The given level attainment problem and material destruction

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In frameworks of percolation scenario of material destruction, the general stochastic model is proposed. It is connected with the so-called attainment problem of the given level $E > 0$ in the case of independent, equally distributed, non-negative random variables ε_k, $k = 1, 2, \ldots$. The asymptotic formula for the probability distribution of random instant of the level $E > 0$ attainment is proved when the absorbed destructive energy is large. It is done in the case when ε_k possess a finite second moment and their sums have an absolutely continuous probability distribution with the bounded density.

В рамках представления о перколяционном сценарии разрушения материала, предложена стохастическая модель, которая связана с т.н. задачей достижения заданного уровня $E > 0$. Эта модель проанализирована в случае независимых, одинаково распределённых, неотрицательных случайных величин ε_k, $k = 1, 2, \ldots$, представляющих порции поглощённой энергии. Доказана асимптотическая формула для распределения вероятностей случайного момента достижения заданного уровня $E > 0$, когда величина поглощённой энергии, затраченной на разрушение очень велика. Эта формула имеет смысл, если второй момент ε_k является конечной величиной и их суммы имеют абсолютно непрерывное распределение вероятностей с ограниченной плотностью.

The following problem which is arisen in the statistical theory of material destruction is considered in this work. It is required to find the probability distribution of the random destruction time τ of the material when the definite energy level defining its degradation is attained [1].

Earlier, it was proposed to solve this problem considering the destruction process as the formation of the defective cluster in the material piece. The destruction scenario consisting of the formation of the defective cluster which is covered the sample was named the percolation one [1, 2]. If the percolation scenario is realized, it is possible to characterize the material degradation by the absorbed energy $J(t)$ in the material which is the random function. Then, for the description of the destruction process, it is necessary to determine the probability distribution of the random time $\tau(E)$ when $J(t)$ attains the given level E. Such a problem is named as the given level attainment one. The natural approach which has been analyzed in papers [2, 3] assumes that the energy is pumped to the system with the constant average intensity $\varepsilon(t) = dJ(t)/dt$. The simplest physical picture consists of the following. There are not any external influence on the system during some random temporal intervals and so the material destruction is absent during them. Therefore, the function $J(t)$ does not change during these intervals. Otherwise, there are some random time instants of τ_k, $k = 1, 2...$, when the absorption of appreciable energy portions occurs so quickly that it is reasonable, from the mathematical point of view, to neglect the duration of this process. We may model such kind of energy absorption in the form $\sum_k \varepsilon_k \delta(t - \tau_k)$ of the process intensity. In this case, one may consider the random time τ which is defined by the formula

$$\tau(E) = \min\{t : J(t) \geq E\} . \quad (1)$$

If the level E is very large, it is possible to expect that probability distribution of the random variable $\tau(E)$ will have the universality at the limit $E \to \infty$. This requires the rigorous justification. Earlier, we have analyzed the problem pointed out in the particular case [4] when $J(t)$ is the sum of independent and equally distributed variables. They were discrete and their general probability distribution was the poissonian one. In this case, the energy absorbed in the system is modelled by the random sequence $\{J_n[\varepsilon_k]; n \in \mathbb{N}\}$ with realizations

$$J_n[\varepsilon_k] \;=\; \sum_{k=1}^{n} \varepsilon_k\,. \tag{2}$$

Besides, the attaining time $\tau(E)$ is determined by the integer random variable

$$\nu(E) \;=\; \min\{n\,;\, J_n[\varepsilon_k] \geq E\}\,, \tag{3}$$

$\tau(E) = t_0\nu(E)$. The asymptotic probability distribution of the random variable $\nu(E)$ at the limit $E \to \infty$ has been calculated. In the present work, we solve this problem for the arbitrary continuous probability distribution of random variables ε_k, $k = 1, 2, \ldots$.

Let $P_E(n)$ is the probability of the random event when the given level E is attained at the moment nt_0. Then, due to the independence of the variables ε_k, $k = 1, 2, \ldots$, the following formula takes place

$$P_E(n) = \int_0^{E-0} \Pr\{\varepsilon_n > E - x\} d\Pr\{J_{n-1}[\varepsilon_k] < x\}\,. \tag{4}$$

We solve the above-mentioned problem when the probability distribution of random ε_k, $k = 1, 2, \ldots$ has the finite dispersion. We notice that there exists the Fourier transformation on the parameter E at such a condition,

$$\psi(t, n) = \int_0^{\infty} e^{itE} P_E(n) dE\,. \tag{5}$$

Therefore, it is proved easily that the function $\psi(t, n)$ is given by the formula

$$\psi(t, n) = it^{-1}(1 - \varphi(t))\,[\varphi(t)]^{n-1} \tag{6}$$

where

$$\varphi(t) = \int_0^{\infty} e^{itx} d\Pr\{\varepsilon < x\} \tag{7}$$

is the characteristic function of the random variable ε. Moreover, if the probability distribution $\Pr\{\varepsilon < x\}$ has the bounded density, there exists the absolutely converged inverse transformation of the function $\psi(t, n)$ due to the Linnik lemma. Then, we have

$$P_E(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t, n) e^{-itE} dt\,. \tag{8}$$

Basing on the integral representation (8), we prove now the asymptotic formula for the function $P_E(n)$. Really, the following statement is true.

Let $\varepsilon_1, \varepsilon_2 \ldots$ be the sequence of independent, equally distributed, nonnegative random variables having the finite second moment $\langle \varepsilon_k^2 \rangle$ and the bounded distribution density. Then, the formula

$$\left((2\pi n)^{1/2}(\sigma/a)P_E(n) - \exp\left\{-x^2/2\right\}\right) \to 0 \tag{9}$$

is valid when $E \to \infty$, $E = na + x\sigma n^{1/2}$, $|x| < \infty$ where $a = \langle \varepsilon_k \rangle$, $\sigma^2 = \langle \varepsilon_k^2 \rangle - a^2$.

The proof is based on the integral representation (8) which has the following explicit form

$$P_E(n) = \frac{i}{2\pi}\int\limits_{-\infty}^{\infty}(1-\varphi(t))[\varphi(t)]^{n-1}\frac{e^{-iEt}dt}{t}\,.$$

We introduce the designation

$$\bar\varphi(t) = \varphi(t)e^{-iat}\,,\quad \bar\psi(t) = i\frac{1-\varphi(t)}{t}e^{-iat}$$

and make the change of the variable $t\sigma\sqrt{n} \Rightarrow t$ at the integral. As a result, we obtain

$$\sigma(2\pi n)^{1/2}P_E(n) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\bar\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{n-1}\bar\psi\left(\frac{t}{\sigma\sqrt{n}}\right)e^{itx}dt\,.$$

Further, we use the famous formula

$$\exp\left(-x^2/2\right) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\exp\left(itx - t^2/2\right)dt$$

and represent the expression

$$I = \sqrt{2\pi}\left[(2\pi n)^{1/2}(\sigma/a)P_E(n) - \exp\left(-x^2/2\right)\right]$$

by the following way

$$I = \int\limits_{-\infty}^{\infty}e^{itx}\left[\frac{1}{a}\,\bar\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)^{n-1}\bar\psi\left(\frac{t}{\sigma\sqrt{n}}\right) - \exp\left(-\frac{t^2}{2}\right)\right]dt\,. \qquad (9)$$

It is necessary to prove that the integral (9) tends to zero at $n \to \infty$.

Accurate estimation gives us that there exists the uniform convergence to zero at $n \to \infty$ of the integral part where the integration is fulfilled on fixed compact interval $[-A, A]$. It takes place

$$\int\limits_{|t|>A}\exp\left\{-\frac{t^2}{2}\right\}dt \to 0$$

at $A \to \infty$. At the same time, we have

$$\int\limits_{|t|>\delta\sigma\sqrt{n}}\left|\psi\left(\frac{t}{\sigma\sqrt{n}}\right)\right|\left|\bar\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right|^{n-1}dt \to 0$$

for each fixed $\delta > 0$ at $n \to \infty$ taking into account that ε_k, $k = 1, 2, ...$ are continuous (not discrete) random values. At last, the integral

$$\int\limits_{\delta\sigma\sqrt{n}>|t|>A}\left|\bar\psi\left(\frac{t}{\sigma\sqrt{n}}\right)\left[\bar\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{n-1}\right|dt \to 0$$

due to the inequality

$$\left|\bar\varphi\left(\frac{t}{\sigma\sqrt{n}}\right)\right| < \exp\left(-\frac{t^2}{4n}\right)$$

being valid at sufficiently small $|t|$ and taking into account that

$$\left|\bar\psi\left(\frac{t}{\sigma\sqrt{n}}\right)\right| < \frac{2}{A}\,.$$

On the basis of these inequalities, we obtain

$$\int\limits_{\delta\sigma\sqrt{n}>|t|>A} \left| \bar{\psi}\left(\frac{t}{\sigma\sqrt{n}}\right) \left[\bar{\varphi}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^{n-1} \right| dt < \frac{2}{A^2(1-n^{-1})} \exp\left(-\frac{A^2}{4}(1-n^{-1})\right).$$

Thus, all parts which are included in the integral I tend to zero at $n \to \infty$.

We prove now more simple statement that the probability distribution function of the sum $\sum\limits_{n:N_1<n\leq N_2} P_E(n)$ of $\nu(E)$ tends to the gaussian one at $E \to \infty$, $n \to \infty$. Obviously, it is valid

$$\sum_{n:N_1<n\leq N_2} P_E(n) = \Pr\{J_{N_1}[\varepsilon] < E\} - \Pr\{J_{N_2}[\varepsilon] < E\}$$

Introducing the random variable

$$\frac{J_N[\varepsilon] - Na}{N^{1/2}} = \zeta_N\,, \quad N = 1, 2, ...$$

and parameters

$$A_1 = \frac{E - N_1 a}{N_1^{1/2}}, \quad A_2 = \frac{E - N_2 a}{N_2^{1/2}}\,,$$

we rewrite the obtained equality in the following form

$$\sum_{N_1<n\leq N_2} P_E(n) = \Pr\{\zeta_{N_1} < A_1\} - \Pr\{\zeta_{N_2} < A_2\}\,. \tag{10}$$

Since $\langle\varepsilon_k^2\rangle < \infty$, the classical integral limit theorem (see, for example, [5]) for the corresponding probabilities of sums of independent random variables $\varepsilon_1, \varepsilon_2, ...$ takes place when the threefold limiting transition $E \to \infty$, $N_i \to \infty$ with bounded A_i, $i = 1, 2$ is done in the right hand of the equation (10).

Therefore, it is valid the formula

$$\Pr\{\zeta_{N_i} < A_i\} = \frac{1}{\sqrt{2\pi\sigma}} \int\limits_{-\infty}^{A_i} \exp\{-x^2/2\sigma\} dx\,(1 + o(1))\,, \quad i = 1, 2\,.$$

Substituting these asymptotic expressions in (10), we conclude that, for the probability distribution $P_E(n)$ of the random attainment instant $\nu(E)$ of the level E by the sum $J_n[\varepsilon_k]$ of independent, equally distributed, non-negative random variables $\varepsilon_1, \varepsilon_2, ...$, the limiting formula

$$\sum_{n:N_1<n\leq N_2} P_E(n) = \frac{1}{\sqrt{2\pi\sigma}} \int\limits_{A_1}^{A_2} \exp\{-x^2/2\sigma\} dx\,(1 + o(1))$$

holds when $N_1, N_2, E \to \infty$ in such a way that parameters A_1, A_2 remain bounded.

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Задача досягнення заданого рівня та руйнування матеріалу

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В умовах концепції перколяційного сценарію зруйнування матеріалу доглянуто загальну стохастичну модель руйнування. Вона пов'язана з т.з. проблемой досягнення заданого енергетичного рівня $E > 0$ з незалежними, еквівалентно розподіленними, невід'ємними випадковими величинами поглинутої енергії ε_k, $k = 1, 2, \ldots$. Доведено асимптотичну формулу для розподілу ймовірностей випадкового моменту досягнення рівня $E > 0$, коли повна поглинута енергія велика. Ця функція має сенс, якщо другий момент ε_k є скінченною величиною та їх суми мають абсолютно неперервний розподіл ймовірностей з обмеженою щільністю.