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Spin dynamics of ^{3}He-B with dissipation for the general spin-orbital configurations

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The spatially homogeneous spin dynamics of the superfluid ^{3}He-B with dissipation is considered for the general spin-orbital configurations. It is demonstrated that the possibility of new coherent spin-precessing modes appears explicitly in the equations of motion describing the relaxation of the spin variables towards various attractors (resonance states) found previously as the stationary solutions and observed experimentally.

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1. The order parameter rigidity of the superfluid phases of liquid ^{3}He gives life to a number of the long-lived excitations at the background of the Cooper pair condensate. Among them a great importance, since the discovery of ultralow-temperature ordered states of ^{3}He, has been attributed to the investigation of the coherent spin dynamics. A crucial role here is played by a weak spin-orbital coupling stemming from the dipole-dipole interaction between nuclear magnetic moments of ^{3}He atoms. In the ordered (superfluid) states the dipole-dipole potential U_D lifts the spin-orbital degeneracy and stabilizes the appropriate equilibrium or dynamical spin-orbital configurations of ^{3}He-A and ^{3}He-B.

The spin dynamics of the superfluid phases of ^{3}He is a coupled motion of the nuclear magnetization $\mathbf{M} = g\mathbf{S}$ and the spin part of the order parameter. In the dissipationless approach a starting point is the Leggett Hamiltonian (in what follows we consider a spatially homogeneous spin dynamics)

$$\mathcal{H}_L = \frac{1}{2\chi}\mathbf{M}^2 - \mathbf{M}\mathbf{H}_0 + U_D = \frac{g^2}{2\chi}\mathbf{S}^2 - \omega_0 S_z + U_D, \tag{1}$$

where χ is the magnetic susceptibility, an external magnetic field $\mathbf{H}_0 = -H_0\mathbf{z}$ and the Larmore frequency $\omega_0 = gH_0$. The order parameter here enters through the dipole-dipole potential U_D and introduces the characteristic features of superfluid phases. Below we concentrate on the properties of the spin-precessing modes of ^{3}He-B. For the B phase

$$U_D = \frac{2}{15}\chi_B\left(\frac{\Omega_B}{g}\right)^2\left(\mathrm{Tr}\,\hat{R} - \frac{1}{2}\right)^2, \tag{2}$$

where Ω_B is the frequency of the longitudinal NMR and the orthogonal matrix $\hat{R}$ is the B-phase order parameter describing $3D$ relative rotations of the spin and orbital degrees of freedom. Introducing the triples of Euler angles $(\alpha_S, \beta_S, \gamma_S)$ and $(\alpha_L, \beta_L, \gamma_L)$ describing $3D$ rotations in the spin and orbital spaces, respectively, it can be shown that

$$\mathrm{Tr}\,\hat{R} = s_z l_z + \frac{1}{2}(1 + s_z)(1 + l_z)\cos(\alpha + \gamma) + \frac{1}{2}(1 - s_z)(1 - l_z)\cos(\alpha - \gamma) + \sqrt{(1 - s_z^2)(1 - l_z^2)}\,(\cos\alpha + \cos\gamma), \tag{3}$$

where $s_z = \cos\beta_S$, $l_z = \cos\beta_L$, $\alpha = \alpha_S - \alpha_L$ and $\gamma = \gamma_S - \gamma_L$.

In the strong magnetic field case ($\omega_0 >> \Omega_B$) the spin dynamics is governed by a set of the Hamilton equations for two pairs of the conjugate variables (S_z, α) and (S, γ) with S being the magnitude of $\mathbf{S}$. According to Eq. (1) this set of equations reads as

$$\dot{S}_z = -\frac{\partial U_D}{\partial \alpha},\ \dot{\alpha} = -\omega_0 + \frac{\partial U_D}{\partial S_z}; \qquad (4)$$

$$\dot{S} = -\frac{\partial U_D}{\partial \gamma},\ \dot{\gamma} = (S/S_0)\,\omega_0 + \frac{\partial U_D}{\partial S}; \qquad (5)$$

where $S_0 = \chi\omega_0/g^2$ (the magnitude of equilibrium magnetization $M_0 = gS_0$).

The angle α is a fast variable in the sence that $|\dot{\alpha}| >> \Omega_B$ and the same is true for γ (except the case with $S << S_0$ which we do not considered here). On the other hand, the combination $\Phi = \alpha + (S_0/S)\gamma$ is a slow variable. The significance of this resonance becomes clear when considering the structure of the dipole-dipole potential U_D . Inserting Eq. (3) into Eq. (2) we conclude that

$$U_D/S_0\omega_0 = \varepsilon f(s_z,\ l_z,\ \alpha,\ \gamma) = \varepsilon \sum_{kl} f_{kl}(s_z,\ l_z)\ e^{i(k\alpha + l\gamma)}, \qquad (6)$$

where $\varepsilon \propto (\Omega_B/\omega_0)^2$.

Assuming that $\varepsilon = \frac{1}{10}(\Omega_B/\omega_0)^2$, it follows from Eqs. (2) and (3) that $f_{kl} = f_{lk} = f_{-l-k}$ for the B phase and the non-zero coefficients are given as:

$$f_{00} = 1 + 2s_z^2 l_z^2 + (1 - s_z^2)(1 - l_z^2),$$

$$f_{10} = 2s_z l_z \sqrt{1 - s_z^2}\ \sqrt{1 - l_z^2}\ ,$$

$$f_{20} = \frac{1}{2}(1 - s_z^2)\ (1 - l_z^2)\ ,$$

$$f_{1\pm1} = \frac{1}{3}\ (1 \pm s_z)\ (1 \pm l_z)\ (1 \mp 2s_z)\ (1 \mp 2l_z)\ , \qquad (7)$$

$$f_{1\pm2} = \frac{1}{3}\ (1 \pm s_z)\ (1 \pm l_z)\ \sqrt{1 - s_z^2}\ \sqrt{1 - l_z^2}\ ,$$

$$f_{2\pm2} = \frac{1}{12}\ (1 \pm s_z)^2\ (1 \pm l_z)^2.$$

It is easily verified that at $l_z = 1$, which corresponds to an equilibrium orbital states of ^{3}He-B (the so called Leggett configuration), f_{kl} are nonzero only for $k = l = 0, \pm1, \pm2$. This means that for an orbital state with $l_z = 1$ the dipole-dipole potential depends only on the combination $\Phi = \alpha + \gamma$ and, as we have seen, it is a slow variable at $S \simeq S_0$. This well known resonance is operative even at $l_z \neq 1$ because all other linear combinations of α and γ are fast variables at $S = S_0$ for the strong-field case ($\varepsilon << 1$) and they disappear on the average. The conventional spin dynamics at $S \simeq S_0$ has been explored thoroughly in the past [1,2].

On the other hand, at $l_z \neq 1$ (non-Leggett orbital configuration) an unconventional spin dynamics is also possible since a new resonance regime can develop. Indeed, an inspection of the coefficients f_{kl} shows that a new combination $\Phi = \alpha + 2\gamma$ appears in the expression for U_D which turns out to be a slow variable at a special value of $S = S_0/2$ (another resonance at $S = 2S_0$ is also possible). This has been noticed in Ref. 3 (for more details see Ref. 4) and the corresponding experimental investigations where undertaken recently [5,6].

The stationary solutions for s_z , l_z and Φ corresponding to the particular coherent spin-precessing modes at the fixed resonance values of S are found by minimizing the time-averaged dipole-dipole potential $\bar{U}_D$ (the Van der Pol picture). On the other hand, in order to explore the time evolution of S starting from some initial value, and to find out the routes leading to the mentioned resonance regimes, a full description of the spin dynamics, including the dissipation effects, is necessary. In what follows a theoretical background for the analysis of the relaxation processes in the spin dynamics of ^{3}He-B will be presented. It is a direct generalization of the approach adopted in Ref. 7 and allows us to consider the case of the non-Leggett orbital configurations. It should be noted that using the computer simulation programs (like a package elaborated by A. A. Leman) the spin dynamics including the Leggett–Takagi dissipation mechanism can be explored quite effeciently. At the same time, an analytical approach has the merits of its own and gives, as we shall see, a transparent insight into the essence of the problem.

2. A standard procedure of incorporating the relaxation processes into the homogeneous spin dynamics is based on the introduction of a dissipative function

$$\mathcal{F}_d = \frac{1}{2}\ \kappa\ (\dot{\mathbf{S}} - g\mathbf{S} \times \mathbf{H}_0)^2 =$$

$$= \frac{1}{2}\ \kappa \left[\frac{S^2}{S^2 - S_z^2}\ (\dot{S}^2 + \dot{S}_z^2 - 2\frac{S_z}{S}\ \dot{S}\ \dot{S}_z) + \right.$$

$$\left. +\ (S^2 - S_z^2)\ (\dot{\alpha} + \omega_0)^2 \right], \qquad (8)$$

where κ will be considered as a phenomenological coefficient [7].

During the time interval δt the energy of a dissipative system changes by

$$\delta E = -2\mathcal{F}_d \delta t = \kappa\left[\frac{S^2}{S^2 - S_z^2}\left(\frac{\partial U_D}{\partial \gamma} - \frac{S_z}{S}\frac{\partial U_D}{\partial \alpha}\right)\delta S + \right.$$

$$+ \frac{S^2}{S^2 - S_z^2}\left(\frac{\partial U_D}{\partial \alpha} - \frac{S_z}{S}\frac{\partial U_D}{\partial \gamma}\right)\delta S_z +$$

$$\left. + (S^2 - S_z^2)\frac{1}{\omega_0}\left(\frac{\partial U_D}{\partial S_z}\right)^2 \delta\alpha\right]. \qquad (9)$$

This last relation allows to pass from the Hamiltonian Eqs. (4) and (5) to a set of equations for the spin dynamics with dissipation (from now on the time is measured in units of $1/\omega_0$ and (S_z, S) in units of S_0):

$$\dot{S}_z = \varepsilon X_z, \ \dot{\alpha} = -1 + \varepsilon Y_\alpha, \qquad (10)$$

$$\dot{S} = \varepsilon X_S, \ \dot{\gamma} = S + \varepsilon Y_\alpha, \qquad (11)$$

where

$$X_z(S_z, S, \alpha, \gamma \,|\, \varepsilon) = -\frac{\partial f}{\partial \alpha} + \varepsilon\kappa (S^2 - S_z^2)\left(\frac{\partial f}{\partial S_z}\right)^2, \qquad (12)$$

$$X_S(S_z, S, \alpha, \gamma) = -\frac{\partial f}{\partial \gamma}, \qquad (13)$$

$$Y_\alpha(S_z, S, \alpha, \gamma) = \frac{\partial f}{\partial S_z} - \frac{\kappa S^2}{S^2 - S_z^2}\left(\frac{\partial f}{\partial \alpha} - \frac{S_z}{S}\frac{\partial f}{\partial \gamma}\right), \qquad (14)$$

$$Y_\gamma(S_z, S, \alpha, \gamma) = \frac{\partial f}{\partial S} - \frac{\kappa S^2}{S^2 - S_z^2}\left(\frac{\partial f}{\partial \gamma} - \frac{S_z}{S}\frac{\partial f}{\partial \alpha}\right). \qquad (15)$$

Since $\varepsilon << 1$ a well-known procedure of separating of the slow (S_z, S) and the fast (α, γ) motions can be applied [8] to solve Eqs. (10) and (11). Although the main points are described in Ref. 7, here we show the principle steps for completeness.

Passing to the new variables $\bar{S}_z$, $\bar{S}$, $\bar{\alpha}$ and $\bar{\gamma}$ according to the prescription

$$\begin{aligned} S_z &= \bar{S}_z + \varepsilon u_z + \varepsilon^2 v_z + \dots, \\ S &= \bar{S} + \varepsilon u_S + \varepsilon^2 v_S + \dots, \\ \alpha &= \bar{\alpha} + \varepsilon u_\alpha + \varepsilon^2 v_\alpha + \dots, \\ \gamma &= \bar{\gamma} + \varepsilon u_\gamma + \varepsilon^2 v_\gamma + \dots, \end{aligned} \qquad (16)$$

where $u_i = u_i(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma})$ and $v_i = v_i(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma})$, and adopting that the new variables are subject to a set of equations

$$\begin{aligned} \dot{\bar{S}}_z &= \varepsilon A_z + \varepsilon^2 B_z + \dots, \\ \dot{\bar{S}} &= \varepsilon A_S + \varepsilon^2 B_S + \dots, \\ \dot{\bar{\alpha}} &= -1 + \varepsilon A_\alpha + \varepsilon^2 B_\alpha + \dots, \\ \dot{\bar{\gamma}} &= \bar{S} + \varepsilon A_\gamma + \varepsilon^2 B_\gamma + \dots, \end{aligned} \qquad (17)$$

with $A_i = A_i(\bar{S}_z, \bar{S})$ and $B_i = B_i(\bar{S}_z, \bar{S})$, we arrive at the equations for yet unknown functions u_i, and v_i:

$$-\frac{\partial u_i}{\partial \bar{\alpha}} + \bar{S}\frac{\partial u_i}{\partial \bar{\gamma}} = g_i - A_i, \qquad (18)$$

$$-\frac{\partial v_i}{\partial \bar{\alpha}} + \bar{S}\frac{\partial v_i}{\partial \bar{\gamma}} = h_i - B_i. \qquad (19)$$

In Eqs. (18), in describing the first order effects in ε, the functions g_i are given as follows:

$$\begin{aligned} g_z &= X_z(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma} \,|\, 0), \\ g_S &= X_S(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma}), \\ g_\alpha &= Y_\alpha(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma}), \\ g_\gamma &= X_\gamma(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma}) + u_S(\bar{S}_z, \bar{S}, \bar{\alpha}, \bar{\gamma}). \end{aligned} \qquad (20)$$

The second order effects in ε are governed by Eqs. (19) and the functions h_i contain derivatives of X_i and Y_iˆ, with respect to $\bar{S}_z$, $\bar{S}$, $\bar{\alpha}$, $\bar{\gamma}$ and ε (calculated at $\varepsilon = 0$). In particular

$$h_z = \frac{\partial X_z}{\partial \bar{S}_z}u_z + \frac{\partial X_z}{\partial \bar{S}}u_S + \frac{\partial X_z}{\partial \bar{\alpha}}u_\alpha + \frac{\partial X_z}{\partial \bar{\gamma}}u_\gamma + \frac{\partial X_z}{\partial \varepsilon} -$$

$$-\left(A_z\frac{\partial u_z}{\partial \bar{S}_z} + A_S\frac{\partial u_z}{\partial \bar{S}} + A_\alpha\frac{\partial u_z}{\partial \bar{\alpha}} + A_\gamma\frac{\partial u_z}{\partial \bar{\gamma}}\right). \qquad (21)$$

The other h_i have the similar structure. According to Eqs. (12)–(15) and (6) the functions g_i are periodic in α and γ:

$$g_i = \sum_{kl} g_{kl}^{(i)}(\bar{S}_z, \bar{S})\, e^{i(k\bar{\alpha} + l\bar{\gamma})}, \qquad (22)$$

and the bounded solutions of Eqs. (18) are given as

$$u_i = i \sum_{kl}{}' \frac{g_{kl}^{(i)}}{k - \bar{S}l} \, e^{i(k\bar{\alpha}+l\bar{\gamma})} ,$$
$$A_i = g_{00}^{(i)} (\bar{S}_z , \bar{S}), \tag{23}$$

where a prime in the summation over k and l excludes the contribution of $k = l = 0$. In a similar way can be found the solutions of Eqs. (19).

Performing the above-mentioned procedure it can be established that

$$A_z = A_S = 0, \qquad A_\alpha = \frac{\partial f_{00}}{\partial \bar{S}_z} , \qquad A_\gamma = \frac{\partial f_{00}}{\partial \bar{S}},$$
$$B_z = h_{00}^{(z)}, \qquad B_S = h_{00}^{(S)}. \tag{24}$$

After having calculated $h_{00}^{(S)}$ it can be shown that

$$\dot{\bar{S}} = \varepsilon^2 B_S = \frac{\varepsilon^2\kappa}{1 - s_z^2} \sum_{kl} \frac{l}{k - \bar{S}l} (k^2 + l^2 - 2s_z kl) f_{kl}^2 , \tag{25}$$

where only the dissipative contribution to B_S is retained. In a similar way it is concluded that

$$\dot{\bar{S}}_z = \varepsilon^2 B_z = \frac{\varepsilon^2\kappa}{1 - s_z^2} \sum_{kl} \frac{k}{k - \bar{S}l} (k^2 + l^2 - 2s_z kl) f_{kl}^2 +$$
$$+ \varepsilon^2\kappa (1 - s_z^2) \sum_{kl} \left(\frac{\partial f_{kl}}{\partial s_z}\right)^2 . \tag{26}$$

In Ref. 7 the set of Eqs. (25) and (26) has been used to explore the dissipative processes in the superfluid A and B phases for the special orbital states, the Leggett configurations. For ^{3}He-B, which we consider here, this corresponds to $l_z = 1$. At $l_z = 1$ only the components with $l = k = \pm 1, \pm 2$ contribute to the r.h.s. of Eqs. (25) and (26) and, as mentioned in Ref. 7, irrespective of the initial conditions, $\bar{S}$ is attracted to the resonance value $\bar{S} = 1$.

For a non-Leggett orbital configuration (with $l_z \neq 1$) the new possibilities appear. For the general spin-orbital configurations Eq. (25) can be put in the following form:

$$\dot{\bar{S}} = -\frac{2\varepsilon^2\kappa}{1 - s_z^2} \left[\frac{1}{\bar{S}} (f_{10}^2 + 4f_{20}^2) + 2\,\frac{1 - s_z}{\bar{S} - 1} (f_{11}^2 + 4f_{22}^2) + \left(\frac{5 - 4s_z}{\bar{S} - 1/2} + \frac{5 - 4s_z}{\bar{S} - 2}\right) f_{12}^2 + \right.$$
$$\left. + 2\,\frac{1 + s_z}{\bar{S} + 1} (f_{1-1}^2 + 4f_{2-2}^2) + \left(\frac{5 + 4s_z}{\bar{S} + 1/2} + \frac{5 + 4s_z}{\bar{S} + 2}\right) f_{1-2}^2 \right]. \tag{27}$$

Here (and below) $s_z = \bar{S}_z/\bar{S}$. By using Eq. (26) it can be shown that

$$\dot{\bar{S}}_z = \varepsilon^2\kappa \left\{ \frac{1}{1 - s_z^2} \left[2 (f_{10}^2 + 4f_{20}^2) - 4\,\frac{1 - s_z}{\bar{S} - 1} (f_{11}^2 + 4f_{22}^2) - \left(\frac{5 - 4s_z}{\bar{S} - 1/2} + 4\,\frac{5 - 4s_z}{\bar{S} - 2}\right) f_{12}^2 + \right.\right.$$
$$\left. + 4\,\frac{1 + s_z}{\bar{S} + 1} (f_{1-1}^2 + 4f_{2-2}^2) + \left(\frac{5 + 4s_z}{\bar{S} + 1/2} + 4\,\frac{5 + 4s_z}{\bar{S} + 2}\right) f_{1-2}^2 \right] +$$
$$\left. + (1 - s_z^2) \sum_{kl} \left(\frac{\partial f_{kl}}{\partial s_z}\right)^2 \right\}. \tag{28}$$

From the set of Eqs. (27) and (28) it is seen that, along with a conventional resonance at $\bar{S} = 1$, the new resonances at $\bar{S} = (1/2, 2)$ intervene for the case with $f_{12} \neq 0$. It should be kept in mind that, according to their derivation procedure, Eqs. (27) and (28) are applicable not too close to the mentioned resonance values of S, but the general tendencies of the various relaxation scenarios, leading to the attractors at $\bar{S} = (1, 1/2, 2)$, can still be established.

As an illustration of the content of Eq. (27) we shall consider a non-Leggett orbital state with $l_z = 0$. One can fix this orbital configuration by applying sufficiently strong superfluid counterflow in the transverse direction with respect to the magnetic field. Such a possibility is realized, in particular, in the rotating cryostat in the vortex-free region [9]. From Eq. (27) it is found that at $l_z = 0$ and $s_z \to 1$ $\bar{S}$ is evolving according to the equation

$$\dot{\bar{S}} = -\frac{32}{9}\, \varepsilon^2\kappa\, \frac{(\bar{S} - S_+)(\bar{S} - S_-)}{(\bar{S} - 1)(\bar{S} - 1/2)(\bar{S} - 2)}, \tag{29}$$

where $S_\pm = (19 \pm \sqrt{73})/16$. From Eq. (29) it is immediately concluded that $\bar{S}$ is tending to its resonance value $\bar{S} = 1$ if initially $\bar{S}$ is confined to an interval $S_- < \bar{S} < S_+$. On the other hand, $\bar{S}$ is attracted to $1/2$ if $\bar{S} < S_-$, and $\bar{S}$ approaches 2 for $\bar{S} > S_+$. These conclusions, although rather qualitative, contain interesting hints. More detailed analysis of the solutions of the set of Eqs. (27) and (28) will be given elsewhere.

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