular, a linear relation Θ is an operator if and only if $\mathrm{mul}\,(\Theta)$ is trivial. For the definition of the inverse linear relation, the resolvent set and the spectrum of linear relations we refer to [14]. We recall that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} : (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is said to be *symmetric* if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.

For a symmetric linear relation $\Theta \subseteq \Theta^*$ in $\mathcal{H}$, the multivalued part $\mathrm{mul}\,(\Theta)$ is the orthogonal complement of $\mathrm{dom}(\Theta)$ in $\mathcal{H}$. Setting $\mathcal{H}_{\mathrm{op}} := \overline{\mathrm{dom}(\Theta)}$ and $\mathcal{H}_\infty = \mathrm{mul}\,(\Theta)$, one arrives at the orthogonal decomposition $\Theta = \Theta_{\mathrm{op}} \oplus \Theta_\infty$, where Θ_{op} is a symmetric operator in $\mathcal{H}_{\mathrm{op}}$, the operator part of Θ, and $\Theta_\infty = \big\{ \big(\begin{smallmatrix} 0 \\ f' \end{smallmatrix}\big) : f' \in \mathrm{mul}\,(\Theta) \big\}$ is a "pure" linear relation in $\mathcal{H}_\infty$.

2. Let A be a densely defined closed symmetric operator in the separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $\mathrm{n}_\pm(A) = \dim \mathfrak{N}_{\pm \mathrm{i}} \leq \infty$, $\mathfrak{N}_z := \ker(A^* - z)$.

Definition 2.1 ([17]). *A triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is called a* boundary triplet *for the adjoint operator* A^* *if* $\mathcal{H}$ *is a Hilbert space and* $\Gamma_0, \Gamma_1 : \mathrm{dom}(A^*) \to \mathcal{H}$ *are bounded linear mappings such that the abstract Green identity*

$$(A^* f, g)_{\mathfrak{H}} - (f, A^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}(A^*), \tag{2.1}$$

holds and the mapping $\Gamma := \begin{pmatrix}\Gamma_0\\ \Gamma_1\end{pmatrix} : \mathrm{dom}(A^*) \to \mathcal{H}\oplus\mathcal{H}$ *is surjective.*

First note that a boundary triplet for A^* exists since the deficiency indices of A are assumed to be equal. Moreover, $\mathrm{n}_\pm(A) = \dim(\mathcal{H})$ and $A = A^* \upharpoonright (\ker(\Gamma_0) \cap \ker(\Gamma_1))$ hold. Note also that a boundary triplet for A^* is not unique.

A closed extension $\widetilde{A}$ of A is called *proper* if $A \subseteq \widetilde{A} \subseteq A^*$. Two proper extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ of A are called *disjoint* if $\mathrm{dom}(\widetilde{A}_1) \cap \mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A)$ and *transversal* if , in addition , $\mathrm{dom}(\widetilde{A}_1)\dotplus\mathrm{dom}(\widetilde{A}_2) = \mathrm{dom}(A^*)$. The set of all proper extensions of A is denoted by $\mathrm{Ext}A$. Fixing a boundary triplet Π one can parameterize the set $\mathrm{Ext}A$ in the following way.

Proposition 2.2 ([13])**.** *Let A be as above , and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. Then the mapping $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathrm{dom}(A^*) \to \mathcal{H}\times\mathcal{H}$ establishes a bijective correspondence between the sets Ext_A and $\widetilde{\mathcal{C}}(\mathcal{H})$ as follows:*

$$\Theta \mapsto A_\Theta := A^* \upharpoonright \Gamma^{-1}\Theta = A^* \upharpoonright \big\{f \in \mathrm{dom}(A^*) :\ \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\big\}. \quad (2.2)$$

At the same time, the following relations hold:

(i) $A_\Theta^* = A_{\Theta^*}$.

(ii) The extensions A_Θ and A_0 are disjoint (transversal) if and only if $\Theta \in \mathcal{C}(\mathcal{H})$ $\big(\Theta \in [\mathcal{H}]\big)$. In this case, A_Θ admits a representation $A_\Theta = A^* \upharpoonright \ker(\Gamma_1 - \Theta\Gamma_0)$.

(iii) $A_\Theta \in \mathcal{C}(\mathfrak{H})$ *if and only if* $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$.

(iv) $A_{\Theta_1} \subseteq A_{\Theta_2}$ *if and only if* $\Theta_1 \subseteq \Theta_2$.

(v) A_Θ is symmetric (self–adjoint) if and only if the same is true for Θ, and $\mathrm{n}_\pm(A_\Theta) = \mathrm{n}_\pm(\Theta)$ holds.

(vi) Let $A_\Theta = A_\Theta^*$ and $A_{\widetilde{\Theta}} = A_{\widetilde{\Theta}}^*$. Then for any $p \in (0, +\infty]$ there holds the equivalence:

$$(A_\Theta - \mathrm{i})^{-1} - (A_{\widetilde{\Theta}} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \iff (\Theta - \mathrm{i})^{-1} - (\widetilde{\Theta} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\mathcal{H}).$$

Moreover, if $\mathrm{dom}(\Theta) = \mathrm{dom}(\widetilde{\Theta})$, then the following implication holds:

$$\overline{\Theta - \widetilde{\Theta}} \in \mathfrak{S}_p(\mathcal{H}) \implies (A_\Theta - \mathrm{i})^{-1} - (A_{\widetilde{\Theta}} - \mathrm{i})^{-1} \in \mathfrak{S}_p(\mathfrak{H}).$$

Proposition 2.2 immediately implies that the extensions $A_0 := A^* \upharpoonright \ker(\Gamma_0)$ and $A_1 := A^* \upharpoonright \ker(\Gamma_1)$ are self-adjoint. Clearly, $A_j = A_{\Theta_j}$, $j \in \{0, 1\}$, where the subspaces $\Theta_0 := \{0\} \times \mathcal{H}$ and $\Theta_1 := \mathcal{H} \times \{0\}$ are self-adjoint relations in $\mathcal{H}$. Note that Θ_0 is a “pure” linear relation.

Weyl functions, γ-fields, and Krein type formula for resolvents

1. In [12,13] the concept of the classical Weyl–Titchmarsh m-function from the theory of Sturm–Liouville operators was generalized to the case of symmetric operators with equal deficiency indices. The role of abstract Weyl functions in the extension theory is similar to that of the classical Weyl–Titchmarsh m-function in the spectral theory of singular Sturm–Liouville operators.

Definition 2.3 ([12])**.** *Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator valued functions $\gamma : \rho(A_0) \to [\mathcal{H}, \mathfrak{H}]$ and $M : \rho(A_0) \to [\mathcal{H}]$ defined by*

$$\gamma(z) := \big(\Gamma_0 \upharpoonright \mathfrak{N}_z\big)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \qquad z \in \rho(A_0), \quad (2.3)$$

are called the γ-field *and the* Weyl function*, respectively, corresponding to the boundary triplet* Π.

The γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ in (2.3) are well defined. Moreover, both $\gamma(\cdot)$ and $M(\cdot)$ are holomorphic on $\rho(A_0)$ and the following relations hold (see [12]):

$$\gamma(z) = \big(I + (z - \zeta)(A_0 - z)^{-1}\big)\gamma(\zeta), \quad (2.4)$$

$$M(z) - M(\zeta)^* = (z - \overline{\zeta})\gamma(\zeta)^*\gamma(z), \quad (2.5)$$

$$\gamma^*(\overline{z}) = \Gamma_1(A_0 - z)^{-1}, \qquad z,\ \zeta \in \rho(A_0). \quad (2.6)$$

Identity (2.5) yields that $M(\cdot)$ is an $R_{\mathcal{H}}$-function (or *Nevanlinna function*), that is, $M(\cdot)$ is an ($[\mathcal{H}]$-valued) holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ and

$$\operatorname{Im} z \cdot \operatorname{Im} M(z) \geq 0, \qquad M(z^*) = M(\overline{z}), \qquad z \in \mathbb{C} \setminus \mathbb{R}. \quad (2.7)$$

Besides, it follows from (2.5) that $M(\cdot)$ satisfies $0 \in \rho(\operatorname{Im} M(z))$ for $z \in \mathbb{C} \setminus \mathbb{R}$. Since A is densely defined, $M(\cdot)$ admits an integral representation (see, for instance, [13]):

$$M(z) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{1 + t^2}\right) d\Sigma_M(t), \qquad z \in \rho(A_0), \quad (2.8)$$

where $\Sigma_M(\cdot)$ is an operator-valued Borel measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma_M(t) \in [\mathcal{H}]$ and $C_0 = C_0^* \in [\mathcal{H}]$. The integral in (2.8) is understood in the strong sense.

In contrast to spectral measures of self-adjoint operators, the measure $\Sigma_M(\cdot)$ is not necessarily orthogonal. However, the measure Σ_M is uniquely determined by the Nevanlinna function $M(\cdot)$. The operator-valued measure Σ_M is called *the spectral measure* of $M(\cdot)$. If A is a simple symmetric operator, then the Weyl function $M(\cdot)$ determines the pair $\{A, A_0\}$ up to unitary equivalence (see [13, 26]). Due to this fact, spectral properties of A_0 can be expressed in terms of $M(\cdot)$.

2. The following result provides a description of resolvents and spectra of proper extensions of the operator A in terms of the Weyl function $M(\cdot)$ and the corresponding boundary parameters.

Proposition 2.4 ([12]). *For any $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ the following Krein type formula holds:*

$$(A_\Theta - z)^{-1} - (A_0 - z)^{-1} = \gamma(z)(\Theta - M(z))^{-1}\gamma^*(\overline{z}), \quad z \in \rho(A_0) \cap \rho(A_\Theta). \tag{2.9}$$

Moreover, if $z \in \rho(A_0)$, then

$$z \in \sigma_i(A_\Theta) \quad \Leftrightarrow \quad 0 \in \sigma_i(\Theta - M(z)), \qquad i \in \{\mathrm{p},\ \mathrm{c},\ \mathrm{r}\}.$$

Formula (2.9) is a generalization of the well known Krein formula for canonical resolvents (cf. [2]). We note also that all the objects in (2.9) are expressed in terms of the boundary triplet Π.

The following result is deduced from (2.9).

Proposition 2.5 ([12]). *Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $\Theta_1, \Theta_2 \in \widetilde{\mathcal{C}}(\mathcal{H})$, and let $\mathfrak{S}_p$, $p \in (0, \infty)$, the Neumann–Schatten ideal. Then*

(i) for any $z \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})$, $\zeta \in \rho(\Theta_1) \cap \rho(\Theta_2)$ the following equivalence holds:

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \iff (\Theta_1 - \zeta)^{-1} - (\Theta_2 - \zeta)^{-1} \in \mathfrak{S}_p(\mathcal{H}). \tag{2.10}$$

(ii) If, in addition, $\Theta_1, \Theta_2 \in \mathcal{C}(\mathcal{H})$ and $\mathrm{dom}(\Theta_1) = \mathrm{dom}(\Theta_2)$, then

$$\overline{\Theta_1 - \Theta_2} \in \mathfrak{S}_p(\mathcal{H}) \Longrightarrow (A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}_p(\mathfrak{H}). \tag{2.11}$$

(iii) Moreover, if $\Theta_1, \Theta_2 \in [\mathcal{H}]$, then implication (2.11) *becomes equivalence.*

Extensions of a nonnegative operator

Assume that a symmetric operator $A \in \mathcal{C}(\mathfrak{H})$ is nonnegative. Then the set $\mathrm{Ext}_A(0, \infty)$ of its nonnegative self-adjoint extensions is non-empty (see [2,19]). Moreover, there is a maximal nonnegative extension A_F (also called *Friedrichs'* or *hard* extension), and there is a minimal nonnegative extension A_{K} (*Krein's* or *soft* extension) satisfying

$$(A_F+x)^{-1} \leq (\widetilde{A}+x)^{-1} \leq (A_K+x)^{-1}, \qquad x \in (0,\infty), \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty)$$

(for details we refer the reader to [2, 17]).

Proposition 2.6 ([12]). *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be a boundary triplet for* A^* *such that* $A_0 = A_0^* \geq 0$. *Let* $M(\cdot)$ *be the corresponding Weyl function. Then* $A_0 = A_{\mathrm{F}}$ $(A_0 = A_{\mathrm{K}})$ *if and only if*

$$\lim_{x\downarrow-\infty}(M(x)f,f) = -\infty, \quad \bigl(\lim_{x\uparrow 0}(M(x)f,f) = +\infty\bigr), \quad f \in \mathcal{H} \setminus \{0\}. \tag{2.12}$$

Proposition 2.7 ([12]). *Let* A *be a non-negative symmetric operator in* $\mathfrak{H}$. *Assume that* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is a boundary triplet for* A^* , *and* $M(\cdot)$ *is the corresponding Weyl function. Let also* $A_0 = A_{\mathrm{F}}$ *is the Friedrichs extension. Then the following assertions hold:*

(*i*) *a linear relation* $\Theta \in \widetilde{\mathcal{C}}_{\mathrm{self}}(\mathcal{H})$ *is semibounded below;*

(*ii*) *a self-adjoint extension* A_Θ *is semibounded below;*

are equivalent if and only if $M(\cdot)$ *uniformly tends to* $-\infty$ *as* $x \to -\infty$, *i.e., for any* $a > 0$ *there exists* $x_a < 0$ *such that* $M(x_a) < -a \cdot I_{\mathcal{H}}$.

In this case we will write $M(x) \rightrightarrows -\infty$ *as* $x \to -\infty$.

3. Direct sums of boundary triplets

Let S_n be a densely defined symmetric operator in a Hilbert space $\mathfrak{H}_n$ with $\mathrm{n}_+(S_n) = \mathrm{n}_-(S_n) \leq \infty$, $n \in \mathbb{N}$. Consider the operator $A := \bigoplus_{n=1}^\infty S_n$ acting in $\mathfrak{H} := \bigoplus_{n=1}^\infty \mathfrak{H}_n$, the Hilbert direct sum of Hilbert spaces $\mathfrak{H}_n$. By definition, $\mathfrak{H} = \{f = \oplus_{n=1}^\infty f_n : \ f_n \in \mathfrak{H}_n, \ \sum_{n=1}^\infty \|f_n\|^2 < \infty\}$. Clearly,

$$A^* = \bigoplus_{n=1}^\infty S_n^*,$$

$$\mathrm{dom}(A^*) = \{f = \oplus_{n=1}^\infty f_n \in \mathfrak{H} : \ f_n \in \mathrm{dom}(S_n^*), \ \sum_{n\in\mathbb{N}} \|S_n^* f_n\|^2 < \infty\}. \tag{3.1}$$

We equip the domains $\mathrm{dom}(S_n^*) =: \mathfrak{H}_{n+}$ and $\mathrm{dom}(A^*) =: \mathfrak{H}_+$ with the graph norms $\|f_n\|^2_{\mathfrak{H}_{n+}} := \|f_n\|^2 + \|S_n^* f_n\|^2$ and $\|f\|^2_{\mathfrak{H}_+} := \|f\|^2 + \|A^* f\|^2 = \sum_n \|f_n\|^2_{\mathfrak{H}_{n+}}$, respectively.

Further, let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^*, $n \in \mathbb{N}$. By $\|\Gamma_j^{(n)}\|$ we denote the norm of the linear mapping $\Gamma_j^{(n)} \in [\mathfrak{H}_{n+}, \mathcal{H}_n]$, $j \in \{0,1\}$, $n \in \mathbb{N}$.

Let $\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ be a Hilbert direct sum of $\mathcal{H}_n$. Define mappings Γ_0 and Γ_1 by setting

$$\Gamma_j := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)},$$

$$\operatorname{dom}(\Gamma_j) = \big\{f = \oplus_{n=1}^{\infty} f_n \in \operatorname{dom}(A^*) : \sum_{n\in\mathbb{N}} \|\Gamma_j^{(n)} f_n\|_{\mathcal{H}_n}^2 < \infty\big\}. \tag{3.2}$$

Clearly, $\operatorname{dom}(\Gamma) := \operatorname{dom}(\Gamma_1) \cap \operatorname{dom}(\Gamma_0)$ is dense in $\mathfrak{H}_+$. Define the operators $S_{nj} := S_n^* \upharpoonright \ker \Gamma_j^{(n)}$ and $A_j := \bigoplus_{n=1}^{\infty} S_{nj}$, $j \in \{0,1\}$. Then A_0 and A_1 are self-adjoint extensions of A. Note that A_0 and A_1 are disjoint but not necessarily transversal.

Definition 3.1. *Let Γ_j be defined by* (3.2) *and $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a direct sum of boundary triplets and will be assigned as $\Pi := \bigoplus_{n=1}^{\infty} \Pi_n$.*

The following criterions have been obtained in [9, 21].

Theorem 3.2. *Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* and $M_n(\cdot)$ the corresponding Weyl function, $n \in \mathbb{N}$. A direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for the operator $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if*

$$C_1 = \sup_n \|M_n(i)\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_2 = \sup_n \|(\operatorname{Im} M_n(i))^{-1}\|_{\mathcal{H}_n} < \infty. \tag{3.3}$$

Theorem 3.2 makes it possible to construct a boundary triplet by regularizing an arbitrary direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets.

Theorem 3.3 ([28, 29])**.** *Let S_n be a symmetric operator in $\mathfrak{H}_n$ with deficiency indices $\mathrm{n}_\pm(S_k) = \mathrm{n}_n \le \infty$ and $S_{n0} = S_{n0}^* \in \mathrm{Ext} S_n$, $n \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ there exists a boundary triplet $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for S_n^* such that $\ker \Gamma_0^{(n)} = \operatorname{dom}(S_{n0})$ and $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ satisfying $\ker \Gamma_0 = \operatorname{dom}(\widetilde{A}_0) := \bigoplus_{n=1}^{\infty} \operatorname{dom}(S_{n0})$.*

Next we assume that the operator $A = \bigoplus_{n=1}^{\infty} S_n$ has a regular real point, i.e., there exists an $a = \overline{a} \in \hat{\rho}(A)$. The latter is equivalent to the existence of $\varepsilon > 0$ such that

$$(a - \varepsilon, a + \varepsilon) \subset \cap_{n=1}^{\infty} \widehat{\rho}(S_n). \tag{3.4}$$

Emphasize that condition $a \in \cap_{n=1}^{\infty} \widehat{\rho}(S_n)$ is not enough for the inclusion $a \in \widehat{\rho}(A)$ to hold.

It is known that under condition (3.4) for every $k \in \mathbb{N}$ there exists a selfadjoint extension $\widetilde{S}_k = \widetilde{S}_k^*$ of S_k preserving the gap $(a-\varepsilon, a+\varepsilon)$. Moreover, the Weyl function of the pair $\{S_k, \widetilde{S}_k\}$ is regular within the gap $(a-\varepsilon, a+\varepsilon)$.

Theorem 3.4 ([9, Theorem 2.12])**.** *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying* (3.4)*. Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, and let $M_n(\cdot)$ be the corresponding Weyl function. Then:*

(i) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a B-generalized boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if

$$C_3 := \sup_{n\in\mathbb{N}} \|M_n(a)\|_{\mathcal{H}_n} < \infty \qquad \text{and} \qquad C_4 := \sup_{n\in\mathbb{N}} \|M_n'(a)\|_{\mathcal{H}_n} < \infty, \tag{3.5}$$

where $M_n'(a) := (dM_n(z)/dz)|_{z=a}$.

(ii) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if, in addition to (3.5)*, the following condition is satisfied:*

$$C_5 := \sup_{n\in\mathbb{N}} \|\big(M_n'(a)\big)^{-1}\|_{\mathcal{H}_n} < \infty. \tag{3.6}$$

Corollary 3.5 ([9, Corollary 2.13])**.** *Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying* (3.4)*. Let also $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \upharpoonright \ker(\widetilde{\Gamma}_0^{(n)})$, and $\widetilde{M}_n(\cdot)$ the corresponding Weyl function. Assume also that for some operators R_n such that $R_n, R_n^{-1} \in [\mathcal{H}_n]$, the following conditions are satisfied:*

$$\sup_n \|R_n^{-1}(\widetilde{M}_n'(a))(R_n^{-1})^*\|_{\mathcal{H}_n} < \infty \quad \text{and}$$

$$\sup_n \|R_n^*(\widetilde{M}_n'(a))^{-1} R_n\|_{\mathcal{H}_n} < \infty, \quad n \in \mathbb{N}. \tag{3.7}$$

Then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets

$$\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\} \quad \text{with} \quad \Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)},$$

$$\Gamma_1^{(n)} := (R_n^{-1})^* \big(\widetilde{\Gamma}_1^{(n)} - \widetilde{M}_n(a)\widetilde{\Gamma}_0^{(n)}\big), \tag{3.8}$$

forms a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.

4. First boundary triplet for the operator H_n

In what follows $\mathbb{R}_+ = [0, +\infty)$ and $X = \{x_n\}_{n=0}^\infty \subset \mathbb{R}_+$ is a strictly increasing sequence.

Consider the following symmetric operator in $L^2(x_{n-1}, x_n)$

$$H_n = -\frac{d^2}{dx^2} + q_n, \qquad \mathrm{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n], \tag{4.1}$$

where q_n satisfies (1.5).

Lemma 4.1. *Assume that Hypothesis 1 holds. Then the operator H_n is a symmetric one with deficiency indices $\mathrm{n}_\pm(H_n) = 2$. Its adjoint H_n^* is given by*

$$H_n^* = H_n, \qquad \mathrm{dom}(H_n^*) = W^{2,2}[x_{n-1}, x_n].$$

Moreover, the following assertions hold:

(i) A boundary triplet for the operator H_n^* can be chosen as follows:

$$\mathcal{H} = \mathbb{C}^2, \qquad \widetilde{\Gamma}_0^{(n)} = \begin{pmatrix} f(x_{n-1}) \\ f'(x_n) \end{pmatrix}, \qquad \widetilde{\Gamma}_1^{(n)} = \begin{pmatrix} f'(x_{n-1}) \\ f(x_n) \end{pmatrix}. \tag{4.2}$$

(ii) The corresponding Weyl function $\widetilde{M}_n(\cdot)$ is

$$\widetilde{M}_n(z) = \begin{pmatrix} \sqrt{z-q_n}\tan(d_n\sqrt{z-q_n}) & \frac{1}{\cos(d_n\sqrt{z-q_n})} \\ \frac{1}{\cos(d_n\sqrt{z-q_n})} & \frac{\tan(d_n\sqrt{z-q_n})}{\sqrt{z-q_n}} \end{pmatrix}. \tag{4.3}$$

Proof. It is straightforward. □

Clearly, $H_{\min}$ is closed operator with $\mathrm{n}_\pm(H_{\min}) = \infty$, and

$$H_{\max} := H_{\min}^* = \bigoplus_{n=1}^\infty H_n^*,$$

$$\mathrm{dom}(H_{\max}) \subseteq W^{2,2}(\mathbb{R}_+ \backslash X) = \bigoplus_{n=1}^\infty W^{2,2}[x_{n-1}, x_n].$$

Proposition 4.2. *Assume that Hypothesis 1 holds. Let $X = \{x_n\}_{n=0}^\infty$ be as above and $d^* < +\infty$. Define the mappings $\Gamma_j^{(n)} : W^{2,2}[x_{n-1}, x_n] \to \mathbb{C}^2$, $\quad n \in \mathbb{N}$, $\quad j \in \{0, 1\}$, by setting*

$$\Gamma_0^{(n)} = \begin{pmatrix} d_n^{1/2} f(x_{n-1}) \\ d_n^{3/2} f'(x_n) \end{pmatrix}, \tag{4.4}$$

$$\Gamma_1^{(n)} = \begin{pmatrix} d_n^{-1/2} f'(x_{n-1}) + \sqrt{\frac{q_n}{d_n}} \tanh(d_n\sqrt{q_n}) f(x_{n-1}) - \frac{d_n^{-1/2} f'(x_n)}{\cosh(d_n\sqrt{q_n})} \\ d_n^{-3/2} f(x_n) - \frac{d_n^{-3/2} f(x_{n-1})}{\cosh(d_n\sqrt{q_n})} - \frac{\tanh(d_n\sqrt{q_n}) f'(x_n)}{\sqrt{q_n d_n^3}} \end{pmatrix}. \tag{4.5}$$

Define the function $M_n(z)$ *given by*

$$M_n(z) = \begin{pmatrix} \frac{1}{d_n}(\sqrt{z-q_n}\tan(d_n\sqrt{z-q_n}) + \sqrt{q_n}\tanh(d_n\sqrt{q_n})) & \frac{1}{d_n^2}\left(\frac{1}{\cos(d_n\sqrt{z-q_n})} - \frac{1}{\cosh(d_n\sqrt{z-q_n})}\right) \\ \frac{1}{d_n^2}\left(\frac{1}{\cos(d_n\sqrt{z-q_n})} - \frac{1}{\cosh(d_n\sqrt{z-q_n})}\right) & \frac{1}{d_n^3}\left(\frac{\tan(d_n\sqrt{z-q_n})}{\sqrt{z-q_n}} - \frac{\tanh(d_n\sqrt{q_n})}{\sqrt{q_n}}\right) \end{pmatrix}. \tag{4.6}$$

Then:

(i) For any $n \in \mathbb{N}$ *the triplet* $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ *is the boundary triplet for the operator* H_n^*.

(ii) The Weyl function $M_n(z)$ *corresponding to the triplet* Π_n *takes the form* (4.6).

(iii) The direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *with* $\mathcal{H} = \mathbb{C}^2$ *and* $\Gamma_j = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}$, $j \in \{0,1\}$, *is a boundary triplet for the operator* $H_{\min}^* = \bigoplus_{n=1}^{\infty} H_n^*$.

Proof. (*i*) The proof is straightforward. Note, however, that it follows from Lemma 4.1 since

$$\Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \qquad \Gamma_1^{(n)} := R_n^{-1}(\widetilde{\Gamma}_1^{(n)} - Q_n \widetilde{\Gamma}_0^{(n)}), \qquad n \in \mathbb{N}, \tag{4.7}$$

where

$$R_n := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{3/2} \end{pmatrix},$$

$$Q_n := \widetilde{M}_n(0) = \begin{pmatrix} -\sqrt{q_n}\tanh(d_n\sqrt{q_n}) & \frac{1}{\cosh(d_n\sqrt{q_n})} \\ \frac{1}{\cosh(d_n\sqrt{q_n})} & \frac{\tanh(d_n\sqrt{q_n})}{\sqrt{q_n}} \end{pmatrix}, \quad n \in \mathbb{N}. \tag{4.8}$$

(*ii*) It easily follows from (4.3) and (4.7) that

$$M_n(z) = R_n^{-1}(\widetilde{M}_n(z) - Q_n) R_n^{-1}, \qquad n \in \mathbb{N}. \tag{4.9}$$

(*iii*) We set $v_n := d_n\sqrt{q_n}$. Then

$$\begin{aligned} M_n'(0) &= R_n^{-1} \widetilde{M}_n'(0) R_n^{-1} \\ &= \begin{pmatrix} \frac{\sinh(v_n)\cosh(v_n)+v_n}{2v_n\cosh^2(v_n)} & \frac{\sinh(v_n)}{2v_n\cosh^2(v_n)} \\ \frac{\sinh(v_n)}{2v_n\cosh^2(v_n)} & \frac{\sinh(v_n)\cosh(v_n)-v_n}{2v_n^3\cosh^2(v_n)} \end{pmatrix}, \quad n \in \mathbb{N}. \end{aligned} \tag{4.10}$$

Clearly, (1.5) implies

$$\sinh(v_n) < 2^{-1}\exp(v_n).$$

Since, in addition, $\lim\limits_{x\to 0}\frac{\sinh(x)}{x} = 1$, the matrices $M'_n(0)$ are uniformly bounded

$$\sup_{n\in\mathbb{N}} \|M'_n(0)\| =: c_1 < \infty. \tag{4.11}$$

Further,

$$(M'_n(0))^{-1} = R_n(\widetilde{M}'_n(0))^{-1}R_n$$
$$= \begin{pmatrix} \frac{2v_n(\sinh(v_n)\cosh(v_n)-v_n)}{\sinh^2(v_n)-v_n^2} & \frac{-2v_n^3\sinh(v_n)}{\sinh^2(v_n)-v_n^2} \\ \frac{-2v_n^3\sinh(v_n)}{\sinh^2(v_n)-v_n^2} & \frac{2v_n^3(\sinh(v_n)\cosh(v_n)+v_n)}{\sinh^2(v_n)-v_n^2} \end{pmatrix}, \quad n\in\mathbb{N}. \tag{4.12}$$

Similarly, (1.5) yields uniform boundedness of matrices $(M'_n(0))^{-1}$, i.e.,

$$\sup_{n\in\mathbb{N}} \|(M'_n(0))^{-1}\| =: c_2 < \infty. \tag{4.13}$$

One completes the proof by applying Theorem 3.4. □

Remark 4.3. Assume condition (1.6). Then we have

$$\lim_{n\to\infty} M'_n(0) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}, \qquad n\in\mathbb{N}, \tag{4.14}$$

$$\lim_{n\to\infty} (M'_n(0))^{-1} = \begin{pmatrix} 4 & -6 \\ -6 & 12 \end{pmatrix}, \qquad n\in\mathbb{N}. \tag{4.15}$$

5. Second boundary triplets for the operator H_n

In what follows $\mathbb{R}_+ = [0,\infty) \subseteq \mathbb{R}$ denotes a bounded interval or positive semi-axis, $X = \{x_n\}_{n=0}^{\infty} \subset \mathbb{R}_+$ is a strictly increasing sequence.

Consider the following symmetric operator in $L^2(x_{n-1}, x_n)$:

$$H_n = -\frac{d^2}{dx^2} + q_n, \qquad \mathrm{dom}(H_n) = W_0^{2,2}[x_{n-1}, x_n], \tag{5.1}$$

where q_n satisfies (1.5).

Lemma 5.1. *Assume that Hypothesis 1 holds. Then the operator H_n is a symmetric one with deficiency indices $n_\pm(H_n) = 2$.*
Its adjoint H_n^* is given by

$$H_n^* = H_n, \qquad \mathrm{dom}(H_n^*) = W^{2,2}[x_{n-1}, x_n].$$

Moreover, the following assertions hold:

(i) A boundary triplet for the operator H_n^* can be chosen as follows:

$$\mathcal{H} = \mathbb{C}^2, \qquad \widetilde{\Gamma}_0^{(n)} = \begin{pmatrix} f(x_{n-1}) \\ -f(x_n) \end{pmatrix}, \qquad \widetilde{\Gamma}_1^{(n)} = \begin{pmatrix} f'(x_{n-1}) \\ f'(x_n) \end{pmatrix}; \tag{5.2}$$

(ii) The corresponding Weyl function $\widetilde{M}_n(\cdot)$ is

$$\widetilde{M}_n(z) = \begin{pmatrix} -\sqrt{z-q_n}\cot(d_n\sqrt{z-q_n}) & -\frac{\sqrt{z-q_n}}{\sin(d_n\sqrt{z-q_n})} \\ -\frac{\sqrt{z-q_n}}{\sin(d_n\sqrt{z-q_n})} & -\sqrt{z-q_n}\cot(d_n\sqrt{z-q_n}) \end{pmatrix}. \tag{5.3}$$

Proof. It is straightforward. □

Proposition 5.2. *Assume that Hypothesis 1 holds. Let also $X = \{x_n\}_{n=0}^{\infty}$ be as above , and let $d^* < +\infty$. For any $n \in \mathbb{N}$, define the boundary triplet $\Pi^{(n)} = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for H_n^* by setting*

$$\Gamma_j^{(n)} : W^{2,2}[x_{n-1}, x_n] \to \mathbb{C}^2\,, \quad n \in \mathbb{N}\,, \quad j \in \{0,1\}\,,$$

$$\Gamma_0^{(n)} = \sqrt{d_n}\begin{pmatrix} f(x_{n-1}) \\ -f(x_n) \end{pmatrix}, \tag{5.4}$$

$$\Gamma_1^{(n)} = \frac{1}{\sqrt{d_n}}\begin{pmatrix} f'(x_{n-1}) + \sqrt{q_n}f(x_{n-1})\coth(d_n\sqrt{q_n}) - \frac{\sqrt{q_n}f(x_n)}{\sinh(d_n\sqrt{q_n})} \\ f'(x_n) + \frac{\sqrt{q_n}f(x_{n-1})}{\sinh(d_n\sqrt{q_n})} - \sqrt{q_n}f(x_n)\coth(d_n\sqrt{q_n}) \end{pmatrix}. \tag{5.5}$$

Define the function $M_n(z)$ by

$$M_n(z) = \begin{pmatrix} a_n(z) & b_n(z) \\ b_n(z) & a_n(z) \end{pmatrix}, \tag{5.6}$$

where

$$a_n(z) := \frac{1}{d_n}(-\sqrt{z-q_n}\cot(d_n\sqrt{z-q_n}) + \sqrt{q_n}\coth(d_n\sqrt{q_n})),$$

$$b_n(z) := \frac{1}{d_n}\left(-\frac{\sqrt{z-q_n}}{\sin(d_n\sqrt{z-q_n})} + \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\right).$$

Then:

(i) For any $n \in \mathbb{N}$ the triplet $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ is the boundary triplet for operator H_n^*.

(ii) The Weyl function $M_n(z)$ corresponding to the triplet Π_n takes the form (5.6).

(iii) The direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *with* $\mathcal{H} = \mathbb{C}^2$ *and* $\Gamma_j = \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}$, $j \in \{0,1\}$, *is a boundary triplet for the operator* $H^*_{\min} = \bigoplus_{n=1}^{\infty} H^*_n$.

Proof. (i) The proof is straightforward. Note, however, that it follows from Lemma 5.1 since

$$\Gamma_0^{(n)} := R_n \widetilde{\Gamma}_0^{(n)}, \quad \Gamma_1^{(n)} := R_n^{-1}(\widetilde{\Gamma}_1^{(n)} - Q_n \widetilde{\Gamma}_0^{(n)}), \qquad n \in \mathbb{N}, \tag{5.7}$$

where

$$R_n := \begin{pmatrix} d_n^{1/2} & 0 \\ 0 & d_n^{1/2} \end{pmatrix},$$

$$Q_n := \widetilde{M}_n(0) = \begin{pmatrix} -\sqrt{q_n}\coth(d_n\sqrt{q_n}) & -\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \\ -\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} & -\sqrt{q_n}\coth(d_n\sqrt{q_n}) \end{pmatrix}, \quad n \in \mathbb{N}. \tag{5.8}$$

(ii) It easily follows from (5.3) and (5.7) that

$$M_n(z) = R_n^{-1}(\widetilde{M}_n(z) - Q_n)R_n^{-1}, \qquad n \in \mathbb{N}. \tag{5.9}$$

(iii) We set $v_n := d_n\sqrt{q_n}$. Then

$$\begin{aligned} M_n'(0) &= R_n^{-1} \widetilde{M}_n'(0)\, R_n^{-1} \\ &= \begin{pmatrix} \frac{\cosh(v_n)\sinh(v_n) - v_n}{2v_n\sinh^2(v_n)} & \frac{\sinh(v_n) - v_n\cosh(v_n)}{2v_n\sinh^2(v_n)} \\ \frac{\sinh(v_n) - v_n\cosh(v_n)}{2v_n\sinh^2(v_n)} & \frac{\cosh(v_n)\sinh(v_n) - v_n}{2v_n\sinh^2(v_n)} \end{pmatrix}, \qquad n \in \mathbb{N}. \end{aligned} \tag{5.10}$$

Clearly, (1.5) implies

$$\cosh(v_n) < 2^{-1}\exp(v_n).$$

Since, in addition, $\lim\limits_{x\to 0} \frac{\sinh(x)}{x} = 1$, the matrices $M_n'(0)$ are uniformly bounded

$$\sup_{n\in\mathbb{N}} \|M_n'(0)\| =: c_3 < \infty. \tag{5.11}$$

Further,

$$\begin{aligned} &(M_n'(0))^{-1} = R_n(\widetilde{M}_n'(0))^{-1}R_n \\ &= \frac{1}{\sinh^2(v_n) - v_n^2} \cdot \begin{pmatrix} 2(\cosh(v_n)\sinh(v_n) - v_n) & 2(v_n\cosh(v_n) - \sinh(v_n)) \\ 2(v_n\cosh(v_n) - \sinh(v_n)) & 2(\cosh(v_n)\sinh(v_n) - v_n) \end{pmatrix}, \\ &\qquad n \in \mathbb{N}. \end{aligned} \tag{5.12}$$

Similarly, (1.5) yields uniform boundedness of matrices $(M_n'(0))^{-1}$, i.e.,

$$\sup_{n\in\mathbb{N}} \|(M_n'(0))^{-1}\| =: c_4 < \infty. \tag{5.13}$$

One completes the proof by applying Theorem 3.4. □

Remark 5.3. Assume that condition (1.6) is met. Then we get

$$M_n'(0) = R_n^{-1}\widetilde{M}_n'(0)\, R_n^{-1} \longrightarrow \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}, \qquad n\to\infty, \tag{5.14}$$

$$(M_n'(0))^{-1} = R_n(\widetilde{M}_n'(0))^{-1}R_n \longrightarrow \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \qquad n\to\infty. \tag{5.15}$$

Proposition 5.4. *Assume that Hypothesis 1 holds. Let also* Π *be the boundary triplet for operator* $H_{\min}^*$ *defined in Proposition 5.2, and let* $M(\cdot)$ *be the corresponding Weyl function. If*

$$d^* = \sup_{n\in\mathbb{N}} d_n < +\infty, \tag{5.16}$$

then

$$M(-a^2) \rightrightarrows -\infty \quad \textit{as} \quad a\to+\infty. \tag{5.17}$$

Proof. The Weyl function $M(\cdot)$ has the form $M(z) = \oplus_{n=1}^{\infty} M_n(z)$, where $M_n(\cdot)$ is given by (5.6). Introduce the following matrix-valued function

$$M(-a^2; d_n, q_n) := \begin{pmatrix} F_a(d_n,q_n) & G_a(d_n,q_n) \\ G_a(d_n,q_n) & F_a(d_n,q_n) \end{pmatrix}, \tag{5.18}$$

where

$$\begin{aligned} F_a(d_n,q_n) &:= \frac{1}{d_n}\left[-\sqrt{a^2+q_n}\coth(d_n\sqrt{a^2+q_n}) + \sqrt{q_n}\coth(d_n\sqrt{q_n})\right] \\ &= \frac{1}{d_n^2}\left[-\sqrt{d_n^2a^2+d_n^2q_n}\coth(\sqrt{d_n^2a^2+d_n^2q_n}) + d_n\sqrt{q_n}\coth(d_n\sqrt{q_n})\right], \end{aligned} \tag{5.19}$$

$$\begin{aligned} G_a(d_n,q_n) &:= \frac{1}{d_n}\left[-\frac{\sqrt{a^2+q_n}}{\sinh(d_n\sqrt{a^2+q_n})} + \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\right] \\ &= \frac{1}{d_n^2}\left[-\frac{\sqrt{d_n^2a^2+d_n^2q_n}}{\sinh(\sqrt{d_n^2a^2+d_n^2q_n})} + \frac{d_n\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\right]. \end{aligned} \tag{5.20}$$

Let us check that

$$G_a(d_n,q_n) > 0 \quad \text{and} \quad F_a(d_n,q_n) < 0 \quad \text{for} \quad a^2 > 1.$$

Consider the function $f_1(x) := \frac{\sinh(\sqrt{x})}{\sqrt{x}}$. Since

$$f_1'(x) = \left(\frac{\sinh(\sqrt{x})}{\sqrt{x}}\right)' = \frac{\sqrt{x}\cosh(\sqrt{x}) - \sinh(\sqrt{x})}{2x\sqrt{x}}$$

$$= \frac{e^{\sqrt{x}}(\sqrt{x}-1) + e^{-\sqrt{x}}(\sqrt{x}+1)}{4x\sqrt{x}} > 0 \quad \text{for} \quad x > 1,$$

then we have that $f_1(x)$ grows if $x > 1$. Hence the function $f_1^{-1}(x) = \frac{\sqrt{x}}{\sinh(\sqrt{x})}$ decreases for $x > 1$. This implies that $G_a(d_n, q_n) > 0$ whenever $a^2 > 1$.

Further, consider function $f_2(x) := \sqrt{x}\coth(\sqrt{x})$. Since

$$f_2'(x) = \left(\sqrt{x}\coth(\sqrt{x})\right)' = \frac{\cosh(\sqrt{x})\sinh(\sqrt{x}) - \sqrt{x}}{2\sqrt{x}\sinh^2(\sqrt{x})}$$

$$= \frac{\sinh(2\sqrt{x}) - 2\sqrt{x}}{2\sqrt{x}\sinh^2(\sqrt{x})} > 0 \qquad \text{for} \quad x > 1,$$

then we have that $f_2(x)$ grows if $x > 1$. Hence $F_a(d_n, q_n) < 0$ for $a^2 > 1$.

According to Hypothesis 1, we have $d_n\sqrt{q_n} < c$. Since

$$\begin{aligned} &\begin{pmatrix} F_a(d_n,q_n) & G_a(d_n,q_n) \\ G_a(d_n,q_n) & F_a(d_n,q_n) \end{pmatrix} - (F_a(d_n,q_n) + G_a(d_n,q_n))I_2 \\ &= G_a(d_n,q_n)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned} \tag{5.21}$$

and $G_a(d_n, q_n) > 0$, we get the following inequality:

$$M(-a^2; d_n, q_n) \le (F_a(d_n,q_n) + G_a(d_n,q_n))I_2.$$

Further, consider the function

$$\begin{aligned} F_a(d_n,q_n) + G_a(d_n,q_n) &= \frac{1}{d_n^2}\left[\frac{d_n\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\{\cosh(d_n\sqrt{q_n}) + 1\}\right. \\ &\left.- \frac{\sqrt{a^2d_n^2 + d_n^2q_n}}{\sinh(\sqrt{a^2d_n^2 + d_n^2q_n})}\left\{\cosh(\sqrt{a^2d_n^2 + d_n^2q_n}) + 1\right\}\right]. \end{aligned} \tag{5.22}$$

Consider the function $g(x) := \frac{\sqrt{x}}{\sinh(\sqrt{x})}(1 + \cosh(\sqrt{x}))$. Since

$$g'(x) = \left(\frac{\sqrt{x}}{\sinh(\sqrt{x})}\left(1 + \cosh(\sqrt{x})\right)\right)' = \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sqrt{x}\sinh^2(\frac{\sqrt{x}}{2})} > 0, \tag{5.23}$$

the function $g(\cdot)$ grows. Applying the Lagrange theorem to the right-hand side of (5.22), we get

$$F_a(d_n, q_n) + G_a(d_n, q_n) = -\frac{1}{d_n^2}\big(g(a^2 d_n^2 + d_n^2 q_n) - g(d_n^2 q_n)\big) = -a^2 g'(\xi_n), \tag{5.24}$$

where $\xi_n \in (d_n^2 q_n, a^2 d_n^2 + d_n^2 q_n)$. Further, since $\lim_{x\to 0} g'(x) = \frac{1}{6} > 0$, there exists $\varepsilon > 0$ such that

$$g'(x) > \frac{1}{12}, \qquad x \in [\varepsilon, \infty). \tag{5.25}$$

On the other hand,

$$\lim_{x\to\infty} \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sinh^2(\frac{\sqrt{x}}{2})} = \frac{1}{2}. \tag{5.26}$$

Combining this relation with the obvious inequality $\sinh(\sqrt{x}) > \sqrt{x}$, $x > 0$, one arrives at the two-sided estimate

$$C_1 < \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sinh^2(\frac{\sqrt{x}}{2})} < C_2, \qquad x \in [\varepsilon, \infty). \tag{5.27}$$

It follows with account of (5.23) that

$$\frac{C_1}{\sqrt{x}} < g'(x) = \frac{\sinh(\sqrt{x}) - \sqrt{x}}{4\sqrt{x}\sinh^2(\frac{\sqrt{x}}{2})} < \frac{C_2}{\sqrt{x}}, \qquad x \in [\varepsilon, \infty). \tag{5.28}$$

Using $d_n^2 q_n < c^2$ (see Hypothesis 1) and (5.16), we derive

$$\frac{C_1}{\sqrt{a^2 d_n^2 + d_n^2 q_n}} > \frac{C_1}{\sqrt{a^2 d_n^2 + c^2}} > \frac{C_1}{\sqrt{a^2 (d^*)^2 + c^2}}. \tag{5.29}$$

Combining the latter with (5.25) and (5.28), one has

$$\inf_{x\in(d_n^2 q_n, a^2 d_n^2 + d_n^2 q_n)} g'(x) > \beta(a), \quad \text{where} \quad \beta(a) = \min\left\{\frac{1}{12}, \frac{C_1}{\sqrt{a^2 (d^*)^2 + c^2}}\right\}. \tag{5.30}$$

Choosing $a > \frac{\sqrt{3}c}{d^*}$, we continue this inequality as

$$\inf_{x\in(d_n^2 q_n, a^2 d_n^2 + d_n^2 q_n)} g'(x) > \frac{C_1}{2ad^*}, \qquad a > \frac{\sqrt{3}c}{d^*}. \tag{5.31}$$

Combining this estimate with (5.24) yields

$$\sup_n (F_a(d_n, q_n) + G_a(d_n, q_n)) \le -a^2 \cdot \frac{C_1}{2ad^*} = -a\frac{C_1}{2d^*}. \tag{5.32}$$

Since $M_n(-a^2) = M(-a^2, d_n)$, the preceding inequality implies

$$M(-a^2) = \oplus_{n=1}^{\infty} M_n(-a^2) \le -a\frac{C_1}{2d^*}, \qquad a > \frac{\sqrt{3}c}{d^*}. \tag{5.33}$$

Relation (5.17) is obviously yields. □

Combining Proposition 5.2 with Proposition 2.2, we arrive at the following parametrization of the set Ext$H_{\min}$ of closed proper extensions of the operator $H_{\min}$:

$$\widetilde{H} = H_\Theta := H_{\min}^* \lceil \mathrm{dom}(H_\Theta),$$

$$\mathrm{dom}(H_\Theta) = \{f \in \mathrm{dom}(H_{\min}^*) : \ \{\Gamma_0 f, \Gamma_1 f\} \in \Theta\}, \tag{5.34}$$

where $\Theta \in \widetilde{\mathcal{C}}(l^2)$ and $\Gamma_0,\ \Gamma_1$ are defined by (5.4)-(5.5).

Theorem 5.5. *Let $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ be a boundary triplet for $H_{\min}^*$ defined in Proposition 5.2, $\Theta, \widetilde{\Theta} \in \widetilde{\mathcal{C}}(\mathcal{H})$, and let $H_\Theta, H_{\widetilde{\Theta}} \in$ Ext$H_{\min}$ be proper extensions of $H_{\min}$ defined by* (5.34)*. Then:*

(i) The operator H_Θ is symmetric (self-adjoint) if and only if so is Θ, and $\mathrm{n}_\pm(H_{\min}) = \mathrm{n}_\pm(\Theta)$.

(ii) The self-adjoint (symmetric) operator H_Θ is lower semibounded if and only if so is Θ.

(iii) Let $\Theta = \Theta^*$. Then $\kappa_-(H_\Theta) = \kappa_-(\Theta)$. In particular $H_\Theta \ge 0$ if and if and only if $\Theta \ge 0$.

(iv) For any $p \in (0, \infty]$, $z \in \rho(H_\Theta) \cap \rho(H_{\widetilde{\Theta}})$, and $\zeta \in \rho(\Theta) \cap \rho(\widetilde{\Theta})$ the following equivalence holds

$$(H_\Theta - z)^{-1} - (H_{\widetilde{\Theta}} - z)^{-1} \in \mathfrak{S}_p \quad \Longleftrightarrow \quad (\Theta - \zeta)^{-1} - (\widetilde{\Theta} - \zeta)^{-1} \in \mathfrak{S}_p.$$

(v) The operator $H_\Theta = H_{\Theta}^*$ has discrete spectrum if and only if $d_n \searrow 0$ and Θ has discrete spectrum.

Proof. (i) is immediate from Proposition 2.2.

$(ii), (iii)$ Combining Proposition 2.7 with Proposition 5.4 yields the first statement.

(iv) is implied by Proposition 2.5.

(v) First we show that conditions are sufficient. Indeed, the operator

$$H_0 := H^*_{\min}\lceil\ker(\Gamma_0) = \oplus_{n\in\mathbb{N}} H_{n0}, \qquad H_{n0} := H^*_n\lceil\ker(\Gamma^{(n)}_0), \qquad (5.35)$$

has discrete spectrum if $\lim_{n\to\infty} d_n = 0$. Moreover, the Krein resolvent formula and discreteness of $\sigma(\Theta)$ implies $\mathcal{R}_{H_\Theta}(z) - \mathcal{R}_{H_0}(z) \in \mathfrak{S}_\infty$, $z \in \mathbb{C}_+$, and hence $\mathcal{R}_{H_\Theta}(z) \in \mathfrak{S}_\infty$.

Let us show that condition $d_n \searrow 0$ is necessary for discreteness of $\sigma(H_\Theta)$. Without loss of generality assume that $0 \in \rho(H_\Theta)$. Assume also that $\limsup_{n\to\infty} d_n > 0$ and H_Θ has discrete spectrum. Then there exists a sequence $\{d_{n_k}\}_{k=1}^\infty$ such that $d_{n_k} \geq d_*/2 > 0$. For $\varepsilon \in (0, d_*/2)$, define the function

$$\varphi_\varepsilon(\cdot) \in W^2_2(\mathbb{R}), \qquad \varphi_\varepsilon(x) = \begin{cases} 1, & \varepsilon \leq x \leq d_* - \varepsilon, \\ 0, & x \notin [0, d_*]. \end{cases}$$

Note that $\varphi_k(x) := P_\mathcal{I}\varphi_\varepsilon(x + x_{n_k}) \in \operatorname{dom}(H_\Theta)$, where $P_\mathcal{I}$ is the orthoprojection in $L^2(\mathbb{R})$ onto $L^2(\mathcal{I})$. Moreover, $\|\varphi_k\|_{L^2} \equiv \text{const}$ and $\|H_\Theta\varphi_k\|_{L^2} \equiv \text{const}$. Since the functions $\varphi_k(\cdot)$ have disjoint supports, the operator $(H_\Theta)^{-1}$ is not compact. Contradiction.

□

Remark 5.6. Clearly, all statements of Theorem 5.5 with exception of *(ii)*–*(iii)* remain valid for the boundary triplet $\Pi = \oplus_1^\infty \Pi_n$ with Π_n defined by (4.4)–(4.5) in place of (5.4)–(5.5).

Corollary 5.7. *If a is large enough, then $H_\Theta \geq -a^2$ whenever $\Theta \geq -\frac{a}{2d^*} I_{l_2}$.*

6. Schrödinger operators with δ-interactions

Now we return to the symmetric differential operator $H^0_{X,\alpha,q}$ in $L^2(\mathbb{R}_+)$

$$H^0_{X,\alpha,q} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_n,$$
$$\operatorname{dom}(H^0_{X,\alpha,q}) = \left\{ f \in W^{2,2}_{\text{comp}}(\mathcal{I} \setminus X) : \begin{array}{c} f'(0) = 0,\ f(x_n+) = f(x_n-) \\ f'(x_n+) - f'(x_n-) = \alpha_n f(x_n) \end{array} \right\}. \qquad (6.1)$$

As above, we denote by $H_{X,\alpha,q}$ the closure of $H^0_{X,\alpha,q}$, $H_{X,\alpha,q} = \overline{H^0_{X,\alpha,q}}$.

6.1. Parametrization of the operator $H_{X,\alpha,q}$

Let $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ and $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplets defined in Propositions 5.2 and 4.2, respectively. According to Proposition 2.2, the extension $H_{X,\alpha}(\in \mathrm{Ext}H_{\min})$ admits two representations

$$H_{X,\alpha,q} = H_{\Theta_j} := H^*_{\min}\lceil \mathrm{dom}(H_{\Theta_j}),$$

$$\mathrm{dom}(H_{\Theta_j}) = \{f \in \mathrm{dom}(H^*_{\min}) : \ \{\Gamma_0^j f, \Gamma_1^j f\} \in \Theta_j\}, \quad j \in \{1,2\}. \tag{6.2}$$

where $\Theta_j \in \widetilde{\mathcal{C}}(\mathcal{H})$ $(j \in \{1,2\})$ are closed symmetric linear relations. In this section we show that Θ_2 as well as the operator part Θ_1' of Θ_1 is a Jacobi matrix.

1. The first parametrization. At first we consider the triplet $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ constructed in Proposition 4.2. For any α the operators $H_{X,\alpha}$ and $H_0^{(1)} := H^*_{\min}\lceil \ker(\Gamma_0^1)$ are disjoint. Hence Θ_1 in (6.2) is a (closed) operator in $\mathcal{H} = l^2(\mathbb{N})$, $\Theta_1 \in \mathcal{C}(l^2)$. More precisely, consider the Jacobi matrix

$$B_{X,\alpha,q} := \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & b_4 & a_4 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{6.3}$$

where

$$b_{2k-1} = d_k^{-1}(\alpha_{k-1} + \sqrt{q_k}\tanh(d_k\sqrt{q_k})), \qquad b_{2k} = -\frac{\tanh(d_k\sqrt{q_k})}{d_k^3\sqrt{q_k}},$$

$$a_{2k-1} = -\frac{1}{d_k^2\cosh(d_k\sqrt{q_k})}, \qquad a_{2k} = d_k^{-3/2}d_{k+1}^{-1/2}.$$

Let $\tau_{X,\alpha,q}$ be a second order difference expression associated with (6.3). One defines the corresponding minimal symmetric operator in l^2 by (see [1, 6])

$$B^0_{X,\alpha,q}f := \tau_{X,\alpha,q}f, \qquad f \in \mathrm{dom}(B^0_{X,\alpha,q}) := l^2_0 \quad \text{and} \quad B_{X,\alpha,q} = \overline{B^0_{X,\alpha,q}}. \tag{6.4}$$

Recall that $B_{X,\alpha,q}$ has equal deficiency indices and $\mathrm{n}_+(B_{X,\alpha,q}) = \mathrm{n}_-(B_{X,\alpha,q}) \le 1$.

In addition, note that $B_{X,\alpha,q}$ admits a representation

$$B_{X,\alpha,q} = R_X^{-1}(\widetilde{B}_\alpha - Q_X)R_X^{-1},$$

$$\text{where} \quad \widetilde{B}_\alpha := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 1 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.5}$$

and $R_X = \oplus_{n=1}^{\infty} R_n$, $Q_X = \oplus_{n=1}^{\infty} Q_n$ are given by (4.8).

Proposition 6.1. *Let* $\Pi^1 = \{\mathcal{H}, \Gamma_0^1, \Gamma_1^1\}$ *be the boundary triplet for* $H_{\min}^*$ *constructed in Proposition 4.2 and let* $B_{X,\alpha,q}$ *be the minimal Jacobi operator defined by* (6.3)–(6.4). *Then* $\Theta_1 = B_{X,\alpha,q}$, *i.e.,*

$$H_{X,\alpha,q} = H_{B_{X,\alpha,q}} = H_{\min}^* \lceil \operatorname{dom}(H_{B_{X,\alpha,q}}),$$

$$\operatorname{dom}(H_{B_{X,\alpha,q}}) = \{f \in \operatorname{dom}(H_{\min}^*) : \Gamma_1^1 f = B_{X,\alpha,q} \Gamma_0^1 f\}.$$

Proof. Let $f \in W_{\text{comp}}^{2,2}(\mathbb{R}_+ \setminus X)$. Then $f \in \operatorname{dom}(H_{X,\alpha,q})$ if and only if $\widetilde{\Gamma}_1^1 f = \widetilde{B}_\alpha \widetilde{\Gamma}_0^1 f$. Here $\widetilde{\Gamma}_j^1 := \oplus_{n\in\mathbb{N}} \widetilde{\Gamma}_j^{(n)}$ where $\widetilde{\Gamma}_j^{(n)}$, $j \in \{0, 1\}$, are defined by (4.5), and $\widetilde{B}_\alpha$ is defined by (6.5). Combining (5.7) and (5.8) with (6.5), we rewrite the equality $\widetilde{\Gamma}_1^1 f = \widetilde{B}_\alpha \widetilde{\Gamma}_0^1 f$ as $\Gamma_1^1 f = B_{X,\alpha} \Gamma_0^1 f$.

Taking the closures one completes the proof. □

Remark 6.2. Note that the matrix (6.3) has negative off-diagonal entries, although in the classical theory of Jacobi operators, off-diagonal entries are assumed to be positive. But it is known (see, for instance, [35]) that the (minimal) operator $B_{X,\alpha,q}$ is unitarily equivalent to the minimal Jacobi operator associated with the matrix

$$B'_{X,\alpha,q} := \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ 0 & 0 & a_3 & b_4 & a_4 & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{6.6}$$

where

$$b_{2k-1} = d_k^{-1}(\alpha_{k-1} + \sqrt{q_k} \tanh(d_k \sqrt{q_k})), \qquad b_{2k} = -\frac{\tanh(d_k \sqrt{q_k})}{d_k^3 \sqrt{q_k}},$$

$$a_{2k-1} = \frac{1}{d_k^2 \cosh(d_k \sqrt{q_k})}, \qquad a_{2k} = d_k^{-3/2} d_{k+1}^{-1/2}.$$

In the sequel we will identify the operators $B_{X,\alpha,q}$ and $B'_{X,\alpha,q}$ when investigating those spectral properties of the operator $H_{X,\alpha,q}$, which are invariant under unitary transformations.

2. The second parametrization. Let us consider the boundary triplet $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ constructed in Proposition 5.2. Now the operators $H_{X,\alpha,q}$ and $H_0^{(2)} := H_{\min}^*\lceil\ker(\Gamma_0^2)$ are not disjoint, hence by Proposition 2.2(ii), the corresponding linear relation Θ_2 in (6.2) is not an operator, i.e., it has a nontrivial multivalued part, $\operatorname{mul}\Theta_2 := \{f \in \mathcal{H} : \{0, f\} \in \Theta_2\} \neq \{0\}$.

Let $f \in W^{2,2}_{\mathrm{comp}}(\mathbb{R}_+ \setminus X)$. Then $\Gamma_0^2 f, \Gamma_1^2 f \in l_0^2$ and $f \in \operatorname{dom}(H_{X,\alpha,q})$ if and only if $C_{X,\alpha,q}\Gamma_1 f = D_{X,\alpha,q}\Gamma_0 f$, where

$$C_{X,\alpha,q} := CR_X, \qquad D_{X,\alpha,q} := (D_\alpha - CQ_X)R_X^{-1}, \tag{6.7}$$

$$C := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -1 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$D_\alpha := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & \alpha_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \alpha_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.8}$$

and $R_X = \oplus_{n=1}^\infty R_n$, $Q_X = \oplus_{n=1}^\infty Q_n$ are defined by (5.8);

$$C_{X,\alpha,q} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -d_1^{1/2} & d_2^{1/2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -d_2^{1/2} & d_3^{1/2} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{6.9}$$

$$D_{X,\alpha,q} := (a_{i,j})_{i,j=1}^\infty, \tag{6.10}$$

where

$$a_{1,1} = d_1^{-1/2},$$

$$a_{2k,2k} = d_k^{-1/2}, \quad a_{2k,2k+1} = d_{k+1}^{-1/2},$$

$$a_{2k+1,2k-1} = -\frac{d_k^{-1/2}\sqrt{q_k}}{\sinh(d_k\sqrt{q_k})}, \quad a_{2k+1,2k} = -d_k^{-1/2}\sqrt{q_k}\coth(d_k\sqrt{q_k}),$$

$$a_{2k+1,2k+1} = d_{k+1}^{-1/2}(\alpha_k + \sqrt{q_{k+1}}\coth(d_k\sqrt{q_k})),$$

$$a_{2k+1,2k+2} = -\frac{d_{k+1}^{-1/2}\sqrt{q_{k+1}}}{\sinh(d_{k+1}\sqrt{q_{k+1}})},$$

$$a_{i,j} = 0 \quad \text{otherwise.}$$

Define a linear relation Θ_2^0 by

$$\Theta_2^0 = \{\{f, g\} \in l_0^2 \oplus l_0^2 : \ D_{X,\alpha,q}f = C_{X,\alpha,q}g\}. \tag{6.11}$$

Hence we obviously get

$$H^0_{X,\alpha,q} = H^*_{\min}\lceil \operatorname{dom}(H^0_{X,\alpha,q}),$$

$$\operatorname{dom}(H^0_{X,\alpha,q}) = \{f \in W^{2,2}_{\text{comp}}(\mathbb{R}_+ \setminus X) : \ \{\Gamma_0^2 f, \Gamma_1^2 f\} \in \Theta_2^0\}. \tag{6.12}$$

Direct calculations show that Θ_2^0 is symmetric. Moreover, (6.12) implies that the closure of Θ_2^0 is Θ_2. Hence Θ_2 is a closed symmetric linear relation. Therefore (see Subsection 2.1), Θ_2 admits the representation

$$\Theta_2 = \Theta_2^{\text{op}} \oplus \Theta_2^{\infty}, \quad \mathcal{H} = \mathcal{H}_{\text{op}} \oplus \mathcal{H}_{\infty},$$

$$\mathcal{H}_{\text{op}} = \overline{\operatorname{dom}(\Theta_2)} = \overline{\operatorname{dom}(\Theta_2^{\text{op}})}, \quad \mathcal{H}_\infty := \operatorname{mul}\Theta_2, \tag{6.13}$$

where $\Theta_2^{\text{op}} (\in \mathcal{C}(\mathcal{H}_{\text{op}}))$ is the operator part of Θ_2. Moreover, it follows from (6.7) that

$$\operatorname{mul}\Theta_2 = \ker(C_{X,\alpha}) = \overline{R_X^{-1}(\ker C)}, \qquad \Theta_2^\infty = \{\{0, f\} : \ f \in \operatorname{mul}\Theta_2\}. \tag{6.14}$$

Since $\mathcal{H}_{\text{op}} = \overline{\operatorname{ran}(R_X C^*)}$, the system $\{\mathbf{f}_n\}_{n=1}^\infty$, $\mathbf{f}_n := \frac{\sqrt{d_n}e_{2n} - \sqrt{d_{n+1}}e_{2n+1}}{\sqrt{d_n + d_{n+1}}}$, forms the orthonormal basis in $\mathcal{H}_{\text{op}}$. Next we show that the operator part Θ_2^{op} of Θ_2 is unitarily equivalent to the minimal Jacobi operator

$$B_{X,\alpha,q} = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{6.15}$$

where

$$b_n = r_n^{-2}(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})),$$

$$a_n = -\frac{\sqrt{q_{n+1}}}{r_n r_{n+1}\sinh(d_{n+1}\sqrt{q_{n+1}})},$$

and $r_n := \sqrt{d_n + d_{n+1}}$, $n \in \mathbb{N}$. We show that $\{\mathbf{f}_n\}_{n=1}^{\infty} \subset \mathrm{dom}(\Theta_2^{\mathrm{op}})$. Assume that there exists $\mathbf{g}_n$ such that $\{\mathbf{f}_n, \mathbf{g}_n\} \in \Theta_2^{\mathrm{op}}$, i.e., $\mathbf{g}_n = \Theta_2^{\mathrm{op}}\mathbf{f}_n$. The latter yields $\mathbf{g}_n \in \mathcal{H}_{\mathrm{op}}$ and hence $\mathbf{g}_n = \sum_{k=1}^{\infty} g_{n,k}\mathbf{f}_k$. Moreover, after direct calculations we obtain

$$\begin{aligned}
D_{X,\alpha,q}\mathbf{f}_1 &= r_1^{-1}\big(-(\alpha_1 + \sqrt{q_1}\coth(d_1\sqrt{q_1}) + \sqrt{q_2}\coth(d_2\sqrt{q_2}))\mathrm{e}_3 \\
&\qquad + \sqrt{q_2}\sinh^{-1}(d_2\sqrt{q_2})\mathrm{e}_5\big), \\
D_{X,\alpha,q}\mathbf{f}_n &= r_n^{-1}\left(\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}\mathrm{e}_{2n-1} - (\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n})\right. \\
&\qquad + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\mathrm{e}_{2n+1} \\
&\qquad \left. + \frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})}\mathrm{e}_{2n+3}\right), \quad n \geq 2, \\
C_{X,\alpha,q}\mathbf{g}_n &= -\sum_{k=1}^{\infty} g_{n,k} r_k \mathrm{e}_{2k+1}, \quad n \geq 1.
\end{aligned}$$

Hence $\{\mathbf{f}_n, \mathbf{g}_n\} \in \Theta$, i.e., equality $D_{X,\alpha,q}\mathbf{f}_n = C_{X,\alpha,q}\mathbf{g}_n$ holds if and only if

$$g_{n,n-1} = -\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n},$$

$$g_{n,n} = \frac{1}{r_n^2}\big(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\big),$$

$$g_{n,n+1} = -\frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})r_n r_{n+1}}, \qquad n \geq 2,$$

and $g_{n,k} = 0$ for all $k \notin \{n-1, n, n+1\}$. Hence $\mathbf{f}_n \in \mathrm{dom}(\Theta_2^{\mathrm{op}})$, and in the basis $\{\mathbf{f}_n\}_{n=1}^{\infty}$ the matrix representation of the operator Θ_2^{op} coincides with the matrix $B_{X,\alpha,q}$ defined by (6.15).

Since the operator $B_{X,\alpha,q}$ of the form (6.3) and (6.15) is closed, we conclude that Θ_1^{op} and $B_{X,\alpha,q}$ are unitarily equivalent.

Proposition 6.3. *Let $\Pi^2 = \{\mathcal{H}, \Gamma_0^2, \Gamma_1^2\}$ be the boundary triplet constructed in Proposition 5.2, and let the linear relation Θ_2 be defined by* (6.2). *Then Θ_2 admits representation* (6.13), *where the "pure" relation Θ_2^{∞} is determined by* (6.14) *and* (6.8), *and the operator part Θ_2^{op} is unitarily equivalent to the minimal Jacobi operator $B_{X,\alpha,q}$ of the form* (6.4) *and* (6.15).

6.2. Self–adjontness

Theorem 6.4. *The operator $H_{X,\alpha,q}$ has equal deficiency indices $\mathrm{n}_+(H_{X,\alpha,q}) = \mathrm{n}_-(H_{X,\alpha,q}) \leq 1$. Moreover, $\mathrm{n}_{\pm}(H_{X,\alpha,q}) = \mathrm{n}_{\pm}(B_{X,\alpha,q})$,* *where $B_{X,\alpha,q}$ is the minimal operator associated with the Jacobi matrix either* (6.3) *or* (6.15)*. In particular, $H_{X,\alpha,q}$ is self-adjoint if and only if $B_{X,\alpha,q}$ is.*

Proof. Combining Theorem 5.5 (i) with Propositions 6.1 and 6.3, we arrive at the equality $\mathrm{n}_\pm(H_{X,\alpha,q}) = \mathrm{n}_\pm(B_{X,\alpha,q})$. It remans to note that for Jacobi matrices $\mathrm{n}_\pm(B_{X,\alpha,q}) \le 1$ (see [1,6]). □

Corollary 6.5. *Let $B^{(1)}_{X,\alpha,q}$ and $B^{(2)}_{X,\alpha,q}$ be the minimal Jacobi operators associated with* (6.3) *and* (6.15)*, respectively. Then $\mathrm{n}_\pm(B^{(1)}_{X,\alpha,q}) = \mathrm{n}_\pm(B^{(2)}_{X,\alpha,q})$. In particular, $B^{(1)}_{X,\alpha,q}$ is self-adjoint if and only if so is $B^{(2)}_{X,\alpha,q}$.*

Proof. It immediately follows from Theorem 6.4. □

Proposition 6.6. *Assume Hypothesis 1. Then the Hamiltonian $H_{X,\alpha,q}$ is self-adjoint for any $\alpha = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{R}$ provided that*

$$\sum_{n=1}^\infty d_n^2 = \infty. \tag{6.16}$$

Proof. Consider the Jacobi matrix $B_{X,\alpha,q}$ (6.6). By Carleman's theorem [1], [6, Chapter VII.1.2], $B_{X,\alpha,q}$ is self-adjoint whenever

$$\sum_{n=1}^\infty (d_n^2 \cosh(d_n\sqrt{q_n}) + d_n^{3/2} d_{n+1}^{1/2}) = \infty. \tag{6.17}$$

Obviously,

$$d_n^2 \cosh(d_n\sqrt{q_n}) \sim d_n^2, \text{ and } d_n^2 < d_n^2 + d_n^{3/2} d_{n+1}^{1/2} \le \frac{7}{4} d_n^2 + \frac{1}{4} d_{n+1}^2 \text{ as } n \to \infty,$$

and hence relations (6.16) and (6.17) are equivalent.

Now, the result is implied by Theorem 6.4. □

Corollary 6.7 ([16])**.** *If* $\limsup_n d_n > 0$ *(in particular,* $d_* = \liminf_n d_n > 0$*), then $H_{X,\alpha}$ is self-adjoint.*

Let us present sufficient conditions for self-adjointness in the case when (6.16) does not hold.

Proposition 6.8. *Let $\{d_n\}_{n=1}^\infty \in l^2$,*

$$c_1 \le d_n\sqrt{q_n} \le c_2, \quad c_1, c_2 > 0, \tag{6.18}$$

and let

$$d_{n-1} \cdot d_{n+1} \ge d_n^2, \quad n \in \mathbb{N}. \tag{6.19}$$

If, in addition, the strengths α_n of δ-interactions satisfy

$$\sum_{n=1}^{\infty} d_{n+1}\,|\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}| < \infty, \tag{6.20}$$

then the operator $H_{X,\alpha,q}$ is symmetric with $\mathrm{n}_\pm(H_{X,\alpha,q}) = 1$.

Proof. Consider the Jacobi matrix (6.15). To apply [25, Theorem 1], we denote $a_n := r_n^{-2}|\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}|$ and $b_n := \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n}$, $n \in \mathbb{N}$, and define a sequence $\{c_n\}_{n=1}^{\infty}$ as follows:

$$c_1 := b_1, \qquad c_2 := 1, \qquad c_{n+1} := -\frac{b_{n-1}}{b_n}c_{n-1}, \qquad n \in \mathbb{N}.$$

It is easily seen that

$$c_{n+1} = (-1)^{n+1} r_{n+1} \frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}}$$

$$\times \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \ \ldots \ \cdot \widetilde{c}, \quad n \in \mathbb{N};$$

$$\widetilde{c} := \begin{cases} c_1 r_1^{-1}, & n = 2k, \\ c_2 r_2^{-1}, & n = 2k+1. \end{cases}$$

Using both the conditions (6.24)–(6.19) and the obvious inequality $\sinh(x) > x$, $x > 0$, we obtain

$$\frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}$$

$$\times \frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}} \cdot \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \ \ldots$$

$$= \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}}$$

$$\times \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})} \cdot \ \ldots$$

$$= \sqrt{\frac{\sinh(d_{n+2}\sqrt{q_{n+2}})}{\sqrt{q_{n+2}}}}$$

$$\times \left(\frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}} \cdot \sqrt{\frac{\sqrt{q_{n+2}}}{\sinh(d_{n+2}\sqrt{q_{n+2}})} \cdot \frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})}} \right)$$

$$\times \left(\frac{\sinh(d_{n-1}\sqrt{q_{n-1}})}{\sqrt{q_{n-1}}} \cdot \sqrt{\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})} \cdot \frac{\sqrt{q_{n-2}}}{\sinh(d_{n-2}\sqrt{q_{n-2}})}} \right)$$
$$\leq C\sqrt{d_{n+2}}, \qquad n \in \mathbb{N}. \quad (6.21)$$

Therefore,

$$|c_{n+1}| \leq C\widetilde{c}r_{n+1}\sqrt{d_{n+2}} = \sqrt{2}C\widetilde{c}(d_{n+2} + \sqrt{d_{n+1}d_{n+2}})$$
$$\leq \sqrt{2}C\widetilde{c}\left(\frac{3}{2}d_{n+2} + \frac{1}{2}d_{n+1}\right),$$

and hence $\{c_n\}_{n=1}^{\infty} \in l^2$. On the other hand, it follows from (6.20) and (6.21) that $\sum_{n=1}^{\infty} |a_n| c_n^2 < \infty$, i.e.,

$$\sum_{n=1}^{\infty} \frac{\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_{n+1}}}$$
$$\times |\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}| < \infty.$$

Since $\sinh(x) > x$, $x > 0$, we easily get conditions (6.20). By [25, Theorem 1], this inequality together with the inclusion $\{c_n\}_{n=1}^{\infty} \in l^2$ yields $\mathrm{n}_{\pm}(B_{X,\alpha,q}) = 1$. It remains to apply Theorem 6.4. □

Corollary 6.9. *Let the assumptions of Proposition 6.8 be satisfied. If*

$$d_n(q_n)^{\frac{3}{2}} \leq c, \qquad c > 0, \quad (6.22)$$

then condition (6.20) *is equivalent to*

$$\sum_{n=1}^{\infty} d_{n+1} \left| \alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}} + \frac{1}{3}\left(d_n\sqrt{q_n} + d_{n+1}\sqrt{q_{n+1}}\right) \right| < \infty. \quad (6.23)$$

Proof. Using

$$\coth(x) = \frac{1}{x} + \frac{x}{3} - O(x^3),$$

$\{d_n\}_{n=1}^{\infty} \in l^2$ and (6.22) we prove the claim. □

Remark 6.10. Let the assumptions of Proposition 6.8 be satisfied. Note that condition (6.20) is automatically satisfied whenever

$$\alpha_n = -(\sqrt{q_n} + \sqrt{q_{n+1}}).$$

Proposition 6.11. *Assume Hypothesis 1 , and assume that* (6.16) *does not hold. Let also* $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ *and* $X = \{x_n\}_{n=1}^{\infty}$ *satisfy one of the following conditions:*

(i)

$$c_1 \le d_n\sqrt{q_n} \le c_2, \quad c_1, c_2 > 0, \tag{6.24}$$

and also

$$\sum_{n=1}^{\infty} |\alpha_n| d_n d_{n+1} r_{n-1} r_{n+1} = \infty, \tag{6.25}$$

where $r_n = \sqrt{d_n + d_{n+1}}$

(ii) There exists a positive constant $C_1 > 0$ *such that*

$$\alpha_n + \sqrt{q_n}\left(M_1 + \frac{r_n}{Mr_{n-1}}\right) + \sqrt{q_{n+1}}\left(M_1 + \frac{r_n}{Mr_{n+1}}\right) \le C_1(d_n + d_{n+1}), \qquad n \in \mathbb{N}, \tag{6.26}$$

where

$$M = \liminf_{n\to\infty} \sinh(d_n\sqrt{q_n}), \qquad M_1 = \limsup_{n\to\infty} \coth(d_n\sqrt{q_n}). \tag{6.27}$$

(iii) There exists a positive constant $C_2 > 0$ *such that*

$$\alpha_n + \sqrt{q_n}\left(M_2 - \frac{r_n}{Mr_{n-1}}\right) + \sqrt{q_{n+1}}\left(M_2 - \frac{r_n}{Mr_{n+1}}\right) \ge -C_2(d_n + d_{n+1}), \qquad n \in \mathbb{N}, \tag{6.28}$$

where

$$M = \liminf_{n\to\infty} \sinh(d_n\sqrt{q_n}), \qquad M_2 = \liminf_{n\to\infty} \coth(d_n\sqrt{q_n}). \tag{6.29}$$

Then the operator $H_{X,\alpha,q}$ *is self-adjoint in* $L^2(\mathbb{R}_+)$.

Proof. *(i)* Applying the Dennis–Wall test ([1, p. 25, Problem 2]) to matrix (6.15), we obtain that the condition

$$\sum_{n=1}^{\infty} |\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}|$$

$$\times\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}r_{n-1}r_{n+1}=\infty \tag{6.30}$$

yields self-adjointness of the minimal operator $B_{X,\alpha,q}$ associated with (6.15).

Obviously,

$$\begin{aligned}&|\alpha_n+\sqrt{q_n}\coth(d_n\sqrt{q_n})+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})|\\&\qquad\times\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}\\&\geq(|\alpha_n|-|\sqrt{q_n}\coth(d_n\sqrt{q_n})+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})|)\\&\qquad\times\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}.\end{aligned} \tag{6.31}$$

Since $\sinh(x)>x,\ x>0$, we get

$$|\alpha_n|\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}\geq|\alpha_n|d_nd_{n+1}. \tag{6.32}$$

Condition (6.24) implies that

$$|\sqrt{q_n}\coth(d_n\sqrt{q_n})+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})|$$

$$\times\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}\leq\frac{\sinh(2c_2)}{2c_1}\left(d_n+d_{n+1}\right). \tag{6.33}$$

Since $\{d_n\}_{n=1}^\infty\in l^2$, from the latter we get

$$\sum_{n=1}^\infty|\sqrt{q_n}\coth(d_n\sqrt{q_n})+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})|$$

$$\times\frac{\sinh(d_n\sqrt{q_n})\sinh(d_{n+1}\sqrt{q_{n+1}})}{\sqrt{q_n}\sqrt{q_{n+1}}}<\infty. \tag{6.34}$$

Combining (6.31)–(6.32) with (6.34) we get, that relations (6.25) and (6.30) are equivalent. By Theorem 6.4, $H_{X,\alpha,q}=H^*_{X,\alpha,q}$.

(ii) – *(iii)* Applying [6, Theorem VII.1.4] (see also [1, Problem 3, p. 37]) to the Jacobi matrix (6.15), we obtain that conditions

$$\begin{aligned}&-\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n}+\frac{1}{r_n^2}\big(\alpha_n+\sqrt{q_n}\coth(d_n\sqrt{q_n})\\&+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\big)-\frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})r_nr_{n+1}}\leq C_1\end{aligned} \tag{6.35}$$

and

$$-\frac{\sqrt{q_n}}{\sinh(d_n\sqrt{q_n})r_{n-1}r_n} - \frac{1}{r_n^2}\big(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) \\ +\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\big) - \frac{\sqrt{q_{n+1}}}{\sinh(d_{n+1}\sqrt{q_{n+1}})r_n r_{n+1}} \leq C_2 \tag{6.36}$$

guarantee self-adjointness of $B_{X,\alpha,q}$. Since $d_n\sqrt{q_n}$ is bounded , then, using conditions (6.27) and (6.29), we easily get conditions (6.26) and (6.38). Theorem 6.4 completes the proof. □

Corollary 6.12. *Let the assumptions of Proposition 6.11 be satisfied. If, in addition,* $\lim_{n\to\infty} d_n\sqrt{q_n} = 0$, *then conditions* (6.26)–(6.38) *are equivalent to*

$$\alpha_n + \frac{1}{d_n}\left(1 + \frac{r_n}{r_{n-1}}\right) + \frac{1}{d_{n+1}}\left(1 + \frac{r_n}{r_{n+1}}\right) \leq C_1(d_n + d_{n+1}), \qquad n \in \mathbb{N} \tag{6.37}$$

and

$$\alpha_n + \frac{1}{d_n}\left(1 - \frac{r_n}{r_{n-1}}\right) + \frac{1}{d_{n+1}}\left(1 - \frac{r_n}{r_{n+1}}\right) \geq -C_2(d_n + d_{n+1}), \qquad n \in \mathbb{N}, \tag{6.38}$$

respectively.

Example 6.13. Let $d_n := \frac{1}{n}$, $n \in \mathbb{N}$. Consider the operator

$$H_A := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x) + \sum_{n=1}^{\infty} \alpha_n \delta(x - x_n). \tag{6.39}$$

Clearly, $\{d_n\}_{n=1}^{\infty} \in l^2$, i.e., condition (6.16) is violated. Applying Propositions 6.8 and 6.11, after direct calculations we obtain:

(i) If $\sum_{n=1}^{\infty} \frac{|\alpha_n|}{n^3} = \infty$, then the operator H_A is self-adjoint (cf. Proposition 6.11 *(i)*).

(ii) If $\alpha_n \leq -2(c_2M_1 + \frac{c_2}{M})n - (c_2M_1 + \frac{c_2}{M}) + O(n^{-1})$, then H_A is self-adjoint (cf. Proposition 6.11 *(ii)*).

(iii) If $\alpha_n \geq -\frac{K}{n}$, $n \in \mathbb{N}$, $K \equiv \mathrm{const} > 0$, then H_A is self-adjoint (cf. Proposition 6.11 *(iii)*).

(iv) If $\alpha_n = -(\sqrt{q_n} + \sqrt{q_{n+1}}) + O(n^{-\varepsilon})$, then $\mathrm{n}_\pm(H_A) = 1$ (cf. Proposition 6.8).

6.3. Operators with discrete spectrum

Theorem 6.14. *Assume Hypothesis 1. Let* $B_{X,\alpha,q}$ *be the minimal Jacobi operator defined either by* (6.3) *or* (6.15)*.*

(i) If $\mathrm{n}_\pm(B_{X,\alpha,q}) = 1$*, then any self-adjoint extension of* $H_{X,\alpha,q}$ *has discrete spectrum.*

(ii) If $B_{X,\alpha,q} = B^*_{X,\alpha,q}$*, then the Hamiltonian* $H_{X,\alpha,q} (= H^*_{X,\alpha,q})$ *has discrete spectrum if and only if*

- $\lim_{n\to\infty} d_n = 0$ *and*
- $B_{X,\alpha,q}$ *has discrete spectrum.*

Proof. 1) To be precise, let $B_{X,\alpha,q}$ be defined by (6.3). Since $\mathrm{n}_\pm(B_{X,\alpha,q}) = 1$, any self-adjoint extension of $B_{X,\alpha,q}$ has discrete spectrum (see [1, 6]). Moreover, by Corollary 6.7, $\lim_{n\to\infty} d_n = 0$. Hence the operator H_0 defined by (5.35) has discrete spectrum too. The Krein resolvent formula (2.9) implies that any self-adjoint extension of $H_{X,\alpha,q}$ is discrete.

2) It follows from Theorem 5.5 (*iv*) and Remark 5.6. □

Proposition 6.15. *Assume Hypothesis 1. Let the operator* $B_{X,\alpha,q}$ *defined by* (6.15) *be self-adjoint, and let* $\lim_{n\to\infty} d_n = 0$*. Assume also that* $\alpha_n < 0$ *and exist*

$$\liminf_{n\to\infty} \sinh(d_n\sqrt{q_n}) = C, \qquad \limsup_{n\to\infty} \coth(d_n\sqrt{q_n}) = C_2 > 0, \tag{6.40}$$

and also

$$\begin{gathered} \lim_{n\to\infty} \frac{\left|\alpha_n + C_2(\sqrt{q_n} + \sqrt{q_{n+1}})\right|}{\left(d_n + d_{n+1}\right)} = \infty, \\ \lim_{n\to\infty} q_{n+1} C^{-2} \left(\alpha_n + C_2(\sqrt{q_n} + c\sqrt{q_{n+1}})\right)^{-1} \\ \times \left(\alpha_{n+1} + C_2(\sqrt{q_{n+1}} + c\sqrt{q_{n+2}})\right)^{-1} < \frac{1}{4}. \end{gathered} \tag{6.41}$$

Then the operator $H_{X,\alpha,q}$ *has purely discrete spectrum.*

Proof. Applying [10, Theorem 8] to the Jacobi matrix $B_{X,\alpha,q}$ of the form (6.15) we get sufficient conditions for the discreteness of spectrum:

$$\lim_{n\to\infty} \frac{1}{r_n^2} \left(\alpha_n + \sqrt{q_n}\coth(d_n\sqrt{q_n}) + \sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}}\right) = \infty \tag{6.42}$$

and

$$\lim_{n\to\infty} q_{n+1}\sinh^{-2}(d_{n+1}\sqrt{q_{n+1}})$$
$$\times\left(\alpha_n+\sqrt{q_n}\coth(d_n\sqrt{q_n})+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})\right)^{-1}\times$$
$$\left(\alpha_{n+1}+\sqrt{q_{n+1}}\coth(d_{n+1}\sqrt{q_{n+1}})+\sqrt{q_{n+2}}\coth(d_{n+2}\sqrt{q_{n+2}})\right)^{-1}<\frac{1}{4}. \tag{6.43}$$

Since $\alpha_n < 0$, then, taking into account that $d_n\sqrt{q_n}$ is bounded, we get that conditions (6.42) and (6.43) are equivalent to (6.40) and (6.41), respectively. And since $\lim_{n\to\infty} d_n = 0$, by Theorem 6.14 so is $H_{X,\alpha,q}$. □

Remark 6.16. If $H_{X,\alpha,q}$ is semibounded operator, in particular, if $\alpha_n > 0$, then claim of the Proposition 6.15 follows immediately from analogous classical A. M. Molchanov discreteness criterion (see [4]).

Proposition 6.17. *Assume Hypothesis 1, and assume that* $\lim_{n\to\infty} d_n = 0$ *and* $d_n\sqrt{q_n}\to 0$ *as* $n\to\infty$. *Let also the operator* $B_{X,\alpha,q}$ *defined by* (6.3)–(6.4) *be self-adjoint. If the following conditions are satisfied:*

$$\lim_{n\to\infty}\left|\frac{\alpha_{n-1}}{d_n}+q_n\right|=\infty,\qquad \lim_{n\to\infty}\frac{1}{d_n(\alpha_n+q_{n+1}d_{n+1})}>-\frac{1}{4}$$

$$\text{and}\qquad \lim_{n\to\infty}\frac{1}{d_n\alpha_{n-1}}>-\frac{1}{4}, \tag{6.44}$$

then the operator $H_{X,\alpha,q}$ *has discrete spectrum.*

Proof. Apply [10, Theorem 8] to the operator $B'_{X,\alpha,q}$ of the form (6.6). We prove the statement in at least two steps.

At first, we consider the case $b_n = b_{2k-1}$ and $a_n = a_{2k-1}$. We obtain the following sufficient conditions for the discreteness of spectrum of $B'_{X,\alpha}$:

$$\lim_{k\to\infty}\left|\frac{\alpha_{k-1}}{d_k}+q_k\right|=\infty\qquad\text{and}\qquad\lim_{k\to\infty}\frac{1}{d_k\alpha_{k-1}}>-\frac{1}{4}. \tag{6.45}$$

Similarly, if $b_n = b_{2k}$ and $a_n = a_{2k}$, we obtain

$$\lim_{k\to\infty}\left|\frac{1}{d_k^2}\right|=\infty\qquad\text{and}\qquad\lim_{k\to\infty}\frac{1}{d_k(\alpha_k+q_{k+1}d_{k+1})}>-\frac{1}{4}. \tag{6.46}$$

Since $\lim_{n\to\infty} d_n = 0$ and q_n is unbounded, then the conditions (6.45)–(6.46) are equivalent to (6.44). Theorem 6.14 completes the proof. □

Remark 6.18. In the case of $q \in L^\infty(\mathbb{R}_+)$, Proposition 6.17 was obtained in [21].

Corollary 6.19. *Let the assumptions of Proposition 6.17 be satisfied. Assume that $\alpha_n + q_{n+1}d_{n+1} < 0$ and the following conditions are met:*

$$\lim_{n\to\infty} \frac{1}{d_n(\alpha_n + q_{n+1}d_{n+1})} > -\frac{1}{4} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{1}{d_n\alpha_{n-1}} > -\frac{1}{4}. \tag{6.47}$$

Then the operator $H_{X,\alpha,q}$ has discrete spectrum.

Proof. If $\alpha_n + q_{n+1}d_{n+1} < 0$, then the condition

$$\lim_{n\to\infty} \frac{1}{d_n(\alpha_n + q_{n+1}d_{n+1})} > -\frac{1}{4}$$

implies the relation

$$|\alpha_n + q_{n+1}d_{n+1}| > \frac{4}{d_n}.$$

Combining the latter with the condition $\lim_{n\to\infty} d_n = 0$, one gets

$$\lim_{n\to\infty} \left| \frac{\alpha_{n-1}}{d_n} + q_n \right| = \infty.$$

In this case conditions (6.44) are reduced to (6.47). □

Remark 6.20. Note that if $\alpha_n + q_{n+1}d_{n+1} > 0$, then the condition $\lim_{n\to\infty} \frac{1}{d_n(\alpha_n+q_{n+1}d_{n+1})} > -\frac{1}{4}$ in (6.44) is automatically satisfied and can be omitted.

6.4. Resolvent comparability

Proposition 6.21. *Assume Hypothesis 1. Suppose also that $H_{X,\alpha,q}$ and $H_{X,\widetilde{\alpha},q}$ are self-adjoint, and $B_{X,\alpha,q}$ and $B_{X,\widetilde{\alpha},q}$ are the corresponding (self-adjoint) Jacobi operators defined either by* (6.3) *or* (6.6). *Then for any $p \in (0, \infty]$ the inclusion*

$$(H_{X,\alpha,q} - z)^{-1} - (H_{X,\widetilde{\alpha},q} - z)^{-1} \in \mathfrak{S}_p \tag{6.48}$$

is equivalent to the inclusion

$$(B_{X,\alpha,q} - \mathrm{i})^{-1} - (B_{X,\widetilde{\alpha},q} - \mathrm{i})^{-1} \in \mathfrak{S}_p. \tag{6.49}$$

Proof. From Theorem 2.5 we get the result with $B_{X,\alpha,q}$ and defined by (6.6). The result with the matrices defined by (6.3) is implied by Proposition 6.1. □

Corollary 6.22. *Assume Hypothesis 1. If* $\left\{\frac{\alpha_n-\widetilde{\alpha_n}}{d_{n+1}}\right\}_{n=1}^{\infty} \in l^p,\ p\in(0,\infty)$ *(*$\in c_0,\ p=\infty$*), then inclusion* (6.48) *holds.*

Proof. Note that the condition $B_{X,\widetilde{\alpha},q} - B_{X,\alpha,q} \in \mathfrak{S}_p$ is sufficient for the inclusion (6.48) to hold. Clearly, $l_0^2 \subset \mathrm{dom}(B_{X,\alpha,q}) \cap \mathrm{dom}(B_{X,\widetilde{\alpha},q})$. On the other hand, for any $f \in l^{2,0}$, (6.5) yields the inclusion $(B_{X,\widetilde{\alpha},q} - B_{X,\alpha,q}) \in \mathfrak{S}_p$, i.e.,

$$B_{X,\widetilde{\alpha},q}f - B_{X,\alpha,q}f = R_X^{-1}\big(\widetilde{B}_{\widetilde{\alpha}} - \widetilde{B}_{\alpha}\big)R_X^{-1}f = \oplus_{n=1}^{\infty}\begin{pmatrix} \frac{\alpha_n-\widetilde{\alpha_n}}{d_{n+1}} & 0 \\ 0 & 0 \end{pmatrix} f$$

for all finite sequences $f \in l^2(\mathbb{N})$. Hence due to the assumption, Corollary 6.22, we get $\overline{B_{X,\widetilde{\alpha},q} - B_{X,\alpha,q}} \in \mathfrak{S}_p \subset [\mathcal{H}]$ and $\mathrm{dom}(B_{X,\alpha,q}) = \mathrm{dom}(B_{X,\widetilde{\alpha},q})$. It remains to apply Proposition 2.5. Finally, Proposition 6.21 completes the proof. □

Proposition 6.23. *Assume Hypothesis 1. Let also* $d^* < \infty$*. If*

$$\sum_{n=1}^{\infty} \frac{|\alpha_n|}{d_{n+1}} < \infty, \tag{6.50}$$

then

$$\sigma_{ac}(H_{X,\alpha,q}) = \sigma_{ac}(H_{X,0,q}). \tag{6.51}$$

If, in addition $q(\cdot) \in L^1(\mathbb{R}_+)$*, then* $\sigma_{ac}(H_{X,\alpha,q}) = \mathbb{R}_+$*.*

Proof. Applying Corollary 6.22 for $p = 1$ to the Hamiltonians $H_{X,\alpha,q}$ and $H_{X,0,q}$ and using (6.50), we get that inclusion (6.48) holds. Now the result is implied by the Kato-Rozenblum theorem (cf. [32, Theorem XI.9]) we prove the claim.

If $q(\cdot) \in L^1(\mathbb{R}_+)$, then $\sigma_{ac}(H_{X,0,q}) = \mathbb{R}_+$. Hence,

$$\sigma_{ac}(H_{X,\alpha,q}) = \sigma_{ac}(H_{X,0,q}) = \mathbb{R}_+.$$

□

Remark 6.24. In the case of $q \in L^\infty(\mathbb{R}_+)$, Proposition 6.23 was established in [4].

Example 6.25. Let $x_0 = 0$,

$$x_n := \begin{cases} k, & n = 2k-1, \\ k+\frac{1}{k^3}, & n = 2k, \end{cases} \qquad k\in\mathbb{N}, \tag{6.52}$$

and let

$$d_n := \begin{cases} 1 - \frac{1}{(k-1)^3}, & n = 2k-1, \\ \frac{1}{k^3}, & n = 2k, \end{cases} \quad k \in \mathbb{N}. \tag{6.53}$$

Set

$$q(x) := \begin{cases} k, & x \in [x_{2k-1}, x_{2k}], \\ 0, & \text{otherwise}, \end{cases} \quad k \in \mathbb{N}. \tag{6.54}$$

Consider the minimal symmetric operator $H_{X,\alpha,q}$ associated with (1.1) in $L^2(\mathbb{R}_+)$.

Define

$$q_n(x) := \begin{cases} k, & n = 2k, \\ 0, & n = 2k-1. \end{cases} \quad k \in \mathbb{N}. \tag{6.55}$$

In addition, suppose that

$$\sum_{k=1}^{\infty} \left(k^3 \alpha_{2k-1} + \alpha_{2k}\right) < \infty.$$

Since d_n and $q_n(\cdot)$ satisfy Hypothesis 1 and $q(\cdot) \in L^1(\mathbb{R}_+)$, Proposition 6.23 immediately yields

$$\sigma_{ac}(H_{X,\alpha,q}) = \mathbb{R}_+. \tag{6.56}$$

Acknowledgment. The author wishes to express gratitude to M. M. Malamud for setting the problem and help and to A. S. Kostenko for useful comments.

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Contact information

Aleksandra Yuriivna Ananieva

Institute of Applied Mathematics and Mechanics,
Donetsk, Ukraine
E-Mail: `ananeva89@gmail.com`