RELATIVISTIC CHARGED-PARTICLE BEAM SCALAR POTENTIAL CALCULATIONS FOR COAXIAL DRIFT TUBE OF INFINITE LENGTH

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We receive the first order analytical expression for the scalar potential transversal distribution of finite width uniform density axisymmetric relativistic charged-particle beam propagating in the unbounded in the longitudinal direction coaxial drift tube in the strong guide magnetic field, which exhibits nonlinear dependence on the beam injection current. Ranges of the injection current values that show non-uniqueness of the solutions to the nonlinear ordinary differential equation describing the transverse distribution of the scalar potential in unbounded coaxial drift tube are also found numerically.

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INTRODUCTION

There has been recently a number of articles in the literature dedicated to analytical estimation of space-charge limited (SCL) current of a charged-particle beam transported in unbounded in the direction of the beam propagation coaxial drift tube [1 - 5]. It turned out that the question of the scalar potential transverse distribution is of a paramount importance to such estimates. In papers [6, 7] the nonlinear ordinary differential equation (ODE) that describes the scalar potential distribution is studied for the planar diode and circular cylindrical geometry analytically and numerically, respectively.

In this paper, we receive the first order (in addition to the zeros order usually presented in the literature, c.f. [8 - 10]) analytical expression of the scalar potential transverse distribution for finite width relativistic charged-particle beam transported in the strong guide magnetic field. The injection current values for which the nonlinear ODE provides one, two or no solutions are identified numerically with the aid of the shooting method as applied to the boundary-value problem under the investigation.

1. PROBLEM SETUP

Under the approximations stated above the scalar potential transversal distribution is given by the following nonlinear ODE and boundary conditions:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right) = -\frac{4}{c\beta_{\|}(r_o^2 - r_i^2)}\begin{cases}0, & r_1 \le r < r_i;\\ I_0, & r_i \le r < r_o;\\ 0, & r_o \le r \le r_2;\end{cases} \quad (1)$$

$$\varphi(r_1) = \varphi(r_2) = 0; \quad (2)$$

$$\varphi(r_i - 0) = \varphi(r_i + 0), \quad \varphi(r_o - 0) = \varphi(r_o + 0). \quad (3)$$

Here r_1, r_2 and r_i, r_o are the inner and outer radii of the drift tube and beam, respectively;

$$\beta_{\|} = \left(1 - \frac{\gamma_0^2}{\gamma_{\|0}^2}\left[\gamma_0 - \frac{q\varphi}{m_q c^2}\right]^{-2}\right)^{1/2} \quad (4)$$

is the longitudinal dimensionless velocity of the relativistic charged-particle beam; I_0 is the injection current (for the electrons $I_0 < 0$); $\gamma_{\|0} = \gamma_0(1+\gamma_0^2\beta_{\perp 0}^2)^{-1/2}$ is the initial beam dimensionless energy in the longitudinal motion; $\beta_{\perp 0} = \upsilon_{\perp 0}/c$; q and m_q are the particles charge and mass, respectively. The continuity conditions for the scalar potential and its first derivatives on the inner and outer edges of the beam have the form

$$\left.\frac{d\varphi(r)}{dr}\right|_{r_i-0} = \left.\frac{d\varphi(r)}{dr}\right|_{r_i+0}, \quad (5)$$

$$\left.\frac{d\varphi(r)}{dr}\right|_{r_o-0} = \left.\frac{d\varphi(r)}{dr}\right|_{r_o+0}. \quad (6)$$

We neglect the Larmor rotation of the charged-particle beam (assuming that it moves along the lines of a strong longitudinal (guide) magnetic field) and the beam self-magnetic field because of the same assumption.

2. SCALAR POTENTIAL CALCULATIONS

2.1. FIRST ORDER ANALYTICAL FORMULAS FOR SCALAR POTENTIAL

It is well-known that for small values of the injection current I_0 the normalized potential $f(\rho) = q\varphi(r)/(m_q c^2)$ ($\rho \equiv r/r_2$; c is the speed of light in vacuum) is also small, thus, we can expand the right hand side of Eq. (1) (see (4)) in powers of $f(\rho)$. Then, to the first order it reads

$$f_1''(\rho) + \frac{1}{\rho}f_1'(\rho) = -\begin{cases}0, & \rho_1 \le \rho < \rho_i, \quad \rho_o < \rho \le \rho_2;\\ M_0 + M_1 f_1(\rho), & \rho_i \le \rho \le \rho_o;\end{cases} \quad (7)$$

where the prime "′" denotes $d/d\rho$; $f_1(\rho)$ is the first order approximations to $f(\rho)$; $I_A = m_q c^3/q$ is the Alfvén current ($|I_A|$=17.05 kA for the electrons); $\beta_{\|0} = (1-1/\gamma_{\|0}^2)^{1/2}$ is the initial longitudinal dimensionless beam velocity; $\rho_1 = r_1/r_2$, $\rho_i = r_i/r_2$, $\rho_o = r_o/r_2$; $M_0 = 4I_0/(I_A\beta_{\|0}[\rho_o^2 - \rho_i^2])$, $M_1 = M_0/(\gamma_0\gamma_{\|0}^2\beta_{\|0}^2)$. The dimensionless boundary and continuity conditions for $f_1(\rho)$ take the form

$$f_1(\rho_1) = f_1(1) = 0,$$

$$f_1(\rho_i - 0) = f_1(\rho_i + 0), \quad f_1(\rho_o - 0) = f_1(\rho_o + 0), \quad (8)$$

$$\left.\frac{\partial f_1(\rho)}{\partial \rho}\right|_{\rho_i - 0} = \left.\frac{\partial f_1(\rho)}{\partial \rho}\right|_{\rho_i + 0}, \quad \left.\frac{\partial f_1(\rho)}{\partial \rho}\right|_{\rho_o - 0} = \left.\frac{\partial f_1(\rho)}{\partial \rho}\right|_{\rho_o + 0}.$$

In earlier works [4, 5] we derived analytically the zeros order (assuming $\beta_\parallel \equiv \beta_{\parallel 0}$) scalar potential transverse distribution and with help of it (apparently for the first time) received the analytical estimate of the SCL current for a finite width charged-particle beam in the unbounded coaxial drift tube. The solution to linear inhomogeneous ODE (7) has the following analytical form:

$$f_1(\rho) = \begin{cases} C_{i;1} \ln(\rho/\rho_1), & \rho_1 \le \rho < \rho_i; \\ C_{1;1} J_0(\sqrt{M_1}\rho)/M_1 + \\ \quad + C_{2;1} Y_0(\sqrt{M_1}\rho)/M_1 - M_0/M_1, & \rho_i \le \rho \le \rho_o; \\ C_{o;1} \ln \rho, & \rho_o < \rho \le 1; \end{cases} \quad (9)$$

where

$$C_{1;1} = \frac{1}{\sqrt{M_1}} \frac{M_1 D_i - M_0 D_o}{N_o D_i - N_i D_o}, C_{2;1} = -\frac{1}{\sqrt{M_1}} \frac{D_i (N_i M_1 - N_0 M_0)}{D_o (N_o D_i - N_i D_o)},$$

$$C_{i;1} = \frac{\rho_i}{M_1}[C_{1;1} J_1(\sqrt{M_1}\rho_i) + C_{2;1}\sqrt{M_1} Y_1(\sqrt{M_1}\rho_i)],$$

$$C_{o;1} = -\frac{\rho_o}{M_1}[C_{1;1} J_1(\sqrt{M_1}\rho_o) + C_{2;1}\sqrt{M_1} Y_1(\sqrt{M_1}\rho_o)].$$

Here

$$D_{i,o} = J_0(\sqrt{M_1}\rho_{i,o})/\sqrt{M_1} + \rho_{i,o} \ln \rho_{i,o} J_1(\sqrt{M_1}\rho_{i,o})/\sqrt{M_1},$$

$$N_{i,o} = Y_0(\sqrt{M_1}\rho_{i,o})/\sqrt{M_1} + \rho_{i,o} \ln \rho_{i,o} Y_1(\sqrt{M_1}\rho_{i,o})/\sqrt{M_1}.$$

The scalar potential distribution (7) obviously can attain the extremal value only inside ($\rho_i \le \rho_{\text{ext}} \le \rho_o$) the charged-particle beam, which immediately results in the following transcendental equation for its dimensionless radial position:

$$D_o J_1(\sqrt{M_1}\rho_{\text{ext};1}) - D_i Y_1(\sqrt{M_1}\rho_{\text{ext};1}) = 0 \quad (10)$$

that needs to be solved numerically.

For the coaxial drift tube geometry and relativistic electron beam parameters below, the analytical estimate of the SCL current presented elsewhere (see for details [4, 5]) gives $|I_{\text{lim}}^{\text{an}}| = 30.28$ kA with the corresponding extremal normalized scalar potential value $f_{\text{lim}}^{\text{an}}(\rho_{\text{ext}}^{\text{an}}) = 0.52$ ($r_{\text{ext}}^{\text{an}} = 1.4748$ cm) while the nonlinear numerical estimate is $|I_0| \equiv |I_{\text{lim}}^{(\text{t})}| = 32.89$ kA with the corresponding extremal normalized scalar potential value $f_{\text{lim}}^{\text{num}}(\rho_{\text{ext}}^{\text{num}}) = 0.76$ ($r_{\text{ext}}^{\text{num}} = 1.4724$ cm) for $r_1 = 1$ cm, $r_2 = 2$ cm, $r_i = 1.25$ cm, $r_o = 1.75$ cm, $\gamma_0 = 2$ and $\beta_{\perp 0} = 0$.

In Fig. 1,*a* numerical solution to Eq. (1) by an iterative method, the zeros and first order analytical solutions for the scalar potential transverse distribution in the coaxial drift tube for different injection currents I_0 are shown for the stated above relativistic electron beam parameters. One can see that the first order analytical solution for the scalar potential distribution provides a better approximation to the solutions to nonlinear Eq. (1).

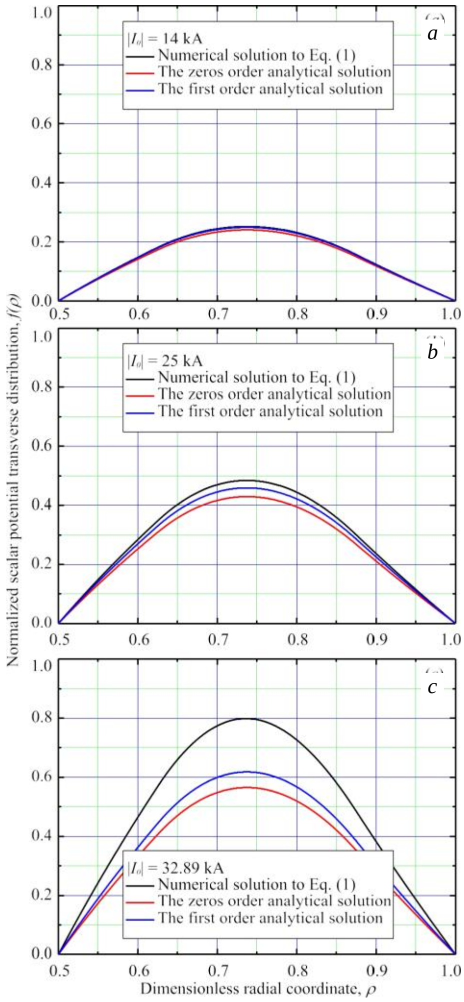

Fig. 1. Normalized scalar potential, $f(\rho) = q\varphi/(m_q c^2)$ *(*$\rho \equiv r/r_2$*), transverse distribution of relativistic electron beam for different values of injection current* I_0 *(*r_1*=1 cm,* r_2*=2 cm,* r_i*=1.25 cm,* r_o*=1.75 cm,* γ_0*=2,* $\beta_{\perp 0}$*=0)*

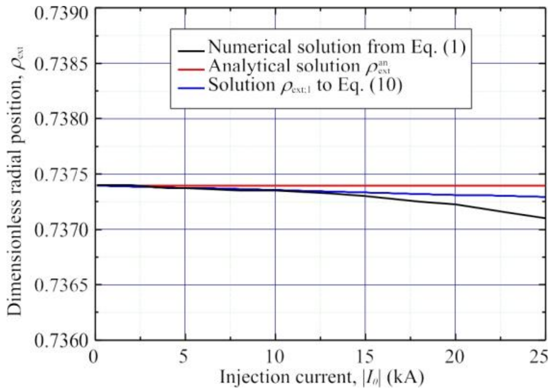

Fig. 2. Dimensionless radial position $\rho_{\text{ext}} \equiv r_{\text{ext}}/r_2$*, at which extremal values of the scalar potential distribution are attained in the coaxial drift tube, as a function of injection current* I_0 *for relativistic electron beam (*r_1*=1 cm,* r_2*=2 cm,* r_i*=1.25 cm,* r_o*=1.75 cm,* γ_0*=2,* $\beta_{\perp 0}$*=0)*

This is less essential for smaller injection currents and becomes more substantial for larger ones. In Fig. 2 we plot the dimensionless radial position $\rho_{\text{ext}} \equiv r_{\text{ext}}/r_2$, at which extremal values of the scalar potential distribution are attained in the coaxial drift tube, as a function of injection current I_0 obtained from the numerical solution to nonlinear Eq. (1), the analytical solution

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$\rho_{\rm ext}^{\rm an} \equiv r_{\rm ext}^{\rm an} / r_2$ ([4, 5], i.e. derived from the zeros order scalar potential transverse distribution) and the solution $\rho_{\rm ext;1}$ to transcendental Eq. (10) (i.e. derived from the first order scalar potential transverse distribution).

2.2. NON-UNIQUENESS OF SOLUTIONS FOR THE SCALAR POTENTIAL

It turns out that for nonlinear ODE (1) the unique solution exists only locally in ρ and only for small injection currents (see [6, 7]). Also, a standard iterative method, as applied to the boundary-value problem (1) -(6), allows us easily find numerically only one solution even for large injection current values, where, perhaps, would exist more than one solution. These solutions would obey the same boundary conditions (Eqs. (2) - (3)) on the drift tube walls. However, they, obviously, would possess there different values of the one-sided first derivatives. For this reason, we consider instead a substitute Cauchy problem, which is set in correspondence to the boundary-value problem (7), (8) (or (1) - (6) under the consideration, in the form

$$f(\rho_1) = 0, \quad f'(\rho_1) = f_0'. \qquad (11)$$

Here f_0' is a constant and the task is to find its value(s), for which the boundary condition $f(1) = 0$ is also satisfied (this is essentially the shooting method as applied to the boundary-value problem under the consideration).

In Fig. 3 we present the numerical solutions by the shooting method to the dimensionless Cauchy problem (1), (3) - (6), (11) for different injection currents I_0 and some values of the initial first derivatives $f'(\rho_1) = f_0'$ (dash-dotted and dotted curves) as well as the numerical solution by an iterative method to the boundary-value problem (1) - (6) (solid curve; $\rho_1 = 0.5$). Thus, similar to the planar diode and circular cylindrical geometry [6, 7], we can distinguish the three ranges of injection current. For lower injection currents nonlinear ODE (1) with boundary conditions (2) possess only one (unique) solution (this situation is illustrated in Fig. 3,*a*). This is also verified through solution to Eq. (1) by use of the shooting method (11): it gives the initial first derivative value $f'(\rho_1) = f_0' = 1.6255$ corresponding to the fulfillment of the second boundary condition $f(1) = 0$ in (2) for $|I_0| = 14$ kA. With a further growth of the injection current, one obtains the second its range, in which there are two distinct (non-uniqueness) solutions both obeying boundary conditions (2) with two different values for initial first derivatives $f'(\rho_1) = f_0'$ of the Cauchy problem (1), (3) - (6), (11) (see this is shown in Figs. 3,*b*,*c* and 4). For the chosen coaxial drift tube geometry and the relativistic electron beam parameters two-fold solution starts appearing at injection current $|I_0| = 18$ kA. At the beginning of this range of injection currents this two solutions differ a lot; they become closer to one another with the increase of the injection current and at a certain (limiting) value of injection current ($|I_0| \equiv |I_{\rm lim}^{(t)}| = 32.89$ kA for the geometry and parameters under consideration) they merge into a single entity (c.f. Fig. 4,*b*). For the injection currents I_0 greater than this SCL current ($I_{\rm lim}^{(t)}$) there is no solution to boundary-value problem (1) - (6) (or, equivalently, to substitute Cauchy problem (1), (3) - (6), (11).

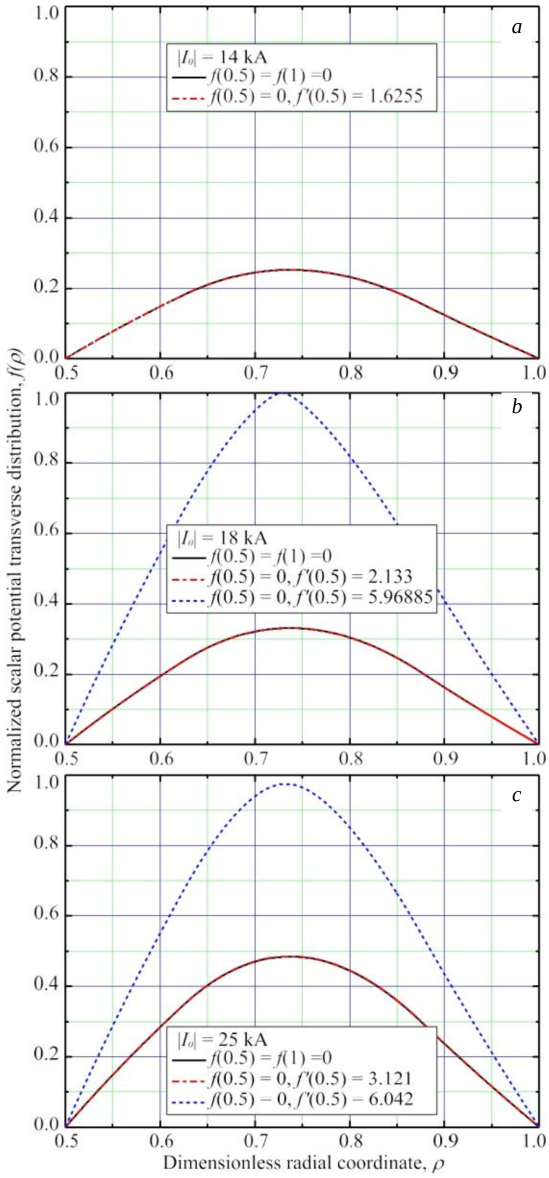

Fig. 3. Normalized scalar potential, $f(\rho) = q\varphi/(m_q c^2)$ *(*$\rho \equiv r/r_2$*), transverse distribution of relativistic electron beam for different values of injection current* I_0 *obtained as the numerical solution to Eq. (1) with the help of the shooting method (dash-dotted (red) and dotted (blue) curves) and that of with the help of an iterative method (solid (black) curve): the drift tube geometry and relativistic electron beam parameters are as above*

Fig. 5 contains the plots of extremal values of normalized scalar potential distribution, $f(\rho_{\rm ext})$, induced by relativistic electron beam in the unbounded coaxial drift tube (*a*); dimensionless radial positions, $\rho_{\rm ext}$, at which these extremal values of scalar potential distribution are attained (*b*); and initial first derivative values, $f'(\rho_1)$, for the substitute Cauchy problem (11) as functions of injection current I_0 for the first and second types of solutions to nonlinear ODE (1).

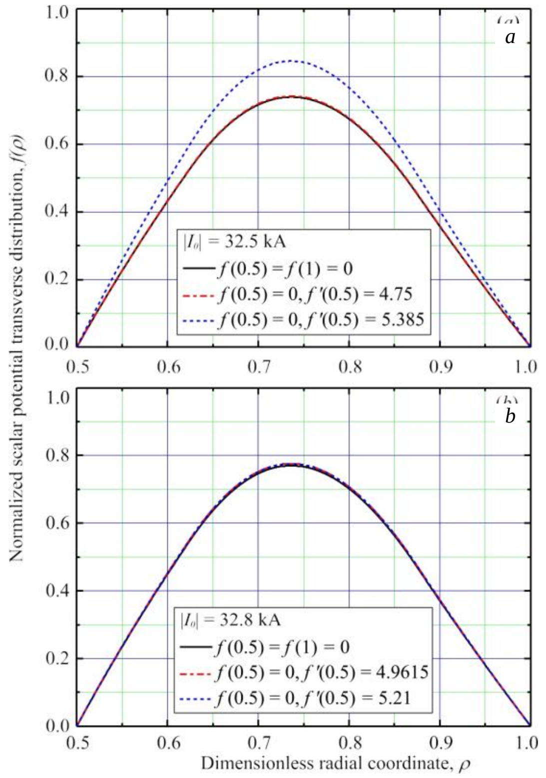

Fig. 4. Normalized scalar potential, $f(\rho) = q\varphi/(m_q c^2)$ ($\rho \equiv r/r_2$), transverse distribution of relativistic electron beam in the second range of values of injection current I_0 obtained as the numerical solution to Eq. (1) with the help of the shooting method (dash-dotted (red) and dotted (blue) curves) and that of with the help of an iterative method (solid (black) curve): the drift tube geometry and relativistic electron beam parameters are as above

The first type of solutions exists for all values of the injection current that render nonlinear ODE (1) solvable. The second type of solutions exhibits a threshold behavior with respect to the injection current. At the SCL current value the both types of solutions merge.

CONCLUSIONS

Conducted analytical and numerical investigation demonstrates that analytical estimate of the radial position, at which the scalar potential distribution for a finite width relativistic charge-particle beam propagating in an unbounded coaxial drift tube attains its extremum, to the best of our knowledge presented for the first time in [4, 5] is, in fact, quite accurate (see Fig. 2), which explains an unexpectedly good precision of the SCL current estimates based on it. It is also worth mentioning that already in the first order ('consistently' linear) approximation (7) to the scalar potential distribution induced by a relativistic charged-particle beam in an unbounded coaxial drift tube the extremal radial position, $r_{\text{ext};1}$, becomes transcendentally dependent (see Eq. (10)) on the beam injection current parameter I_0/I_A; that diminishes its usefulness for possible utilization in analytical estimations of the SCL current.

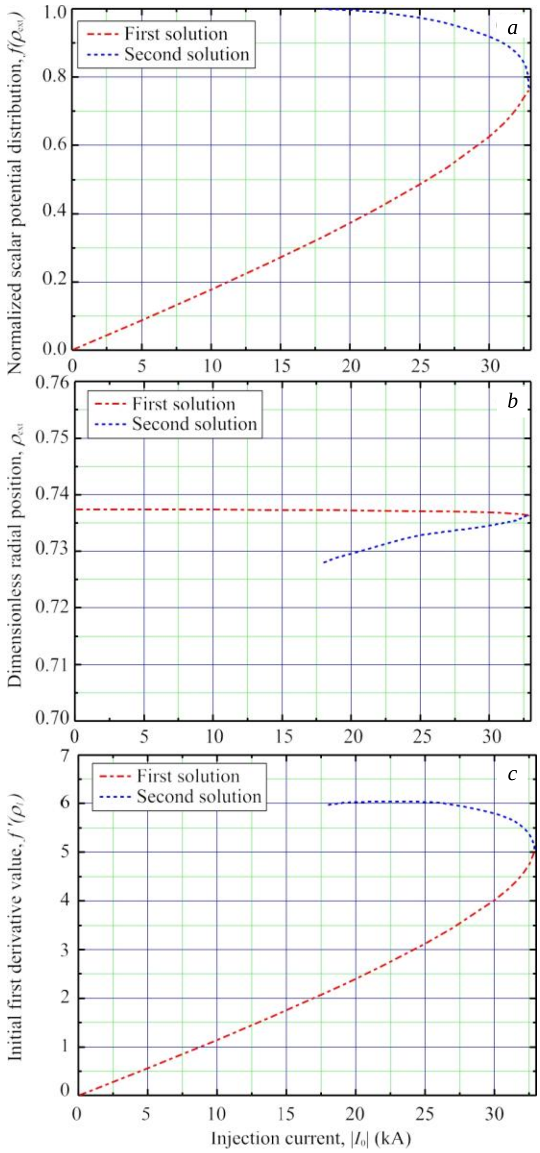

Fig. 5. Extremal values of normalized scalar potential distribution, $f(\rho_{\text{ext}})$, induced by relativistic electron beam in the unbounded coaxial drift tube (a); dimensionless radial positions, ρ_{ext}, at which these extremal values of scalar potential distribution are attained (b); and initial first derivative values, $f'(\rho_1)$ $(\rho_1 = 0.5)$, for the substitute Cauchy problem (11) as functions of injection current I_0 for the first and second types of solutions to nonlinear ODE (1): the drift tube geometry and relativistic electron beam parameters are as above

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РАСЧЕТЫ СКАЛЯРНОГО ПОТЕНЦИАЛА РЕЛЯТИВИСТСКОГО ПУЧКА ЗАРЯЖЕННЫХ ЧАСТИЦ В НЕОГРАНИЧЕННОЙ КОАКСИАЛЬНОЙ КАМЕРЕ ДРЕЙФА

Татьяна Яценко и Константин Ильенко

В приближении сильного внешнего магнитного поля получены аналитические выражения первого порядка для скалярного потенциала, создаваемого пучком заряженных частиц, который распространяется в неограниченной в продольном направлении коаксиальной камере дрейфа, учитывающие нелинейное влияние тока инжекции пучка. С помощью численных методов найдены зоны значений тока инжекции пучка заряженных частиц, распространяющегося в неограниченной в продольном направлении коаксиальной камере, соответствующие наличию нескольких решений нелинейного обыкновенного дифференциального уравнения, которое описывает скалярный потенциал, создаваемый пучком в этой камере.

РОЗРАХУНКИ СКАЛЯРНОГО ПОТЕНЦІАЛУ РЕЛЯТИВІСТСЬКОГО ПУЧКА ЗАРЯДЖЕНИХ ЧАСТИНОК У НЕОБМЕЖЕНІЙ КОАКСІАЛЬНІЙ КАМЕРІ ДРЕЙФУ

Тетяна Яценко та Костянтин Ільєнко

У наближенні сильного зовнішнього магнітного поля отримано аналітичні вирази першого порядку для скалярного потенціалу, що створюється пучком заряджених частинок, який розповсюджується в необмеженій в повздовжньому напрямку коаксіальній камері дрейфу, які враховують нелінійний вплив струму інжекції пучка. За допомогою чисельних методів знайдено зони значень струму інжекції пучка заряджених частинок, що розповсюджується в необмеженій в поздовжньому напрямку коаксіальній камері, які відповідають наявності декількох рішень нелінійного звичайного диференційного рівняння, котре описує скалярний потенціал, що створює пучок у цій камері.