

Nonlinear spin waves in antiferromagnets with uniaxial magnetic anysotropy in a magnetic field

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It was shown in this paper that nonlinear spin waves can propagate in antiferromagnets with uniaxial magnetic anisotropy in an external magnetic field. The nonlinear waves are characterized by simultaneous oscillations of great amplitude of polar and azimuth angles of antiferromagnetism vector. It means that nonlinear wave can propagate being the exact dynamic 3D solution of Landau-Lifshitz equations in antiferromagnet. The dispersion of polar and azimuth angle oscillations are found for antiferromagnetism vector. In particular, the dispersion for oscillation of azimuth angle in a long wavelength approximation coinsides with known spin wave dispersion in antiferromagnet when the polar angle is constant. Besides, the new type of spin waves obtained in this work is characterized by possibility to form front structure at the wide class of 2D harmonic functions, possibility of two-frequency and amplitude modulation.

Показано, что в антиферромагнетиках с одноосной магнитной анизотропией во внешнем магнитном поле могут распространяться нелинейные спиновые волны, такие, что одновременно происходят колебания с большой амплитудой полярного и азимутального углов вектора антиферромагнетизма, т.е. может распространяться нелинейная волна, которая является точным динамическим трехмерным решением уравнений Ландау-Лифшица для антиферромагнетика. Найдены законы дисперсии колебаний полярного и азимутального углов вектора антиферромагнетизма. В частности, в длинноволновом приближении для колебаний полярного угла закон дисперсии колебаний азимутального угла совпадает с известным законом дисперсии спиновых волн в антиферромагнетиках в приближении, в котором полярный угол постоянен. Кроме того, новый тип спиновых волн, полученный в данной работе, характеризуется возможностью формирования структуры фронта на широком классе гармонических двухмерных функций, формы колебания, а также возможностью двухчастотной и амплитудной модуляции.

The Landau-Lifshitz equation which was proposed by Landau for description of magnetization dynamics in [1] is the basic one in scientific research of temporary evolution of magnetization [2], propagation of spin waves including magnetostatic ones [3, 4] that is studying intensely in connection with new applications of magnetic materials for magnetic memory nanoelements and sensors for fabrication of which it is important to understand deeply remagnetization processes [2]. Progress in this field occurs simultaneously in two directions: numerical micromagnetism and a search for exact dynamic solutions of Landau-Lifshitz equation. For example, in works [5 – 9], excitation of magnetostatic spin waves was investigated numerically, in work [10], analysis of magnetostatic spin wave excitation after switching on a rotating magnetic field was conducted in the framework of numerical micromagnetism. Achievements in ω_H finding exact solutions of Landau-Lifshitz equations, in particular, are mentioned in the work [11]. Every approach that's mean both numerical modeling and exact solving of Landau-Lifshitz equation have its own advantages and restrictions. Thus, methods of numerical modeling of systems having realistic sizes leads to the system of ordinary differential equations that have about one million variables [2, 9]. Besides, Landau-Lifshitz equation can not be solved for certain problems by means of numerical methods because of existence of an infinite number of exact solutions under the same boundary conditions [12]. That is why such solutions are important for an serve as the reference solutions testing numerical algorithms. However, multidimensional exact dynamic solutions of Landau-Lifshitz equation as a rule can be found only under strict restrictions for their form. In this connection direction of search for quasi-exact solutions is also developing. Such solutions are exact enough for great intervals of time while numerical method are especially adapted for visualization of solutions of Landau-Lifshitz equation when time approaches to zero [13].

New exact dynamic solutions of Landau-Lifshitz equations are foud and analyzed in this work for an antiferromagnet with uniaxial magnetic anisotropy in applied external magnetic field. The model without damping is usually used for magnetic materials exposed to influence of short pulses of magnetic field so that order parameter dynamics occurs so rapidly that it is possible to neglect dissipative effects during the pulse action [4]. Nonlinear oscillations of antiferromagnetism vector calculated in this work as an exact solution of Landau-Lifshitz equations provide possibility to describe propagation of oscillation with nonuniform vector antiferromagnetism distribution at its front. And the spectrum of nonlinear waves in long wave length approximation obtained in this work coincides with known spectrum of spin waves in an antiferromagnet. As it is well known that uniform movement of magnetization represents traditional interest in the connection with ferromagnetic resonance and that as a rule homogeneous magnetic field induces magnetization movement. However, it have been revealed in the resent works that taking on account significantly nonlinear nature of Landau-Lifshitz equations spatially homogeneous distributions of order parameter can be accompanied by spin-wave modes under high enough power of magnetic field pulses. In this case complicated magnetization distributions arise. Their investigation are important for modern applications in the field of nanomagnetism. In this connection the results of this work are also of interest.

Let us consider antiferromagnet with uniaxial magnetic anisotropy. Landau-Lifshitz equations for antiferromagnetism vector without damping have the following form in a spherical coordinate system [14, 15]:

$$\begin{cases} \dfrac{\partial}{\partial t}\left[\sin^2\theta(\dot{\varphi}-\omega_H)\right]-c^2 div((\nabla\varphi)\sin^2\theta)=0 \\ \ddot{\theta}-c^2\nabla^2\theta+[\omega_0^2+c^2(\nabla\varphi)^2-(\dot{\varphi}-\omega_H)^2]\sin\theta\cos\theta=0 \end{cases}, \tag{1}$$

where θ – is the polar angle of antiferromagnetism vector $\vec{L}$, φ is the azimuth angle of antiferromagnetism vector, $\omega_H=gH_0$, $g=\frac{2\mu_0}{\hbar}$ (μ_0 is Bohr magneton, $\hbar$ is Plank constant), $\omega_0=\frac{4\mu_0 M_0}{\hbar}\sqrt{A\beta_1}$, $c=\frac{4\mu_0 M_0}{\hbar}\sqrt{A\alpha_1}$ (M_0 is magnetization of antiferromagnet sublattice), A is energy constant of uniform exchange, α_1, α_2 are constants of non-uniform exchange, β_1, β_2 are constants of uniaxial magnetic anisotropy which are included in the expression of antiferromagnet energy [14–17]:

$$W=\frac{1}{2}A\vec{M}^2+\frac{1}{2}\alpha_1\left(\frac{\partial\vec{L}}{\partial x_i}\right)^2+\frac{1}{2}\alpha_2\left(\frac{\partial\vec{M}}{\partial x_i}\right)^2-\frac{1}{2}\beta_1 L_z^2-\frac{1}{2}\beta_2 M_z^2-\vec{M}\vec{H}_0, \tag{2}$$

where M is vector of magnetization of an antiferromagnet.

Similarly to the method explained in [12], we'll find the unknown functions θ, φ in the form:

$$\begin{cases} \theta = 2arctg[H(P(x,y,z-\gamma t))] \\ \varphi = Q(x,y,z) + \tilde{\varphi}(t) \end{cases}, \tag{3}$$

where $\tilde{\varphi}(t) = \int \omega_H(t)dt + \omega t$, ω – is an arbitrary constant. In the case when $\omega_H(t) = \omega_H = const$ $\tilde{\varphi}(t) = \omega' \cdot t$ where ω' – is a constant.

It is not difficult to become convinced that connection between the trigonometric functions that are included in Landau-Lifshitz equation (1) and function $H(P)$ can be expressed by formulae:

$$\begin{cases} \cos\theta = \dfrac{1-H^2}{1+H^2} \\ \sin\theta = \pm\dfrac{2H}{1+H^2} \end{cases}, \tag{4}$$

Further the sign "+" is defined in the right part of the expressions (4) considering the function θ to be positive because the polar angle θ is changing in the range $[0,\pi]$.

The partial derivative is equal to

$$\frac{\partial\varphi}{\partial t} = \omega'. \tag{5}$$

where the definition $\omega = \omega' - \omega_H$ is introduced.

The solution of Landau-Lifshitz equations will be found according to the parametrization (3) describing uniform motion of nonlinear static solution of Landau-Lifshitz equation with a velocity γ along the coordinate axis OZ.

It is not difficult to verify that

$$\frac{\partial\theta}{\partial t} = \frac{2H'}{1+H^2}(-\gamma)\frac{\partial P}{\partial z}, \tag{6}$$

where $H' = \dfrac{dH}{dP}$ and also

$$div[(\nabla\varphi)\sin\theta] = \frac{4H^2}{\left(1+H^2\right)^2}\sum_i \frac{\partial^2 Q}{\partial x_i^2} + \frac{8H\left(1-H^2\right)H'}{\left(1+H^2\right)^3}\sum_i \frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial x_i}, \tag{7}$$

$$\frac{\partial}{\partial t}\left(\sin^2\theta\right) = \frac{8H\left(1-H^2\right)H'}{\left(1+H^2\right)^3}(-\gamma)\cdot\frac{\partial P}{\partial z}, \tag{9}$$

where x_i – are Cartesian coordinates of the radius-vector, i.e. $x_1 = x$, $x_2 = y$, $x_3 = z$.

Substituting the expressions (3) – (9) into the first equation of the system (1), it is possible to obtain:

$$c^2\left\{\frac{4H^2}{\left(1+H^2\right)^2}\sum_i \frac{\partial^2 Q}{\partial x_i^2} + \frac{8H\left(1-H^2\right)H'}{\left(1+H^2\right)^3}\sum_i \frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial x_i}\right\} + \frac{8H\left(1-H^2\right)H'}{\left(1+H^2\right)^3}\omega\gamma\frac{\partial P}{\partial z} = 0 \tag{10}$$

The derivatives and functions are calculated for substitution into the second equation of the system (1):

$$\frac{\partial^2\theta}{\partial t^2} = -2\gamma\left\{\frac{-H'}{1+H^2}\gamma\frac{\partial^2 P}{\partial z^2} - \left(\frac{H''}{1+H^2} - \frac{2HH'^2}{\left(1+H^2\right)^2}\right)\gamma\left(\frac{\partial P}{\partial z}\right)^2\right\}, \tag{11}$$

$$\frac{\partial\theta}{\partial x_i} = \frac{2H'}{1+H^2}\frac{\partial P}{\partial x_i}, \tag{12}$$

$$\sum_i \frac{\partial^2\theta}{\partial x_i^2} = \frac{2H'}{1+H^2}\sum_i \frac{\partial^2 P}{\partial x_i^2} + 2\left(\frac{H''}{1+H^2} - \frac{2HH'^2}{\left(1+H^2\right)^2}\right)\sum_i\left(\frac{\partial P}{\partial x_i}\right)^2. \tag{13}$$

Substituting the expressions (11) – (13) into the second equation of the system (1) it is possible to obtain:

$$\begin{gathered} 2\gamma^2\left\{\frac{H'}{1+H^2}\frac{\partial^2 P}{\partial z^2} + \left(\frac{H''}{1+H^2} - \frac{2HH'^2}{\left(1+H^2\right)^2}\right)\left(\frac{\partial P}{\partial z}\right)^2\right\} - \\ -2c^2\left\{\frac{H'}{1+H^2}\sum_i\frac{\partial^2 P}{\partial x_i^2} + \left(\frac{H''}{1+H^2} - \frac{2HH'^2}{\left(1+H^2\right)^2}\right)\sum_i\left(\frac{\partial P}{\partial x_i}\right)^2\right\} + \\ +\left\{\omega_0^2 + c^2\sum_i\left(\frac{\partial Q}{\partial x_i}\right)^2 - \omega^2\right\}\frac{2H\left(1-H^2\right)}{\left(1+H^2\right)^2} = 0 \end{gathered} \tag{14}$$

It is possible to derive the following system of equations for functions P, Q, H starting from (10), (14):

$$\frac{8H\left(1-H^2\right)H'}{\left(1+H^2\right)}\left\{c^2\sum_i\frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial x_i} + \gamma\omega\frac{\partial P}{\partial z}\right\} = \frac{-4c^2H^2}{\left(1+H^2\right)^2}\sum_i\frac{\partial^2 Q}{\partial x_i^2} \tag{15}$$

$$\begin{gathered} \frac{2H'}{1+H^2}\left\{\gamma^2\frac{\partial^2 P}{\partial z^2} - c^2\sum_i\frac{\partial^2 P}{\partial x_i^2}\right\} + \\ +2\left(\frac{H''}{1+H^2} - \frac{2HH'^2}{\left(1+H^2\right)^2}\right)\left\{\gamma^2\left(\frac{\partial P}{\partial z}\right)^2 - c^2\sum_i\left(\frac{\partial P}{\partial x_i}\right)^2\right\} + \\ +\left\{\omega_0^2 - \omega^2 + c^2\sum_i\left(\frac{\partial Q}{\partial x_i}\right)^2\right\}\frac{2H\left(1-H^2\right)}{\left(1+H^2\right)^2} = 0 \end{gathered} \tag{16}$$

The system of equations (15) – (16) can be transformed to the separate system of equations for functions P, Q

$$\begin{cases} c^2 \sum_i \frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial x_i} = -\gamma\omega \frac{\partial P}{\partial z} \\ \sum_i \frac{\partial^2 Q}{\partial x_i^2} = 0 \end{cases}, \tag{17}$$

$$\begin{cases} \sum_i \frac{\partial^2 P}{\partial x_i^2} = \frac{\gamma^2}{c^2} \frac{\partial^2 P}{\partial z^2} \\ \gamma^2 \left(\frac{\partial P}{\partial z} \right)^2 - c^2 \sum_i \left(\frac{\partial P}{\partial x_i} \right)^2 + \omega_0^2 - \omega^2 + c^2 \sum_i \left(\frac{\partial^2 Q}{\partial x_i} \right) = 0, \end{cases} \tag{18}$$

And also to the separate equation for the function H:

$$\frac{H''}{1+H^2} - \frac{2HH'^2}{(1+H^2)^2} = \frac{H(1-H^2)}{(1+H^2)^2}. \tag{19}$$

If the functions P, Q satisfy the Caushy-Riemann conditions:

$$\begin{cases} \frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \\ \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \end{cases}, \tag{20}$$

then the 2D Laplace operator applied to the functions P, Q gives identically zero:

$$\begin{cases} \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0 \\ \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = 0. \end{cases} \tag{21}$$

Similarly it is possible to derive the following relations starting from the Caushy-Riemann conditions (20):

$$\begin{cases} \frac{\partial P}{\partial x} \frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y} \frac{\partial Q}{\partial y} \\ \left(\frac{\partial P}{\partial x} \right)^2 + \left(\frac{\partial P}{\partial y} \right)^2 = \left(\frac{\partial Q}{\partial x} \right)^2 + \left(\frac{\partial Q}{\partial y} \right)^2. \end{cases} \tag{22}$$

By the way if the Caushy-Riemann conditions (20) are satisfied then the system of equations (17) – (18) can be simplified significantly:

$$\begin{cases} c^2 \frac{\partial P}{\partial z} \frac{\partial Q}{\partial z} = -\gamma\omega \frac{\partial P}{\partial z} \\ \frac{\partial^2 Q}{\partial z^2} = 0 \end{cases}, \tag{23}$$

$$\begin{cases} \frac{\partial^2 P}{\partial z^2} = \left(\frac{\gamma}{c} \right)^2 \frac{\partial^2 P}{\partial z^2} \\ \gamma^2 \left(\frac{\partial P}{\partial z} \right)^2 - c^2 \left(\frac{\partial P}{\partial z} \right)^2 + c^2 \left(\frac{\partial Q}{\partial z} \right)^2 + \omega_0^2 - \omega^2 = 0. \end{cases} \tag{24}$$

The system of equations (23) – (24) can be transformed to the form:

$$\begin{cases} \dfrac{\partial^2 P}{\partial z^2}\left[1-\left(\dfrac{\gamma}{c}\right)^2\right]=0 \\ \dfrac{\partial^2 Q}{\partial z^2}=0 \end{cases}, \tag{25}$$

$$\begin{cases} \dfrac{\partial P}{\partial z}\left(c^2\dfrac{\partial Q}{\partial z}+\gamma\omega\right)=0 \\ \left(\dfrac{\partial P}{\partial z}\right)^2\left(\gamma^2-c^2\right)+c^2\left(\dfrac{\partial Q}{\partial z}\right)^2+\omega_0^2-\omega^2=0 \end{cases}. \tag{26}$$

There are two possible solutions of the system of equations (25) – (26):

1) If $\omega_0=0$ and the conditions are satisfied

$$\begin{cases} \gamma=c \\ \dfrac{\partial P}{\partial \mathrm{z}}\neq 0 \end{cases} \tag{27}$$

Then it is possible to obtain

$$\begin{cases} \dfrac{\partial Q}{\partial z}=-\dfrac{\omega}{c} \\ Q=-\dfrac{\omega}{c}z+g(x,y) \end{cases} \tag{28}$$

And the solution has the form:

$$P(x,y,z)=P(z-\gamma t)+f(x,y)$$
$$Q(x,y,z)=-\frac{\omega}{c}z+g(x,y),$$

where $P(z-\gamma t)$ is an arbitrary function and functions $f(x,y)$ and $g(x,y)$ are interrelated by the Caushy-Riemann conditions.

2) If $\gamma\neq A$ then the unknown functions can be found in the form:

$$\begin{aligned} P(x,y,z-\gamma t)&=p(z-\gamma t)+f(x,y) \\ Q(x,y,z)&=qz+g(x,y) \end{aligned}, \tag{29}$$

where p and q are the certain constants and $f(x,y)$, $g(x,y)$ also satisfy the Caushy-Riemann conditions.

The substitution of (29) into (25) – (26) gives the following conditions for the parameters:

$$\begin{cases} p(c^2q+\gamma\omega)=0 \\ p^2(\gamma^2-c^2)+c^2q^2+\omega_0^2-\omega^2=0 \end{cases}. \tag{30}$$

Solving the algebraic equations (30) it is possible to obtain two different solutions depending the value of the parameter p. The fist solution has the form:

$$\begin{cases} p=0 \\ q=\dfrac{\sqrt{\omega^2-\omega_0^2}}{c}, \quad \omega>\omega_0 \end{cases}. \tag{31}$$

It is obvious for this solution that the polar angle of the antiferromagnetism vector θ doesn't depend on time. That is why such solution describes the well-known precession of the azimuth angle φ.

The second solution of the equation (30) is specified by the expression:

$$\begin{cases} q = -\dfrac{\gamma\omega}{c^2} \\ p = \pm\dfrac{\omega_0}{c}\sqrt{\dfrac{1-\left(\dfrac{\omega}{\omega_0}\right)^2\left[1-\left(\dfrac{\gamma}{c}\right)^2\right]}{1-\left(\dfrac{\gamma}{c}\right)^2}} \end{cases}. \tag{32}$$

The change of sign «+» onto «–» in the last formula corresponds to the substitution $z \to -z$ in the expression for oscillation of the polar angle of the antiferrtomagnetism vector θ.

Such nonlinear waves can propagate in the range of phase velocities γ

$$c\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2} < \gamma < c$$

and for the case when $\omega \ge \omega_0$. The last two inequalities provide values of the parameter p in the expression (32) to be real. Thus $\gamma = c$ is the limit velocity of propagation of the interrelated nonlinear spin waves of the type (30). If $\omega = \omega_0$ then the lower limit of allowed phase velocities $\gamma = 0$ If $\omega > \omega_0$ then the lower limit of propagation phase velocity of the nonlinear spin waves is different from zero $\gamma_{\min} = c\sqrt{1-\left(\frac{\omega_0}{\omega}\right)^2}$ and "slow" nonlinear waves with the velocities $\gamma < \gamma_{\min}$ of the mentioned type can not propagate.

The notation $\frac{\omega_0}{c} = \frac{1}{\delta_0}$ is introduced where δ_0 is the characteristic length. That is why the parameter p has the form:

$$p = \frac{\delta(\gamma)}{\delta_0}, \tag{33}$$

where

$$\delta(\gamma) = \sqrt{\frac{1-\left(\dfrac{\omega}{\omega_0}\right)^2\left[1-\left(\dfrac{\gamma}{c}\right)^2\right]}{1-\left(\dfrac{\gamma}{c}\right)^2}}. \tag{34}$$

$\delta(\gamma)$ is the dependence of the characteristic size of nonlinear wave as a soliton on velocity γ. Thus $\lim\limits_{\gamma\to c}\delta(\gamma) = \infty$ and $\lim\limits_{\gamma\to\gamma_{\min}}\delta(\gamma) = 0$.

The solution (29) with this notations is transformed to the form:

$$\begin{cases} P(x,y,z) = \dfrac{\delta(\gamma)}{\delta_0}(z-\gamma t) + f(x,y) \\ Q(x,y,z) = -\dfrac{\gamma\omega}{c^2}z + g(x,y) \end{cases}. \tag{35}$$

The equation (19) for the function H can be transformed to the form:

$$\left(1+H^2\right)H'' - 2HH'^2 - H\left(1-H^2\right) = 0. \tag{36}$$

The substitution of variables and functions is done in the differential equation (36):

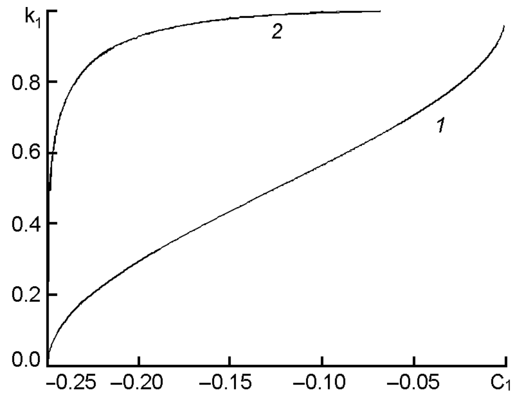

Fig. 1. Dependence of oscillation amplitude $\frac{\theta_0}{\pi}$ (curve *1*) and elliptic function module k_1 (curve *2*) on parameter C_1 .

$$H' = V = \frac{dH}{dP}, \tag{37}$$

$$H'' = \frac{d^2H}{dP^2} = \frac{dV}{dP} = \frac{dV}{dH}\frac{dH}{dP} = \frac{dV}{dH}V. \tag{38}$$

This substitution allows to reduce the order of the differential equation (36) and obtain the linear differential equation of the first order:

$$\left(1+H^2\right)V\frac{dV}{dH} - 2HV^2 - H\left(1-H^2\right) = 0. \tag{39}$$

The equation (39) can be integrated and the general solution of the equation (38) for the function $H(P)$ can be expresses though the elliptical functions

$$P(x,y,z-\gamma t) = \int \frac{\pm dH}{\sqrt{H^2 + C_1\left(1+H^2\right)^2}}. \tag{40}$$

If $-\frac{1}{4} < C_1 < 0$ then the last integral can be transformed to the form:

$$tg\left(\frac{\theta}{2}\right) = \frac{b_0}{dn\left(c_0\sqrt{|C_1|}\cdot P(x,y,z-\gamma t),k_1\right)}, \tag{41}$$

where

$$c_0 = \sqrt{\frac{1+2C_1+\sqrt{1+4C_1}}{2|C_1|}}, \quad b_0 = \sqrt{\frac{1+2C_1-\sqrt{1+4C_1}}{2|C_1|}},$$

$0 < k_1 \le 1$ is the module of the elliptical function,

$$k_1 = \sqrt{\frac{2\sqrt{1+4C_1}}{1+2C_1+\sqrt{1+4C_1}}}.$$

If $C_1 = -\frac{1}{4}$ then the amplitude θ_0 of angle θ oscillations, i.e. the difference between the maximum and minimum values of the angle θ is equal to zero and the elliptic function module $k_1 = 0$. If $C_1 = 0$ then the amplitude of angle oscillation is equal to π and the elliptic function module $k_1 = 1$ The amplitude of oscillation is equal to $\theta_0 = 2arctg(c_0) - 2arctg(b_0)$ in the range of $-\frac{1}{4} < C_1 < 0$. The dependences of oscillation amplitude $\frac{\theta_0}{\pi}$ on the parameter C_1 and elliptic function module k_1 on the parameter C_1 are represented in fig. 1.

It is obvious from Fig. 1 that it is possible to change the oscillation amplitude and the oscillation form by means of choice of the parameters C_1 and k_1.

The following solution was obtained for the case when $C_1 > 0$:

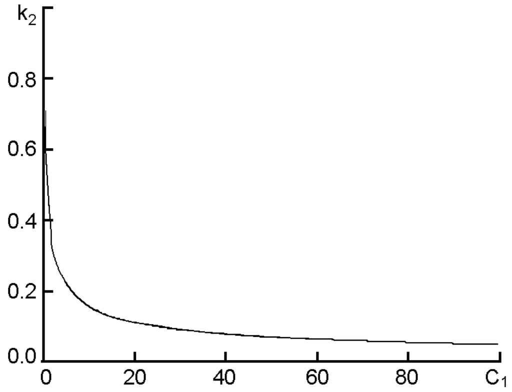

Fig. 2. Dependence of elliptic function module k_2 on the parameter C_1.

$$tg\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-sn\left(\frac{P(x,y,z-\gamma t)}{k_2},k_2\right)}{1+sn\left(\frac{P(x,y,z-\gamma t)}{k_2},k_2\right)}}, \qquad (42)$$

where

$$k_2=\frac{1}{\sqrt{1+4C_1}},$$

$0<k_2\le 1$ is the elliptic function module.

The oscillation amplitude of angle θ which is given by the solution (42) equals π and can not be changed depending on the parameter defining the elliptic function module k_2 and the oscillation form for the angle θ. The dependence of k_2 on C_1 is represented in Fig. 2.

Let us calculate the wavelength λ_1 for the spin wave of the type (41). Taking on account that the period of the elliptic function $dn(u,k)$ equals $2K(k)$ it is not difficult to obtain the wavelength:

$$\lambda_1=\frac{2K(k_1)\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}{c_0\sqrt{|C_1|}\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}} \qquad (43)$$

In this case $\lim\limits_{\gamma\to c}\lambda_1(\gamma)=0$ and $\lim\limits_{\gamma\to\gamma_{\min}}\lambda_1(\gamma)=\infty$.

Similarly it is possible to calculate the oscillation period τ_1 for the spin wave of the type (41):

$$\tau_1=\frac{2K(k_1)\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}{\gamma c_0\sqrt{|C_1|}\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}, \qquad (44)$$

where $K(k)=\int\limits_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\left(1-k^2\cdot\sin^2 x\right)}}$.

The frequency of the spin wave of the type (41) $\Omega_1=\frac{2\pi}{\tau_1}$ is expressed by the formula:

$$\Omega_1 = \frac{2\pi\gamma c_0\sqrt{|C_1|}\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}{2K(k_1)\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}. \quad (45)$$

That is why it is easy to verify that $\lim_{\gamma\to c}\Omega_1(\gamma)=\infty$, and $\lim_{\gamma\to\gamma_{\min}}\Omega_1(\gamma)=0$.

Taking on account that the elliptic function period $sn(u,k)$ equals $4K(k)$ it is possible to calculate similarly the wavelength and the frequency of the wave of the type (42) depending on the phase velocity of the wave γ:

$$\lambda_2 = \frac{4k_2K(k_2)\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}{\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}, \quad (46)$$

$$\Omega_2 = \frac{2\pi\gamma\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}{4k_2K(k_2)\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}. \quad (47)$$

Let us calculate the dispersion relation of the nonlinear spin waves. Let us introduce the following shorthand notations for taking on account wavelengths and frequencies of the waves of both types (41), (42):

$$\lambda_i = \frac{a_i\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}{\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}, \quad (48)$$

$$\Omega_i = \frac{2\pi\gamma\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}{a_i\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}, \quad (49)$$

where $a_1 = \frac{2K(k_1)}{c_0\sqrt{|C_1|}}$, $a_2 = 4k_2K(k_2)$.

Firstly let us calculate the wave vector $\kappa_i = \frac{2\pi}{\lambda_i}$:

$$\kappa_i = \frac{2\pi\sqrt{1-\left(\frac{\omega}{\omega_0}\right)^2\left(1-\left(\frac{\gamma}{c}\right)^2\right)}}{a_i\delta_0\sqrt{1-\left(\frac{\gamma}{c}\right)^2}}. \quad (50)$$

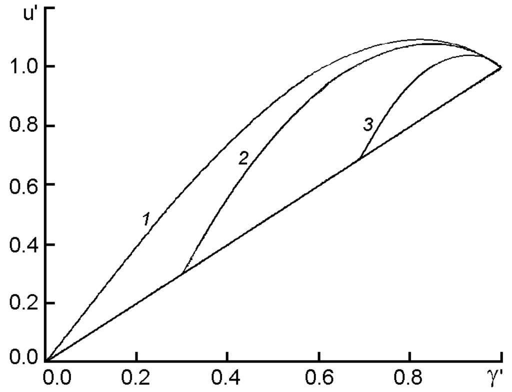

Fig. 3. Dependences of the dimensionless group velocity $u_i' = \frac{u_i}{c}$ on the dimensionless phase velocity $\gamma' = \frac{\gamma}{c}$ at different values of the parameter $\varsigma = \left(\frac{\omega}{\omega_0}\right)^2$. Curve *1* corresponds to the value of the parameter $\varsigma = 1$, curve *2* – the value $\varsigma = 1.1$, curve *3* – $\varsigma = 1.9$ accordingly. The strait line represents the dependence $u' = \gamma'$.

Here $\lim\limits_{\gamma\to c} \kappa_i = \infty$, $\lim\limits_{\gamma\to\gamma_{min}} \kappa_i = 0$

The inverse function has the form $\gamma(\kappa_i)$:

$$\gamma = c\sqrt{1 - \frac{1}{\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2}} \tag{51}$$

Taking on account the last expression it is possible to obtain the dispersion relation for the nonlinear waves of the types (41) and (42):

$$\Omega_i = c\kappa_i\sqrt{1 - \frac{1}{\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2}}\,. \tag{52}$$

Starting from the last formula expressing the dispersion relation it is possible to obtain the group velocity $u_i = \frac{d\Omega_i}{d\kappa_i}$ for the nonlinear spin waves of the types (41), (42):

$$u_i = \frac{c\cdot\left\{1 - \frac{1}{\left(\frac{a_i\delta_0}{2\pi}\right)^2\kappa_i^2 + \left(\frac{\omega}{\omega_0}\right)^2} + \frac{\left(\frac{a_i\delta_0}{2\pi}\right)^2\kappa_i^2}{\left[\left(\frac{a_i\delta_0}{2\pi}\right)^2\kappa_i^2 + \left(\frac{\omega}{\omega_0}\right)^2\right]^2}\right\}}{\sqrt{1 - \frac{1}{\left(\frac{a_i\delta_0}{2\pi}\right)^2\kappa_i^2 + \left(\frac{\omega}{\omega_0}\right)^2}}}\,. \tag{53}$$

It is possible to calculate that $\lim\limits_{\kappa_i\to 0} u_i = c\sqrt{1 - \left(\frac{\omega_0}{\omega}\right)^2} = \gamma_{min}$ in the last formula for the group velocity if the phase velocity is equal to the minimal one $\gamma = \gamma_{min} = c\sqrt{1 - \left(\frac{\omega_0}{\omega}\right)^2}$ when the wave vector value approaches to zero. It follows that the minimal phase velocity coincides with the group velocity of the spin wave. If the phase velocity is equal to the maximal one $\gamma = c$ and the wave vector value approaches infinity $\lim\limits_{\kappa_i\to\infty} u_i = c$ then the maximal phase velocity equals the maximal group

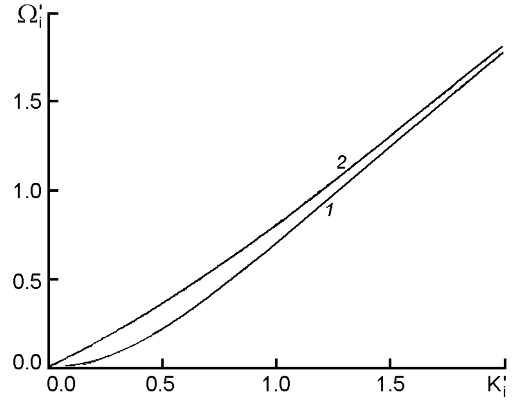

Fig. 4. Dispersion relations of the nonlinear spin waves in the dimensionless variables $\Omega_i' = \kappa_i' \sqrt{1 - \frac{1}{(\kappa_i')^2 + \varsigma}}$. Curve *1* corresponds to the value of the parameter $\varsigma = 1$, curve *2* – the value of the parameter $\varsigma = 1.9$, correspondingly.

velocity. In general it is not difficult to obtain the relation between the group and phase velocities of the spin waves of the types (41), (42) in the range of the phase velocities $\gamma_{\min} < \gamma < c$:

$$u_i = \frac{c^2}{\gamma}\left\{1 - \left(\frac{\omega}{\omega_0}\right)^2 \cdot \left(1 - \left(\frac{\gamma}{c}\right)^2\right)^2\right\}, \qquad (54)$$

It follows that in general the values of the phase and group velocities dpon't coincide for the nonlinear spin waves.

The dependence of dimensionless group velocity $u_i' = \frac{u_i}{c}$ on the dimensionless phase velocity $\gamma' = \frac{\gamma}{c}$ is represented in Fig 3 at different values of the parameter $\varsigma = \left(\frac{\omega}{\omega_0}\right)^2$.

The dispersion relation of the nonlinear spin waves in the dimensionless variables $\Omega_i' = \frac{a_i \delta_0}{2\pi c}\Omega_i$, $\kappa_i' = \frac{a_i \delta_0}{2\pi}\kappa_i$ is represented in Fig 4.

It is obvious from the graphic representation of the dispersion relation for the nonlinear spin waves (Fig. 4) that the frequency approaches to zero at low values of the wave vector so to say that the spectrum has no an activation. While the dependence of frequency on wave vector approaches to the linear dependence at great values of the wave vector. At the same time the known spin waves

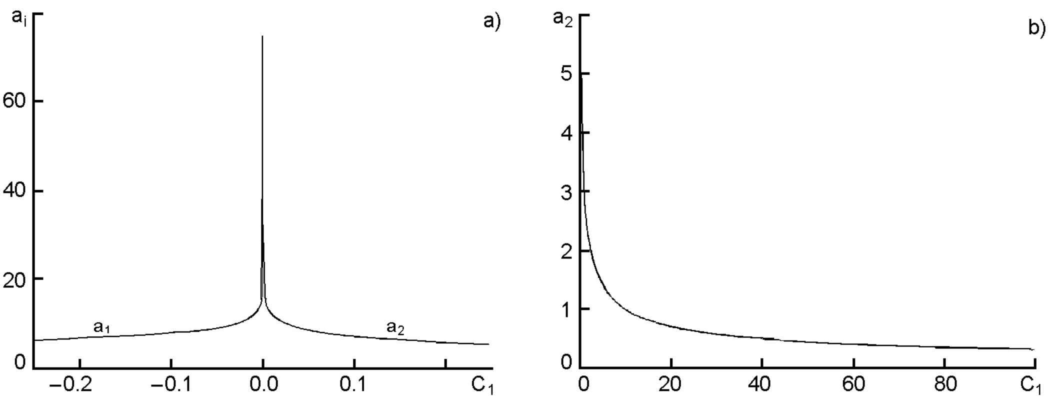

Fig. 5. Dependences of the parameter a_i on C_1: a) in the range of values $C_1 \in \left[-\frac{1}{4}, \frac{1}{4}\right]$; b) in the range $C_1 \in \left[\frac{1}{4}, 100\right]$.

in linear approximation are characterized by the existence of activation [16,17].

Besides, the dimensional frequency and wave vector of polar angle oscillations θ depends on the parameter C_1 while the dimensionless parameter a_i depends on C_1 as it is represented in fig. 5. It means that oscillation frequency for angle θ (41) depends on oscillation amplitude θ_0. The oscillation θ frequency for the solution (42) depends on the elliptic function module k_2.

Taking on account the expressions (3), (29). (32) it is possible to obtain the frequency Ω_φ and the wave vector κ_φ of the azimuth angle oscillations φ of antiferromagnetism vector:

$$\varphi = \kappa_\varphi z - \Omega_\varphi t + g(x,y), \tag{55}$$

$$\kappa_\varphi = -\frac{\gamma\omega}{c^2}, \tag{56}$$

$$\Omega_\varphi = -\omega'. \tag{57}$$

Taking on account the functional dependence (51) of the nonlinear spin wave phase velocity $\gamma(\kappa_i)$ related with the polar angle θ oscillations let us transform the dispersion relation (58) to the following form which takes on account the dependences of azimuth angle oscillation frequency Ω_φ on the wavelengths of both interrelated oscillations of polar and azimuth angles of antiferromagnetism vector. Considering the expressions (56), (57) the following dispersion relation has been obtained for azimuth angle oscillation φ:

$$\Omega_\varphi = \pm\omega_0\sqrt{\frac{1+\left(\frac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2+\sqrt{\left[1+\left(\frac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2\right]^2+4\left(\frac{a_i\delta_0\kappa_i c\kappa_\varphi}{2\pi\omega_0}\right)^2}}{2}}-\omega_H, \tag{58}$$

It is worthy to mention that

$$\varsigma = \left(\frac{\omega}{\omega_0}\right)^2 = \frac{1+\left(\frac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2+\sqrt{\left[1+\left(\frac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2\right]^2+4\left(\frac{a_i\delta_0\kappa_i c\kappa_\varphi}{2\pi\omega_0}\right)^2}}{2} \geq 1. \tag{59}$$

The dispersion relation (58) allows the passage to the limit of usual linear spin waves when the azimuth angle φ oscillates. If the wavelength of polar angle oscillation θ approaches to zero in the formula (58) (i.e. locally $\theta = const$) that corresponds to the wave vector value $\kappa_i = 0$:

$$\Omega_\varphi = \pm\sqrt{\omega_0^2 + \left(c\kappa_\varphi\right)^2} - \omega_H. \tag{60}$$

The spin wave dispersion relation in the limit case (60) coincides with the dispersion relation of usual spin waves of azimuth angle oscillations [16, 17].

Considering (59) it is possible to transform the dispersion relation of nonlinear spin wave oscillations of θ (52) to the following form which takes on account the dependence of the frequency Ω_i on the wave vectors of θ and φ angle oscillations:

$$\Omega_i = c\kappa_i\sqrt{1-\frac{2}{1+\left(\frac{c\kappa_\varphi}{\omega_0}\right)^2+\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2+\sqrt{\left[1+\left(\frac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\frac{a_i\delta_0\kappa_i}{2\pi}\right)^2\right]^2+4\left(\frac{a_i\delta_0\kappa_i c\kappa_\varphi}{2\pi\omega_0}\right)^2}}}. \tag{61}$$

Let us analyze propagation of the interrelated spin waves and the dispersion relations (52), (59). As the direction of order parameter vector $\vec{L}$ is defined by the angles θ and φ:

$$\vec{L} = L_0 \sin\theta\cos\varphi \cdot \vec{e}_x + L_0 \sin\theta\sin\varphi \cdot \vec{e}_y + L_0 \cos\theta \cdot \vec{e}_z\,, \tag{62}$$

where $L_0 = 2M_0$ then the interrelated oscillation of $\vec{L}$ vector which is consisting of θ and φ angle oscillations has a periodic character in time and space only if the ratio of frequencies and wave vectors are the ratios of integers m, n, M, N:

$$\frac{\Omega_i}{\Omega_\varphi} = \frac{m}{M}\,, \tag{63}$$

$$\frac{\kappa_i}{\kappa_\varphi} = \frac{n}{N}\,. \tag{64}$$

In this case the interrelated oscillation of antiferromagnetism vector has the frequency Ω and the wave vector κ that are equal to:

$$\Omega = \frac{\Omega_\varphi}{M} = \frac{\Omega_i}{m}\,, \tag{65}$$

$$\kappa = \frac{\kappa_\varphi}{N} = \frac{\kappa_i}{n}\,. \tag{66}$$

If the relation (61) is satisfied then taking on account the dispersion relations (52), (59) of separate oscillations it is possible to find the restrictions for the wave vectors κ_i, κ_φ for which the relation (60) is satisfied for temporary periodicity of the interrelated oscillation:

$$\begin{cases} \dfrac{m}{M} = \dfrac{\dfrac{c\kappa_i}{\omega_0}\sqrt{1-2\left\{1+\left(\dfrac{c\kappa_\varphi}{\omega_0}\right)^2+\left(\dfrac{a_i\delta_0\kappa_i}{2\pi}\right)^2+\sqrt{\left[1+\left(\dfrac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\dfrac{a_i\delta_0\kappa_i}{2\pi}\right)^2\right]^2+4\left(\dfrac{a_i\delta_0\kappa_i c\kappa_\varphi}{2\pi\omega_0}\right)^2}\right\}^{-1}}}{\pm\sqrt{\dfrac{1}{2}\left\{1+\left(\dfrac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\dfrac{a_i\delta_0\kappa_i}{2\pi}\right)^2+\sqrt{\left[1+\left(\dfrac{c\kappa_\varphi}{\omega_0}\right)^2-\left(\dfrac{a_i\delta_0\kappa_i}{2\pi}\right)^2\right]^2+4\left(\dfrac{a_i\delta_0\kappa_i c\kappa_\varphi}{2\pi\omega_0}\right)^2}\right\}}-\dfrac{\omega_H}{\omega_0}}\,, \\ \dfrac{n}{N} = \dfrac{\kappa_i}{\kappa_\varphi} \end{cases} \tag{67}$$

It is also the ratio of integers or in other words the rational number.

If the relations (61), (62) are not satisfied then the interrelated nonlinear oscillation is not clearly periodic. It means that that the antiferromagnetism vector never has initial direction in the fixed point of an antiferromagnet and it is impossible to find a wavelength of interrelated oscillation for which the antiferromagnetism vector distribution has a spatially periodic change.

Summarizing the results it is worthy to mention that the solutions (3), (56) describe the nonlinear spin wave propagation of order parameter with great amplitude in an antiferromagnet. The results of this work extend the theoretical foundation of functional properties of antiferromagnets and can be useful for invention of spin wave devices.

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Нелінійні спінові хвилі у антиферомагнетиках з одноосною магнітною анізотропією у магнітному полі

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Показано, що у антиферомагнетиках з одноосною магнітною анізотропією в зовнішньому магнітному полі можуть розповсюджуватися нелінійні спінові хвилі, такі, що одночасно відбуваються коливання з великою амплітудою полярного та азимутального кутів вектора антиферомагнетизму, тобто може розповсюджуватися нелінійна хвиля, що є точним динамічним тривимірним розв'язком рівнянь Ландау-Ліфшиця для антиферомагнетику. Знайдено закони дисперсії коливань полярного та азимутального кутів вектора антиферомагнетизму. Зокрема у довгохвильовому наближенні для коливань полярного кута закон дисперсії коливань азимутального кута співпадає з відомим законом дисперсії спінових хвиль у антиферомагнетиках у наближенні, в якому полярний кут є константою. Крім того, новий тип спінових хвиль, отриманий у даній роботі, характеризується можливістю формування структури фронту на широкому класі гармонічних двовимірних функцій, форми коливання, а також можливістю двочастотної та амплітудної модуляції.