Approximate Solving of the Third Boundary Value Problems for Helmholtz Equations in the Plane with Parallel Cuts

V.D. Dushkin

National Academy of the National Guard of Ukraine
3 Zakhystnykiv Ukrainy Sq., Kharkiv, 61001, Ukraine

E-mail: dushkinvd@gmail.com

In the paper, the method of approximate solution of boundary integral equations of the original problem is proposed. The systems of boundary integral equations of the problem are obtained by the method of parametric representation of integral transforms. The convergence of approximate solutions to the exact solution of the original problem is guaranteed by the propositions proved in the paper. Also, the rate of convergence of approximate solutions to exact solutions is found.

Key words: approximate solution of boundary integral equations, singular integral equation, existence of approximate solution, rate of convergence of approximate solution.

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1. Introduction

The modeling of electromagnetic waves diffraction on non-perfectly conducting gratings leads to the consideration of external mixed boundary value problems for the Helmholtz equation [1].

Modern diffraction structures consist of large number of elements located in different planes. It leads to the consideration of boundary value problems in domains of complex shape. An effective way of solving these boundary value problems was proposed by Yu.V. Gandel. It consists of two basic steps. First, the initial boundary value problems are reduced to a system of boundary integral equations by the method of parametric representations of integral transforms [2–4]. Usually the systems of boundary integral equations consist of integral

equations of different types: the first and second kind, singular and hypersingular [5,6]. Second, finite-dimensional approximations of systems of boundary integral equations are constructed by using the method of discrete singularities (the method of discrete vortices [7]).

Methods of approximate solution of the first and second boundary value problems were well studied in [8–10]. Unfortunately, the theory of finding approximate solutions of boundary integral equation systems of the third boundary value problems has not been sufficiently studied yet because of the complexity of these systems. In particular, several types of equations depend on the same unknown functions. This makes it impossible to solve the equations independently from each other. Also, the presence of components with variable upper limit in the integral equations makes it difficult to apply the classical scheme of the method of discrete singularities.

In this article, a version of the approximate solution of boundary integral equations of the third boundary value problem, which was considered in [11], was proposed. The version is based on the ideas of [12,13], where the approximate solution of the diffraction problem of electromagnetic waves on a system of superconductive bands [14] was considered.

2. System of Boundary Integral Equations

The system of boundary integral equations of the initial problem consists of integral equations of two different types.

Equations of the first type are singular integral equations of the first kind:

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\tau - \xi} \frac{\nu_q(\tau)\, d\tau}{\sqrt{1 - \tau^2}} + \frac{1}{\pi} \sum_{p=1}^{2R} \int_{-1}^{1} Q_{q,p}(\xi, \tau) \frac{\nu_p(\tau)\, d\tau}{\sqrt{1 - \tau^2}}$$

$$- \frac{c_q}{\pi} \int_{-1}^{\xi} \frac{\nu_q(\tau)\, d\tau}{\sqrt{1 - \tau^2}} = f_q(\xi), \quad |\xi| < 1, \quad q = 1, \ldots, R; \quad (1)$$

with the additional conditions

$$\frac{1}{\pi} \int_{-1}^{1} \frac{v_q(\tau)\, d\tau}{\sqrt{1 - \tau^2}} = 0, \quad q = 1, \ldots, R. \tag{2}$$

Equations of the second type are Fredholm equations of the second kind:

$$\nu_q(\xi) + \frac{1}{\pi} \sum_{p=1}^{2R} \int_{-1}^{1} Q_{q,p}(\xi, \tau) \frac{\nu_p(\tau)\, d\tau}{\sqrt{1 - \tau^2}}$$

$$-\frac{c_q}{\pi}\sqrt{1-\xi^2}\cdot\int_{-1}^{\xi}\frac{\nu_{q-R}(\tau)\,d\tau}{\sqrt{1-\tau^2}}=f_q(\xi),\quad |\xi|<1,\ q=R+1,\ldots,2R. \quad (3)$$

In equations (1), (3) it is assumed that

$$f_q(\xi)\in C^{0,\frac{1}{2}}[-1,1],\quad Q_{q,p}(\xi,\tau)\in C^{0,\frac{1}{2}}([-1,1]\times[-1,1]),$$
$$p=1,\ldots,2R,\quad q=1,\ldots,2R.$$

In these equations R is the number of plane-parallel slits.

Let $L^2_{\rho,\alpha},\ (\alpha=\pm 1)$, be the Hilbert spaces of measurable functions with respect to the inner product

$$(u,\nu)_\alpha=\int_{-1}^{1}u(\tau)\,\bar{v}(\tau)\,\rho^\alpha(\tau)\,d\tau,\quad \rho(\tau)=\sqrt{1-\tau^2},$$

and the norm $\|\nu\|_\alpha=\sqrt{(v,\nu)_\alpha}$. Also we take under consideration the space

$$L^{2,0}_{\rho,-1}=\left\{u\in L^2_{\rho,-1}\mid (u,1)_{-1}=0\right\}.$$

Define the operators:

$$\Theta_{q,p}:L^2_{\rho,-1}\to L^2_{\rho,1},\quad (\Theta_{q,p}u)(\xi)=\frac{1}{\pi}\int_{-1}^{1}Q_{q,p}(\xi,\tau)\frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},$$
$$|\xi|<1;\quad q=1,\ldots,R;\qquad p=1,\ldots,2R; \quad (4)$$

$$\Theta_{q,p}:L^2_{\rho,-1}\to L^2_{\rho,-1},\quad (\Theta_{q,p}u)(\xi)=\frac{1}{\pi}\int_{-1}^{1}Q_{q,p}(\xi,\tau)\frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},$$
$$|\xi|<1;\quad q=R+1,\ldots,2R;\quad p=1,\ldots,2R; \quad (5)$$

$$\Phi:L^{2,0}_{\rho,-1}\to L^2_{\rho,1},\quad (\Phi u)(\xi)=\frac{1}{\pi}\int_{-1}^{\xi}\frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},\qquad |\xi|<1; \quad (6)$$

$$\Psi:L^{2,0}_{\rho,-1}\to L^2_{\rho,-1},\quad (\Psi u)(\xi)=\frac{\sqrt{1-\xi^2}}{\pi}\int_{-1}^{\xi}\frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},\qquad |\xi|<1; \quad (7)$$

$$\Gamma:L^{2,0}_{\rho,-1}\to L^2_{\rho,1},\quad (\Gamma u)(\xi)=\frac{1}{\pi}\int_{-1}^{-1}\frac{1}{\tau-\xi}\frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},\qquad |\xi|<1. \quad (8)$$

From [15], it is known that

$$\Gamma : T_0\left(\tau\right) \to 0, \qquad T_n\left(\tau\right) \to U_{n-1}\left(\xi\right), \qquad\qquad n \in N; \tag{9}$$

$$\Phi : T_n\left(\tau\right) \to -\frac{U_{n-1}\left(\xi\right)}{n}\sqrt{1-\xi^2}, \qquad\qquad n \in N, \tag{10}$$

where $T_n\left(\tau\right)$ are the Chebyshev polynomials of the first kind, and $U_n\left(\xi\right)$ are the Chebyshev polynomials of the second kind.

Combining (7) and (10), we get

$$\Psi : T_n\left(\tau\right) \to -\frac{U_{n-1}\left(\xi\right)}{n}\left(1-\xi^2\right), \quad n \in N \tag{11}$$

and

$$\|\Gamma\|_{L^{2,0}_{\rho,-1}\to L^2_{\rho,1}} = 1, \quad \|\Psi\|_{L^{2,0}_{\rho,-1}\to L^2_{\rho,-1}} \leq 1, \quad \|\Phi\|_{L^{2,0}_{\rho,-1}\to L^2_{\rho,1}} \leq 1. \tag{12}$$

We introduce the Hilbert spaces H^2_1 and H^2_2:

$$\vec{V} = \left(v_1, v_2, \ldots, v_{2R}\right) \in H^2_1$$
$$\Leftrightarrow \left(v_q \in L^{2,0}_{\rho,-1}, \quad q = 1, \ldots, R; \qquad v_q \in L^2_{\rho,-1}, \quad q = R+1, \ldots, 2R\right);$$
$$\vec{W} = \left(w_1, w_2, \ldots, w_{2R}\right) \in H^2_2$$
$$\Leftrightarrow \left(w_q \in L^2_{\rho,1}, \quad q = 1, \ldots, R; \qquad w_q \in L^2_{\rho,-1}, \quad q = R+1, \ldots, 2R\right),$$

with the inner products

$$\left(\vec{V}, \vec{U}\right)_{H^2_1} = \sum_{i=1}^{2R}\left(v_i, u_i\right)_{L^2_{\rho,-1}},$$

$$\left(\vec{W}, \vec{S}\right)_{H^2_2} = \sum_{i=1}^{R}\left(v_i, s_i\right)_{L^2_{\rho,1}} + \sum_{i=R+1}^{2R}\left(v_i, s_i\right)_{L^2_{\rho,-1}}$$

and the norms $\left\|\vec{V}\right\|_{H^2_1} = \sqrt{\left(\vec{V}, \vec{V}\right)_{H^2_1}}$, $\left\|\vec{W}\right\|_{H^2_2} = \sqrt{\left(\vec{W}, \vec{W}\right)_{H^2_2}}$.

Consider the operators:

$$\mathbf{G} : H^2_1 \to H^2_2, \qquad \left(\vec{W} = \mathbf{G}\vec{V}\right)$$
$$\Leftrightarrow \begin{cases} w_q = \Gamma v_q, & q = 1, \ldots, R \\ w_q = v_q, & q = R+1, \ldots, 2R \end{cases}; \tag{13}$$

$$\mathbf{Z} : H^2_1 \to H^2_2, \qquad \left(\vec{W} = \mathbf{Z}\vec{V}\right)$$
$$\Leftrightarrow \begin{cases} w_q = c_q\Phi v_q, & q = 1, \ldots, R \\ w_q = c_q\Psi v_{q-R}, & q = R+1, \ldots, 2R \end{cases}; \tag{14}$$

$$\mathbf{K} : H_1^2 \to H_2^2, \qquad \left(\vec{W} = \mathbf{K}\vec{V} \right)$$

$$\Leftrightarrow \left(w_q = \sum_{p=1}^{2R} \Theta_{q,p} v_p, \quad q = 1, \ldots, 2R \right); \tag{15}$$

$$\mathbf{A} : H_1^2 \to H_2^2, \qquad \mathbf{A} = \mathbf{G} - \mathbf{Z} + \mathbf{K}. \tag{16}$$

With the notation (13)–(16), the system of equations (1)–(3) can be written as

$$\mathbf{A}\vec{V} = \vec{F}, \tag{17}$$

where $\vec{F} = (f_1, f_2, \ldots, f_{2R})$.

Lemma 1. *The operator* $\mathbf{G} : H_1^2 \to H_2^2$ *is invertible and the operator* $\mathbf{G}^{-1} : H_2^2 \to H_1^2$ *is bounded.*

Proof. We introduce the vector-functions:

$$\vec{X}_{k,q}(\tau) = (x_{k,q,1}(\tau), x_{k,q,2}(\tau), \ldots, x_{k,q,2R}(\tau)),$$

$$x_{k,q,p} = \begin{cases} \delta_{q,p} T_{k+1}(\tau), & q = 1, \ldots, R \\ \delta_{q,p} T_k(\xi), & q = R+1, \ldots, 2R \end{cases}, \tag{18}$$

$$\vec{Y}_{k,q}(\xi) = (y_{k,q,1}(\xi), y_{k,q,2}(\xi), \ldots, y_{k,q,2R}(\xi)),$$

$$y_{k,q,p}(\xi) = \begin{cases} \delta_{q,p} U_k(\xi), & q = 1, \ldots, R \\ \delta_{q,p} T_k(\xi), & q = R+1, \ldots, 2R \end{cases}, \tag{19}$$

where $\delta_{q,p}$ is a Kronecker delta, and $k = 0, 1, 2, \ldots$.

The set of vector functions $\bigcup_{q=0}^{2R} \bigcup_{k=1}^{\infty} \vec{X}_{k,q}$ is an orthogonal basis in H_1^2 and the set of vector functions $\bigcup_{q=1}^{2R} \bigcup_{k=0}^{\infty} \vec{Y}_{k,q}$ is an orthogonal basis in H_2^2. Using (9), (13), (18), and (19), we get

$$\mathbf{G}\vec{X}_{k,q} = \vec{Y}_{k,q}, \quad k = 0, 1, 2, \ldots; \quad q = 1, \ldots, 2R. \tag{20}$$

Thus, the operator $\mathbf{G}$ maps the basis of the space H_1^2 onto the basis of the space H_2^2, and $\|\mathbf{G}\|_{H_1^2 \to H_2^2} = 1$. Therefore, the operator $\mathbf{G}$ is bijective and bounded. So, by the Banach Isomorphism Theorem [16, p. 113] the operator $\mathbf{G}$ has a bounded inverse. $\square$

Lemma 2. *The operator* $\mathbf{A} : H_{\rho,-1}^{2,0} \to H_{\rho,1}^2$ *is invertible and the operator* $\mathbf{A}^{-1}$ *is bounded.*

Proof. The operator $\mathbf{A}$ is the sum of the invertible operator $\mathbf{G} : H_1^2 \to H_2^2$ and the compact operator $-\mathbf{Z} + \mathbf{K} : H_1^2 \to H_2^2$. Hence, by virtue of Nikolsky criterion (see [16, p. 150]),

$$\operatorname{ind}\left(\mathbf{A}\big|_{H_1^2 \to H_2^2}\right) = 0.$$

From the uniqueness of the problem solutions (1)-(3), it follows that

$$\dim \ker\left(\mathbf{A}\big|_{H_1^2 \to H_2^2}\right) = 0.$$

Consequently,

$$\mathbf{A}\left(H_1^2\right) = H_2^2.$$

Finally, by the Banach Isomorphism Theorem (see [16, p. 113]), the operator $\mathbf{A}$ has a bounded inverse. $\qquad\square$

3. Approximate Systems of Integral Equations (1)–(3) and Their Properties

Put

$$t_i^n = \cos\left(\frac{2i-1}{2n}\pi\right), \qquad\qquad i = 1, \ldots, n;$$

$$t_{0,j}^n = \cos\left(\frac{j}{n}\pi\right), \qquad\qquad j = 1, \ldots, n-1.$$

The points t_k^n are the zeros of Chebychev polynomials of the first kind $\mathrm{T}_n(\tau)$ and $t_{0,j}^n$ denote the zeros of Chebychev polynomials of the second kind $U_{n-1}(\xi)$.

Define the basis polynomials:

$$l_{1,n-1,i}(\tau) = \frac{1}{n}\left[1 + 2\sum_{p=1}^{n-1}\mathrm{T}_p(\tau)\,\mathrm{T}_p(t_i^n)\right], \qquad i = 1, \ldots, n;$$

$$l_{2,n-2,j}(\xi) = \frac{U_{n-1}(\xi)}{U_{n-1}'\left(t_{0,j}^n\right)\left(\xi - t_{0,j}^n\right)}, \qquad j = 1, \ldots, n-1.$$

They have the properties:

$$l_{1,n-1,i}(t_m^n) = \delta_{i,m}, \qquad i = 1, \ldots, n; \qquad m = 1, \ldots, n;$$
$$l_{2,n-2,j}(t_{0,s}^n) = \delta_{j,s}, \qquad j = 1, \ldots, n-1; \qquad s = 1, \ldots, n-1.$$

We introduce the Lagrange interpolation polynomials:

$$Q_{q,p,n}(\xi, \tau) = \sum_{j=1}^{n-1}\sum_{i=1}^{n} Q_{q,p}\left(t_{0,j}^n, t_i^n\right) l_{2,n-2,j}(\xi)\, l_{1,n-1,i}(\tau),$$

$$q = 1, \ldots, R; \ p = 1, \ldots, 2R;$$

$$Q_{q,p,n}(\xi, \tau) = \sum_{j=1}^{n} \sum_{i=1}^{n} Q_{q,p}\left(t_j^n, t_i^n\right) l_{1,n-1,j}(\xi) l_{1,n-1,i}(\tau),$$

$$q = R + 1, \ldots, 2R; \ p = 1, \ldots, 2R;$$

$$f_{q,n}(\xi) = \sum_{j=1}^{n-1} f_q\left(t_{0,j}^n\right) l_{2,n-2,j}(\xi), \quad q = 1, \ldots, R;$$

$$f_{q,n}(\xi) = \sum_{i=1}^{n} f_q\left(t_i^n\right) l_{1,n-1,i}(\xi), \quad q = R + 1, \ldots, 2R.$$

In order to solve the problem (1)–(3), we consider the following approximate system of integral equations:

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{\tau - \xi} \frac{\nu_{q,n}(\tau) \, d\tau}{\sqrt{1 - \tau^2}} + \frac{1}{\pi} \sum_{p=1}^{2R} \int_{-1}^{1} Q_{q,p,n}(\xi, \tau) \frac{\nu_{p,n}(\tau) \, d\tau}{\sqrt{1 - \tau^2}}$$

$$- \frac{c_q}{\pi} \sum_{j=1}^{n-1} \frac{l_{2,n-2,j}(\xi)}{\pi} \int_{-1}^{t_{0,j}^n} \frac{\nu_{q,n}(\tau) \, d\tau}{\sqrt{1 - \tau^2}} = f_{q,n}(\xi),$$

$$|\xi| < 1, \quad q = 1, \ldots, R; \tag{21}$$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\nu_{q,n}(\tau) \, d\tau}{\sqrt{1 - \tau^2}} = 0, \qquad q = 1, \ldots, R; \tag{22}$$

$$\nu_{q,n}(\xi) - \frac{c_q}{\pi} \sum_{i=1}^{n} \sqrt{1 - (t_i^n)^2} l_{1,n-1,i}(\xi) \int_{-1}^{t_i^n} \frac{\nu_{q-R,n}(\tau) \, d\tau}{\sqrt{1 - \tau^2}}$$

$$+ \frac{1}{\pi} \sum_{p=1}^{2R} \int_{-1}^{1} Q_{q,p,n}(\xi, \tau) \frac{\nu_{p,n}(\tau) \, d\tau}{\sqrt{1 - \tau^2}} = f_{q,n}(\xi),$$

$$|\xi| < 1, \quad q = R + 1, \ldots, 2R. \tag{23}$$

The functions $\nu_{q,n}$ are sought in the class of polynomials of degree $n - 1$. Subsequently, the reasoning shows the existence of polynomial solutions.

We define the subspaces $L_{\rho,\alpha,n}^2$ of the spaces $L_{\rho,\alpha}^2$. The elements of these subspaces are polynomials of degree n. The subspaces

$$L_{\rho,\alpha,n}^{2,0} = \left\{ u \in L_{\rho,\alpha,n}^2 \mid (u, 1)_\alpha = 0 \right\}$$

of the spaces $L_{\rho,\alpha,n}^2$ are also taken under consideration.

Let us define the operators:

$$\Theta_{q,p,n} : L^2_{\rho,-1,n-1} \to L^2_{\rho,1,n-2}, \quad (\Theta_{q,p,n}u)(\xi) = \frac{1}{\pi} \int_{-1}^{1} Q_{q,p,n}(\xi,\tau) \frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},$$

$$|\xi| < 1; \quad i = 1, \ldots, R; \tag{24}$$

$$\Theta_{q,p,n} : L^2_{\rho,-1,n-1} \to L^2_{\rho,-1,n-1}, \quad (\Theta_{q,p,n}u)(\xi) = \frac{1}{\pi} \int_{-1}^{1} Q_{q,p,n}(\xi,\tau) \frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},$$

$$|\xi| < 1; \quad i = R+1, \ldots, 2R; \tag{25}$$

$$\Phi_n : L^{2,0}_{\rho,-1,n-1} \to L^2_{\rho,1,n-2}, \quad (\Phi_n u)(\xi) = \sum_{j=1}^{n-1} \frac{l_{2,n-2,j}(\xi)}{\pi} \int_{-1}^{t^n_{0,j}} \frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}},$$

$$|\xi| < 1; \tag{26}$$

$$\Psi_n : L^{2,0}_{\rho,-1,n-1} \to L^2_{\rho,-1,n-1}, \quad (\Psi_n u)(\xi) = \sum_{i=1}^{n} \frac{\sqrt{1-(t^n_i)^2}}{\pi} l_{1,n-1,j}(\xi)$$

$$\times \int_{-1}^{t^n_i} \frac{u(\tau)\,d\tau}{\sqrt{1-\tau^2}}, \quad |\xi| < 1. \tag{27}$$

We introduce the subspaces $H^2_{1,n}$ of H^2_1 and the subspaces $H^2_{2,n}$ of H^2_2:

$$\vec{V}_n \in H^2_{1,n} \Leftrightarrow \Big(v_{q,n} \in L^{2,0}_{\rho,-1,n-1}, \quad q = 1, \ldots, R;$$

$$v_{q,n} \in L^2_{\rho,-1,n-1}, \quad q = R+1, \ldots, 2R \Big);$$

$$\vec{W}_n \in H^2_{2,n} \Leftrightarrow \Big(w_{q,n} \in L^2_{\rho,1,n-2}, \quad q = 1, \ldots, R;$$

$$w_{q,n} \in L^2_{\rho,-1,n-1}, \quad q = R+1, \ldots, 2R \Big).$$

Consider the operators:

$$\mathbf{Z}_n : H^2_{1,n} \to H^2_{2,n}, \quad \left(\vec{W}_n = \mathbf{Z}_n \vec{V}_n \right)$$

$$\Leftrightarrow \begin{cases} w_{q,n} = c_q \Phi_n v_{q,n}, & q = 1, \ldots, R; \\ w_{q,n} = c_q \Psi_n v_{q-R,n}, & q = R+1, \ldots, 2R \end{cases} ; \tag{28}$$

$$\mathbf{K}_n : H^2_{1,n} \to H^2_{2,n}, \quad \left(\vec{W}_n = \mathbf{K}_n \vec{V}_n \right)$$

$$\Leftrightarrow \left(w_{q,n} = \sum_{p=1}^{2R} \Theta_{q,p,n} t v_{p,n}, \quad q = 1, \ldots, 2R \right); \tag{29}$$

$$\mathbf{A}_n : H_{1,n}^2 \rightarrow H_{2,n}^2, \quad \mathbf{A}_n = \mathbf{G} - \mathbf{Z}_n + \mathbf{K}_n. \tag{30}$$

With the notation (28)–(30), the system of equations (21)–(23) can be written as

$$\mathbf{A}_n \vec{V}_n = \vec{F}_n, \tag{31}$$

where $\vec{F}_n = (f_{1,n}, f_{2,n}, \ldots, f_{2R,n})$.

For all $n \geq 4$, the following estimates hold true [15]:

$$\|f_{q,n} - f_q\|_{L_{\rho,1}^2} \leq \frac{M_1}{\sqrt{n}}, \quad q = 1, \ldots, R; \tag{32}$$

$$\|f_{q,n} - f_q\|_{L_{\rho,-1}^2} \leq \frac{M_1}{\sqrt{n}}, \quad q = R+1, \ldots, 2R; \tag{33}$$

$$\|\Theta_{q,p,n} - \Theta_{q,p}\|_{L_{\rho,-1}^2 \rightarrow L_{\rho,1}^2} \leq \frac{M_2}{\sqrt{n}}, \quad q = 1, \ldots, R; \quad p = 1, \ldots, 2R; \tag{34}$$

$$\|\Theta_{q,p,n} - \Theta_{q,p}\|_{L_{\rho,-1}^2 \rightarrow L_{\rho,-1}^2} \leq \frac{M_2}{\sqrt{n}}, \quad q = R+1, \ldots, 2R; \quad p = 1, \ldots, 2R, \tag{35}$$

where

$$M_1 = 24\sqrt{2\pi} \max_{q=1,\ldots,2R} \|f_q\|_{C[-1,1]},$$

$$M_2 = 48\sqrt{2\pi} \max_{q=1,\ldots,2R} \|Q_{q,p}\|_{C([-1,1] \times C[-1,1])}.$$

These estimates are the consequences of Jackson's Theorems (see Corollary 1 of Theorem 2 in [17, p. 128]).

Lemma 3. *For all natural n, the inequality*

$$\|\mathbf{A} - \mathbf{A}_n\|_{H_{1,n}^2 \rightarrow H_2^2} \leq \frac{M^*}{\sqrt{n}} \tag{36}$$

holds true. Besides,

$$\|\mathbf{A} - \mathbf{A}_n\|_{H_{1,n}^2 \rightarrow H_2^2} \rightarrow 0, \quad n \rightarrow \infty. \tag{37}$$

Proof. The following inequality clearly holds:

$$\|\mathbf{A} - \mathbf{A}_n\|_{H_{1,n}^2 \rightarrow H_2^2} \leq \|\mathbf{Z}_n - \mathbf{Z}\|_{H_{1,n}^2 \rightarrow H_2^2} + \|\mathbf{K}_n - \mathbf{K}\|_{H_{1,n}^2 \rightarrow H_2^2}. \tag{38}$$

The set of vector functions

$$E_{1,n} = \left(\bigcup_{q=1}^{R} \bigcup_{k=0}^{n-2} \vec{X}_{k,q} \right) \cup \left(\bigcup_{q=R+1}^{2R} \bigcup_{k=0}^{n-1} \vec{X}_{k,q} \right)$$

is an orthogonal basis in $H_{1,n}^2$ and the set of vector functions

$$E_{2,n} = \left(\bigcup_{q=1}^{R} \bigcup_{k=0}^{n-2} \vec{Y}_{k,q} \right) \cup \left(\bigcup_{q=R+1}^{2R} \bigcup_{k=0}^{n-1} \vec{Y}_{k,q} \right)$$

is an orthogonal basis in $H_{2,n}^2$. Let us note that $\dim\left(H_{1,n}^2\right) = \dim\left(H_{2,n}^2\right) = R(2n-3)$.

The estimations

$$\frac{\left\| (\mathbf{K} - \mathbf{K}_n)\, \vec{X}_{k,q} \right\|_{H_2^2}}{\left\| \vec{X}_{k,q} \right\|_{H_1^2}} \leq \frac{2M_2 R}{\sqrt{n}} \tag{39}$$

hold true for all $\vec{X}_{k,q} \in E_{1,n}$. Inequality (39) is the consequence of estimates (34) and (35). Furthermore,

$$\| (\mathbf{K} - \mathbf{K}_n) \|_{H_{1,n}^2 \to H_2^2} \leq \frac{2M_2 R}{\sqrt{n}}. \tag{40}$$

Let $u_n(\tau)$ be a polynomial of degree $(n-1)$ with the property $(u_n, 1)_{-1} = 0$. Properties (10), (11) of the operators Φ and Ψ imply that the function $\Phi u_n \in C^{0,\frac{1}{2}}$ and $\Psi u_n \in L_{\rho,1,n}^2$.

Reasoning as in [15, p. 60], we get

$$\| \Phi - \Phi_n \|_{L_{\rho,-1,n-1}^2 \to L_{\rho,1}^2} \leq \frac{24}{\sqrt{n}},$$

$$\| \Psi - \Psi_n \|_{L_{\rho,-1,n-1}^2 \to L_{\rho,-1}^2} \leq \frac{24}{\sqrt{n}}, \qquad \forall u_n \in L_{\rho,-1,n-1}^{2,0}. \tag{41}$$

Hence, estimations (41) show that the operators

$$\Phi_n : L_{\rho,-1,n-1}^{2,0} \to L_{\rho,1,n-2}^2, \qquad \Psi_n : L_{\rho,-1,n-1}^{2,0} \to L_{\rho,-1,n-1}^2$$

are bounded.

Taking into account (14) and (28), we obtain

$$\left\| (\mathbf{Z} - \mathbf{Z}_n)\, \vec{X}_{k,q} \right\|_{H_1^2} = 0, \quad q = R+1, \dots, 2R,$$

and

$$\left\| (\mathbf{Z} - \mathbf{Z}_n)\, \vec{X}_{k,q} \right\|_{H_1^2} \leq |c_q|\, \| (\Phi - \Phi_n)\, T_k \|_{L_{\rho,-1}^2 \to L_{\rho,1}^2}$$

$$+ |c_{q+R}| \left\| (\Psi - \Psi_n) T_k \right\|_{L^2_{\rho,-1} \to L^2_{\rho,-1}}, \quad q = 1, \ldots, R. \quad (42)$$

Moreover,

$$\left\| \mathbf{Z} - \mathbf{Z}_n \right\|_{H^2_{1,n} \to H^2_2} = \max_{\vec{X}_{k,q} \in E_{1,n}} \frac{\left\| (\mathbf{Z} - \mathbf{Z}_n) \vec{X}_{k,q} \right\|_{H^2_2}}{\left\| \vec{X}_{k,q} \right\|_{H^2_1}} \leq \frac{48C}{\sqrt{n}}, \quad (43)$$

where $C = \max\limits_{q=1,\ldots,2R} |c_q|$. Finally, the validity of Lemma 3 follows from (38), (40), and (43), where $M^* = 2RM_2 + 48C$. $\qquad \square$

In [18], the following theorem can be found.

Theorem 1. *Let X and Y be normed linear spaces and let $\widetilde{X} \subset X$ and $\widetilde{Y} \subset Y$ be finite-dimensional subspaces of the same dimension. We consider two equations.*

The equation for the exact solution of the problem

$$Au = f, \quad u \in X, \quad f \in Y, \quad (44)$$

and the equation for the approximate solution of the problem

$$\widetilde{A}\widetilde{u} = \widetilde{f}, \quad \widetilde{u} \in \widetilde{X}, \quad \widetilde{f} \in \widetilde{Y}, \quad (45)$$

where A and $\widetilde{A}$ are the linear operators
Assume that

1) *the operator A is invertible and the operator $A^{-1} : Y \to X$ is bounded,*

2) *the inequality*

$$p = \left\| A^{-1} \right\|_{Y \to X} \left\| A - \widetilde{A} \right\|_{\widetilde{X} \to Y} < 1$$

holds.
Then

1) *for any function $\widetilde{f} \in \widetilde{Y}$, the equation (45) has the unique solution $\widetilde{u}^* \in \widetilde{X}$;*

2) *let $u^* \in X$ be the solution of equation (44) and let $\delta = \left\| f - \widetilde{f} \right\|_Y$, then*

$$\left\| u^* - \widetilde{u}^* \right\| \leq \left\| A^{-1} \right\|_{Y \to X} (1 - p)^{-1} \left(\delta + p \left\| f \right\|_Y \right).$$

Lemmas 1–3 and Theorem 1 lead us to the following result.

Theorem 2. *Let us denote*

$$M = \left(2M^* \left\| \mathbf{A}^{-1} \right\|_{H_2^2 \to H_1^2} \right)^2.$$ (46)

For all natural $n > M$, the following statements hold true.

1. *Problems (31) have unique solutions.*

2. *The vector-functions $\vec{V}_n \in H_{1,n}^2$.*

3. *The sequence $\left\{ \vec{V}_n \right\}_{n=[M]+1}^{\infty}$ of the approximate solutions of problems (31) converges to the exact solution of problem (17) in the norm of the space H_1^2. Moreover,*

$$\left\| \vec{V} - \vec{V}_n \right\|_{H_1^2} \leq \frac{M^{**}}{\sqrt{n}}.$$ (47)

Proof. Let us define the numbers

$$p_n = \left\| \mathbf{A} - \mathbf{A}_n \right\|_{H_{1,n}^2 \to H_2^2} \left\| \mathbf{A}^{-1} \right\|_{H_2^2 \to H_1^2}.$$ (48)

It follows from estimation (36) and (46) that $p_n \leq \frac{1}{2}$ for $n > M$.

Appealing to Theorem 1 and the previous estimation concludes the uniqueness and existence of solutions of problems (31), where $\vec{V}_n \in H_{1,n}^2$. Also, the estimations

$$\left\| \vec{V} - \vec{V}_n \right\|_{H_1^2} \leq \left\| \mathbf{A}^{-1} \right\|_{H_2^2 \to H_1^2} (1 - p_n)^{-1}$$

$$\times \left(\left\| \vec{F}_n - \vec{F} \right\|_{H_2^2} + p_n \left\| \vec{F} \right\|_{H_1^2} \right), \qquad n > M, \quad (49)$$

follow from the statements of Theorem 1. Thus, inequality (47) holds true for

$$M^{**} = 2 \left\| \mathbf{A}^{-1} \right\|_{H_2^2 \to H_1^2} \left(2RM_1 + M^* \left\| \mathbf{A}^{-1} \right\|_{H_2^2 \to H_1^2} \left\| \vec{F} \right\|_{H_1^2} \right).$$

We obtain the value of M^{**} as a direct consequence of estimations (32), (33), (36), and (49) . This completes the proof of Theorem 2. □

Corollary 1. *For all natural $n > M$, the following statements hold true.*

1. *The systems of integral equations (21)–(23) have a unique solution, where the functions $v_{q,n}$ are the polynomials of degree $(n-1)$.*

2. *For all $q = 1, \ldots, M$, the sequences $\{v_{q,n}\}_{n=[M]+1}^{\infty}$ converge to the functions v_q, which are the exact solutions of problem (1)–(3) in the norm of the space $L_{\rho,-1}^2$. Moreover,*

$$\left\| v_{q,n} - v_q \right\|_{L_{\rho,-1}^2} \leq \frac{M^{**}}{\sqrt{n}}, \qquad \forall n \in N, \quad n > M.$$

4. Conclusions

The justification of the method of approximate solution of the boundary integral equations considered in [19] was given. The results obtained in the paper can be used as basic solutions of other problems considered in [20–22].

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