First passage time problem in the material destruction theory. The poissonian process of energy absorption

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Local limit theorem concerns the probability distribution of the random passage time of the given level m by the sum of independent poissonian random values is proved. It is supposed that m is increased infinitely. The probability distribution with the asymptotic accuracy is represented in the Wald form that is obtained earlier in the case of random statistically independent {0,1}-sequences.

Получена локальная предельная теорема для вероятности момента достижения заданного уровня m суммой независимых пуассоновских случайных величин. Предполагается, что m возрастает неограниченно. Распределение вероятностей с асимптотической точностью представляется распределением Вальда, которое было известно в аналогичной задаче для статистически независимых случайных {0,1}-последовательностей.

1. Problem setting. The following problem arises in the statistical theory of material destruction by the penetrating electromagnetic radiation. It is required to find the probability distribution of the random destruction time τ of the material, i.e. the probability distribution of the destroying of the functional element executed on the basis of this material when the definite energy level defining its degradation is attained [1]. Earlier, it was proposed to solve this problem considering the destruction process as the formation of the defective cluster in the material piece (for example, this cluster consists of some cracks). In this approach, the above mentioned energy level is defined by the size of the defective cluster relative to the piece size when it may be considered as completely destroyed. The destruction scenario consisting of the defective cluster formation covering a macroscopic part of the sample was named the percolation one [1], [2]. When the percolation scenario realizing, it is possible to characterize the degradation level of the material (i.e. the cluster size) by the value of energy absorbed in the material and obtained from destroying disturbances. The value $J(t)$ of the energy is the random temporal function since rendered influences are random. For this reason, the time τ when this function attains the given level E representing the destruction criterion is the random variable and the problem consists of its probability distribution calculation.

The natural problem setting within frameworks of the above described approach which has been analyzed in papers [3],[4] assumes that the energy is pumped to the system with the average intensity $\varepsilon(t)$ being temporally constant. It means that

$$J(t) = \int_0^t \varepsilon(s)ds\,. \quad (1)$$

Here $\{\varepsilon(t); t \in \mathbb{R}_+ = [0,\infty)\}$ is a stationary ergodic random process having nonnegative realizations with the probability one. Physically, one may suppose that the process of random energy absorption may possess some temporal intervals such that all of them have random duration and there is not any external influence on the system. As a consequence, the material destruction is absent during these intervals. For this reason, after the attaining the given level E by the continuous function $J(t)$ at the moment τ, i.e. $J(\tau) = E$, it cannot change during any interval pointed out. Besides, it is necessary to assume that at some random time moments of τ_k, $k = 1, 2...$, the absorption of appreciable portions of energy ε_k, $k = 1, 2...$ can occur so quickly that it is reasonable, from the mathematical point of view, to neglect the duration of this process. Such kind of absorptions give the contribution in the form $\sum_k \varepsilon_k \delta(t - \tau_k)$ to the intensity. In turn, the appropriate contribution to the absorbed energy (1) will be represented by the discontinuous function. Then solution of the equation $J(\tau) = E$ can generally not exist. In connection with the specified reasons, it is more correctly to consider that the time τ is defined by the formula

$$\tau = \min\{t : J(t) \geq E\}\,. \tag{2}$$

This variable is correctly determined since realizations of random process $\{\varepsilon(t)\}$ are nonnegative and, in view of its stationarity, the value of the integral $J(t)$ monotonically increases. This increasing is carried out such that the time average has the linear character. Therefore, with the probability one the integral (1) will be really exceeded any level E at sufficiently great values t.

Exact calculation of the probability distribution of the random variable τ represents itself a sufficiently complicated problem. However, there is not the necessity of the reception of such exact solution. If the random process $\{\varepsilon(t)\}$ has sufficiently fast correlation separation (really, the exponentially fast one), we may present the variable $J(t)$ as the sum of weakly connected random variables, i.e. as the sum of integrals over time intervals with the duration being much more than the correlation time. If the value of the level E is very large in comparison with the correlation time, then the number of summands in this sum are very large too. In this case, the centered and appropriately normed value of this sum will be submitted to the Gaussian distribution. For this reason, it is possible to expect that probability distribution of the random variable τ will have the analogous universality at the limit $E \to \infty$. Certainly, this statement requires the rigorous justification. In the present work we analyze the special problem. Its solution confirms the above stated hypothesis. Namely, we shall solve the model problem by means of the simple analysis. The setting of this problem is closely connected with the representation of $J(t)$ as the sum of equally distributed variables. In the proposed model, we consider the time as the discretely changing one, i.e. it has values multiplied to a time constant t_0, $t = nt_0$, $n = 1, 2, \ldots$. In this case, the random process $\varepsilon(t) \equiv \varepsilon_n$ is represented by the sequence of independent and equally distributed random variables ε_n multiplied to a constant E_0 having the energy dimensionality, $\varepsilon_n = \xi E_0$, $\xi = 0, 1, 2\ldots$. We suppose that the common probability distribution of all variables ε_n, i.e. of the integer variable ξ is the poissonian one since it is naturally to consider the random influence action on the system as a rather rare random event. It is well-known that the Poisson distribution is used in the statistical physics for the registration description of very rare identical random events. Thus, we suppose that

$$\Pr\{\xi = n\} = \frac{\lambda^n}{n!} e^{-\lambda}, \quad \lambda > 0\,. \tag{3}$$

The parameter λ in the Poisson distribution in our model is the inverse value of the average energy injection to the system during one disturbance.

In the above described problem setting, the energy absorbed in the system is modelled by the random sequence $\{J_n[\varepsilon_k]; n \in \mathbb{N}\}$ with realizations

$$J_n[\varepsilon_k] \;=\; \sum_{k=1}^{n} \varepsilon_k\,. \tag{4}$$

The time τ of the attaining of the given level E is determined on the basis of Eq.(2) by the integer random variable

$$\nu_m \;=\; \min\{n\,;\, J_n[\varepsilon_k] \geq E \equiv mE_0\}\,, \tag{5}$$

$\tau = t_0 \nu$. The problem consists of the calculation with the asymptotical accuracy of probability distribution of the random variable ν at the limit $m \to \infty$.

2. The limit probability distribution. We shall obtain the general formula for solution of the formulated problem when the common random variable ξ is lattice, i.e. we suppose that $\xi \in \mathbb{N}_+$.

Let us introduced into consideration the sums η_n, $n = 1, 2, ...$ of independent random variables $\xi_1, \xi_2, ...$, i.e.

$$\eta_n = \sum_{k=1}^{n} \xi_k .$$

Then, in terms of these sums, using Eq.(4) and Eq.(5), we obtain

$$\nu_m = \min\{n; \eta_n \geq m\} .$$

We notice that, using the condition of mutual statistical independence of random variables $\xi_1, \xi_2 ...$ and on the basis of the total probability formula, one can find

$$P_l(n) \equiv \Pr\{\nu_m = l\} = \sum_{k=0}^{m-1} \Pr\{\eta_{l-1} = k\} \Pr\{\xi_l \geq m-k\} . \tag{6}$$

We obtain now the representation of probability distribution $P_m(n)$ being suitable for the further analysis using Eq.(6). In view of the infinite divisibility of the Poisson distribution, we have

$$\Pr\{\xi_1 + ... + \xi_n = k\} = \frac{(n\lambda)^k}{k!} \exp(-\lambda n) . \tag{7}$$

Since following equalities are taken place

$$\sum_{r=m}^{\infty} \frac{\lambda^r}{r!} = \sum_{r=0}^{\infty} \frac{\lambda^{r+m}}{(r+m)!} = \sum_{r=0}^{\infty} \int_0^{\lambda} dx_1 ... \int_0^{x_{m-1}} \frac{x_m^r}{r!} \, dx_m =$$

$$= \int_0^{\lambda} dx_1 ... \int_0^{x_{m-1}} \left(\sum_{r=0}^{\infty} \frac{x_m^r}{r!} \right) dx_m = \int_0^{\lambda} dx_1 ... \int_0^{x_{m-1}} \exp(x_m) \, dx_m =$$

$$= \frac{\lambda^m}{(m-1)!} \int_0^1 (1-y)^{m-1} e^{\lambda y} dy ,$$

it is valid

$$\Pr\{\xi \geq m-k\} = \sum_{r=m-k}^{\infty} \frac{\lambda^r}{r!} e^{-\lambda} = e^{-\lambda} \frac{\lambda^{m-k}}{(m-k-1)!} \int_0^1 (1-y)^{m-k-1} e^{\lambda y} dy .$$

Using the obtained formula and Eq.(7), we transform the right hand side of the expression (6),

$$P_m(n) = \lambda \exp(-\lambda n) \sum_{k=0}^{m-1} \frac{[\lambda(n-1)]^k}{k!} \cdot \frac{\lambda^{m-k-1}}{(m-k-1)!} \int_0^1 (1-y)^{m-k-1} e^{\lambda y} dy =$$

$$= \lambda \exp(-\lambda n) \int_0^1 \left(\sum_{k=0}^{m-1} \frac{[\lambda(n-1)]^k}{k!} \cdot \frac{[\lambda(1-y)]^{m-k-1}}{(m-k-1)!} \right) \exp(\lambda y) \, dy =$$

$$= e^{-\lambda n} \frac{\lambda^m}{(m-1)!} \int_0^1 (n-y)^{m-1} e^{\lambda y} dy . \tag{8}$$

Let us prove the following statement.

If $n, m \to \infty$ like that the variable $(m - \lambda n)n^{-1/2}$ remains bounded, the asymptotically exact formula

$$P_m(n) \sim \lambda \frac{(\lambda n)^{m-1}}{(m-1)!} \exp(-\lambda n) \qquad (9)$$

takes place.

We introduce the variable z, $m - 1 = z(\lambda n)^{1/2} + \lambda n$ which remains bounded at $n \to \infty\,, m \to \infty$ due to the condition of our statement. Then

$$\left(1 - \frac{y}{n}\right)^{m-1} = \left(1 - \frac{y}{n}\right)^{z(\lambda n)^{1/2}} \left(1 - \frac{y}{n}\right)^{\lambda n}$$

and, at the passing to the limit, we have

$$\lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^{m-1} = \lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^{\lambda n} = \exp(-\lambda y)\,,$$

since

$$\lim_{n\to\infty} \left(1 - \frac{y}{n}\right)^{n^{1/2}} = 1\,.$$

Eq.(8) may be written in the form

$$P_m(n) = \lambda e^{-\lambda n} \frac{(\lambda n)^{m-1}}{(m-1)!} \int_0^1 \left(1 - \frac{y}{n}\right)^{m-1} e^{\lambda y} dy\,.$$

For the obtaining of the asymptotically exact formula, we go to the limit $m\,, n \to \infty$ in the factor represented by the integral. Passing to the limit in subintegral expression that is possible in view of the compactness of integration domain, we shall obtain

$$\lim_{m\,,n\to\infty} \int_0^1 \left(1 - \frac{y}{n}\right)^{m-1} e^{\lambda y} dy == \int_0^1 e^{-\lambda y} e^{\lambda y} dy = 1\,.$$

We notice now that, for the Poisson distribution, the local limit theorem like the Moivre-Laplace one is valid [5],

$$\frac{\lambda^l}{l!} e^{-l} \sim \frac{1}{\sqrt{2\pi\lambda}} \exp\left(-\frac{1}{2}x^2\right)\,.$$

Here $l \to \infty$, $\lambda \to \infty$ and $(l - \lambda)\lambda^{-1/2} = x$ is bounded variable (the proof is realized by means of the Stirling formula). If we put $l = m - 1$, $\lambda \to \lambda n$ then application of this asymptotic formula to Eq.(9) gives us the following formula

$$P_m(n) \sim \frac{\lambda}{\sqrt{2\pi\lambda n}} \exp\left(-\frac{x^2}{2}\right) = \sqrt{\frac{\lambda}{2\pi n}} \exp\left(-\frac{m - 1 - \lambda n^2}{2\lambda n}\right)\,.$$

After the assigning $x = (\lambda n)/(m-1)$, $dx = \lambda/(m-1)$, the obtained local limit theorem is represented in the Wald form [6], [7]

$$P_m(n) \sim \sqrt{\frac{m-1}{2\pi x}} \exp\left[-\frac{m-1}{2}\left(x^{1/2} - x^{-1/2}\right)^2\right] dx\,.$$

However, there is the difference from the Wald distribution in this distribution. The degree of the variable x in a pre-exponential factor is equal $(-1/2)$ instead of $(-3/2)$.

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Задача досягнення заданого рівня у теорії перколяційного руйнування матеріалу. Пуассонівський процес поглинання енергії

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Одержано локальну граничну теорему для розподілу імовірностей часу досягнення заданого рівня m сумою незалежних пуассонівських випадкових величин. Припускається, що m зростає необмежено. Розподіл імовірностей з асимптотичною точністю уявляє собою розподіл Вальда, який був одержаний у аналогічній проблемі для статистично незалежних випадкових $\{0,1\}$-послідовностей.