# Intermittency without chaotic phases

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A simple map is proposed where the intermitting regime without chaotic phases is appeared. The main characteristics of that intermittent regime have been studied. The distribution function of laminar phase length, Lyapunov exponent and topologic entropy have been derived analytically. The main characteristics have been compared with the computer simulation results.

Предложено простое отображение, в котором возникает перемежаемый режим без хаотических фаз. Изучены основные характеристики такого перемежаемого режима. Аналитически получены функция распределения длин ламинарных фаз, показатель Ляпунова и топологическая энтропия. Проведено сравнение основных характеристик с результатами численного моделирования.

Intermittent regimes are well known to arise in numerous physical systems [1]. For low-dimensionality systems, such regimes were studied in classical works [2, 3]. The intermittency origin is believed to be associated with the inverse tangent bifurcation and appearing of a local narrow "corridor", or the lamirization region. The intermittent regime arises just when a region of return or re-injection to that "corridor". The classification of intermittent regimes is based on the multiplicator transition across the unit circle, as, for example, in the case of inverse tangent bifurcation (see, e.g., [4]). The bifurcation point is supposed to be inside the system phase space. Recently, new mechanisms were proposed for intermittency arising based on a distortion in smoothness or continuity of maps in the bifurcation point [5, 6]. In this work, a new intermittent regime is considered that has other specific features. In the dynamic system shown below, the lamirization region is not local and is at the system phase space boundary. The intermittent regimes in such a system have some unusual properties. The laminar phase length distribution has been shown to belong to stable distributions and has anomalous properties. One of those is that the Lyapunov exponent is zeroed and that a weak chaos is realized in the system. The topologic entropy in this case is positive ant it is calculated exactly in the work.

## 1. Dynamic system and trajectories

Let us consider the properties of maps $x_{n+1} = f(x_n)$ set as

$$x_{n+1} = \begin{cases} (\frac{1}{x_n^\alpha} - 1) & x_n \leq 1 \\ x_n - 1 & x_n > 1 \end{cases} \tag{1}$$

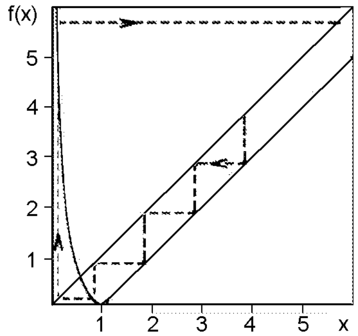


Fig. 1. Plot of $f(x_n)$, function at a fixed value $\alpha$. The evolution character of a certain initial state is also shown.

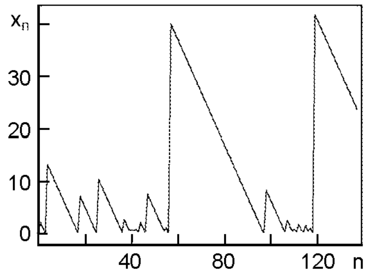


Fig. 2. A typical trajectory of the map (1).

This single-parameter property of continuous maps has a number of specific features (see Fig. 1). The $x > 1$ region is the lamirization one that forms laminar phases. The $0 < x < 1$ region is the re-injection one that provides return to the lamirization region. From the physical viewpoint, it is the chaotic distribution region.

The intermittency arise is believed usually to be associated with the reverse tangent bifurcation [2, 4] in the phase space of the dynamic system and the appearance of a narrow "corridor". The bifurcation point is assumed to be not at the phase space boundary. Note that the intermittency are classified basing on the same assumption. In the map under consideration, such a "corridor"is situated at the phase space boundary and is not local (occupies an infinite region in the phase space $x > 1$), thus, the intermittency character is essentially changed.

Let us consider the dynamic properties of map family (1) . A typical trajectory of that dynamic system is shown in Fig. 2 and it has a saw-like shape. Laminar increase sections and are sharp transitions to new laminar sections are seen. The sharp transitions between the laminar phases are due to short duration of chaotic phases. This is due to instability of the stationary point $x^* = (\sqrt{5}-1)/2$. The trajectory goes from the neighborhood of that point to the lamirization region $x > 1$ during few iterations. Let the chaotic phases be considered in more detail.

As is noted above, the map ($\alpha = 1$) has an unstable stationary point with coordinate satisfying the equation

$$x = \frac{1}{x} - 1$$

with the solution $x^* = (\sqrt{5}-1)/2$ where is the golden section. The closer is the chaotic phase onset coordinate to $x^*$, the more iterations are required to hit the $1 < x$ interval and to onset the new laminar phase. This fact is caused by the exponential departure of the trajectory away from the neighborhood of the unstable stationary point. It is possible to introduce a sequence of intervals from which the chaotic phase exits to laminar one exactly in $n$ steps. The condition for exit from the chaotic phase can be written easily as $x_n = f^n(x) > 1$, where $0 < x < 1$ and $x_{n-1} < 1$. For a n-multiple composition of the map f in the chaotic phase region $x < 1$, a simple relationship follows:

$$x_n = f^n(x) = -\frac{N_n \cdot x - N_{n-1}}{N_{n-1} \cdot x - N_{n-2}}$$

where $N_n$ are Fobinacci numbers meeting the relationship $N_n = N_{n-1} + N_{n-2}$ ($N_1 = 1, N_2 = 2$ see, e.g., [11]). Using the composition obtained, it is easy to find the intervals In from where the exit to laminar region occurs exactly in n time steps. For example, $I_1 = [0, 1/2]$, $I_2 = [2/3, 1]$, $I_3 = [1/2, 3/5]$ и $I_4 = [5/8, 2/3]$. For those interval lengths, the general relation is valid:

$$l_n = \frac{1}{N_{n+1} N_{n-1}}$$

Using the known relationship for Fibonacci numbers $N_{n+1}N_{n-1} = N_n^2 - (-1)^{n+1}$ and the Bine formula [11] $N_n = \frac{1}{\sqrt{5}}\left(\tau^n - \frac{1}{\tau^n}\right)$, it is easy to obtain the asymptotic law of the interval length decrease for $n >> 1$:

$$l_n \approx 5e^{-2n \ln \tau}$$

Taking into account that the probability of falling into that interval is in proportion to its length, it is easy to estimate the provability of a chaotic phase with length $n$. Using the invariant distribution function described below, the average chaotic phase length $\langle n_{ch} \rangle \approx 1.8$ can be estimated. In other words, the chaotic phases of unit length must be observed predominantly in the map.

Thus, all the trajectories can be assumed to consist of a set of long laminar phases of random length, chaotic splashes between those being absent essentially (see Fig. 2). The laminar phase length is defined by the system dynamics in the lamirization region. At $x > 1$, the dynamics is regular and is reduced to sequential subtraction of 1 from the instantaneous coordinate value till $x > 1$ (see Fig. 1). The number of those operations defines the laminar phase length (or better duration). Thus, if the n-th phase initial coordinate is $\tilde{x}_n$, the laminar phase length is $l_n = [\tilde{x}_n]$. Here, $[x]$ is an integral part of $x$. After an evolution within the lamirization region, the point falls into the re-injection region $0 < x < 1$. The map (1) transfers the $0 < x < 1$ segment into the laminar motion region and a new laminar phase arises with the length defined by the scattering distance. The initial data uncertainty increases at each scattering act with a factor $k = |\alpha / x^{\alpha+1}|$ exceeding 1 ($1 < k < \infty$).

Thus, the behavior of the system as a whole can be interpreted as chaotic scattering. The $0 < x < 1$ segment is the scattering region while the rest of the phase space, the return region. Since the system dynamics at the return is regular, the x>1 region can be referred to as "a corridor". However, in contrast to classical intermittency types, this corridor is substantially non-local, therefore, the properties of the regimes observed will depend considerably on the re-injection manner. No such dependence is observed for usual intermittency types. For example, in the 1st kind intermittency, the coordinate values at the "corridor"inlet are assumed to be distributed homogeneously [4]. That approximation is justified only if the whole "corridor"is local and the invariant distribution function has no singularities. In the map under consideration, the parameter varies the re-injection properties, the lamirization region remaining unchanged. It is easy to see that in this case, all the statistical characteristics of the system become changed, beginning with the asymptotics of the laminar phase distribution function and including the Lyapunov exponent. The intermittency example being considered here shows that the properties of intermittent regimes are defined often not only by local characteristics of the system near the tangent bifurcation point. In this case, the intermittent regime characteristics may lose the universal properties and take a dependence on the re-injection region type. In what follows, considered will be the main characteristics of the intermittency for the map (1) at $\alpha = 1$.

## 2. Distribution function of laminar phases

Now let us discuss the laminar phase distribution function for the map (1) trajectories at $\alpha = 1$. That function will be calculated using the reduced description method. In other words, we shall transit from the consideration of the map iteration sequence to the laminar phase lengths themselves. To that end, specific properties of the map (1) will be used(1). Let be taken into account that if the initial coordinate of the laminar phase is $\tilde{x}_n$, its end coordinate is $\{\tilde{x}_n\} = \tilde{x}_n - [\tilde{x}_n]$ and falls into the scattering region. After the re-injection, the initial coordinate of the new laminar phase will coincide with $f(\{\tilde{x}_n\})$. Thus, the system dynamics is defined completely by its behavior within the re-injection region. Therefore, it is convenient to study the iteration statistics on the $0 < x < 1$ segment and determine therefrom the laminar phase length statistics. In this case, it is easy to determine the law for the system behavior on the segment $0 < x < 1$ ($p_n = \{\tilde{x}_n\}$) along a certain trajectory or orbit of the map.

That map has the form

$$p_{n+1} = P(p_n) = \left\{ \frac{1}{p_n} - 1 \right\} \tag{2}$$

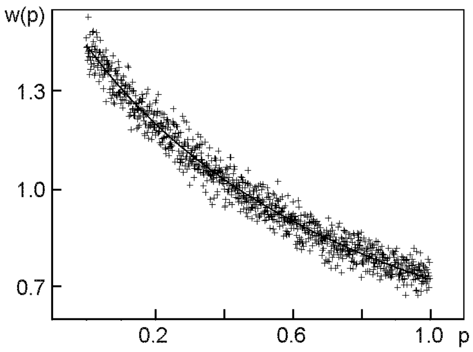


Fig. 3. Numerical simulation results (crosses) for invariant distribution function of end points of laminar phases and analytically calculated function $\frac{1}{ln2(1+x)}$ (solid).

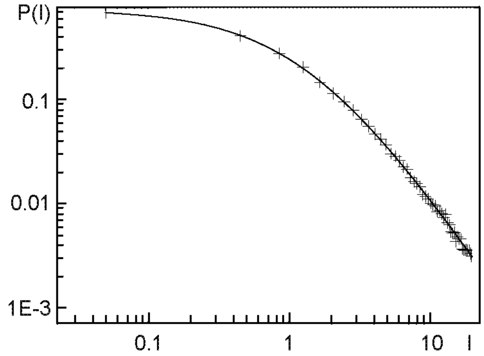


Fig. 4. The laminar phase distribution function $P(l)$ obtained by numerical experiments (crosses) and analytically found function $1/\ln 2\,(l+1)\,(l+1)$ - solid line.

The invariant distribution function for that map is localized naturally on the segment $(0,1)$. The numerical simulation results for that function are shown in Fig. 3. Let the function be calculated analytically basing on the Frobenius-Perron equation ([8]) for the invariant distribution function of the map eqrefpm. The map (2) is similar in form to the Farey map that defines the chain fractions; the invariant distribution function for that map is [7]. Let the Frobenius-Perron equation be written as (3)

$$\rho(x) = \sum_{r=1}^{k} \frac{\rho(\xi_r(x))}{|f'(\xi_r(x))|} \tag{3}$$

where the summation is carried out over all the roots of equation $x = \left\{\frac{1}{\xi} - 1\right\}$, that are defined by the map (2). Those roots are easy to find as

$$\xi_k(x) = \frac{1}{x+1+k} \qquad k = 0, 1, 2, ...$$

After substituting of those roots in Eq. (3), the Frobenius-Perron equation takes the form

$$\rho(x) = \sum_{k=0}^{\infty} \frac{\rho\,(1/(x+1+n))}{(x+1+n)^2} \tag{4}$$

It is easy to check that the function $\rho(x) = \frac{a}{1+x}$ is the exact solution of Eq. (4).

$$\frac{a}{x+1} = \sum_{k=0}^{\infty} \frac{\rho\left(\frac{1}{x+1+k}\right)}{(x+1+k)^2} = a\sum_{k=0}^{\infty} \frac{1}{(x+1+k)^2(1+\frac{1}{x+1+k})} = \frac{a}{x+1}$$

Let the constant a be determined from the normalization condition:

$$\int_0^1 \frac{a}{1+x}dx = 1 \longrightarrow a = \frac{1}{\ln 2}$$

Now the distribution function of the laminar phase lengths $P(l)$ is to be determined from the known distribution function $\rho(p)$. To that end, let us go from the variable $p$ to the laminar phase length $l = \frac{1}{p} - 1$. Then the probability of a laminar phase length $l$ for the map (1) has the form

$$dw(l) = \rho(p(l))\frac{dp}{dl}dl = \frac{1}{\ln 2\,(l+1)\,(l+2)}dl \equiv P(l)dl$$

It is to note that at such a transformation, the laminar phase length is determined formally from 0 to the laminar phase initial point. It can be determined in principle from its initial point to the end one. That determination, however, does not influence the most interesting statistics of long phases, because the length differences in those determinations are small ( $\Delta l < 1$).

Note that $P(l)$ decreases rather slowly as $l$ increases:

$$P(l) \sim \frac{1}{l^2} \quad l \to \infty$$

Such a behavior is typical of stable distributions or the Levy ones [9] with the exponent $\alpha = 1$. Fig. 4 presents the $P(l)$ numerical simulation results compared to the function derived analytically.

### 3. Chaotic properties of the map

Universal dependences of the average laminar phase length on the supercriticality parameter are revealed often in intermittent regimes (see, e.g., [4]). Such dependences are used sometimes to detect the intermittency type, too. There is no such dependence in the intermittency example under discussion free of chaotic phases. The cause is the power-law decrease of the laminar phase length distribution function. It is obvious that the average laminar phase length as calculated using that function diverges logarithmically at large lengths and goes to infinity. The attempts to calculate it numerically result in unrestricted increase of its average length as the number of iterations rises. This follows immediately from the fact that $P(l)$ belongs to stable distributions [9]. In such cases, a length that is determined by lower existing moments of the distribution function as $\bar{l} = \langle l^\alpha \rangle^{1/\alpha}$ (in the case under discussion, $0 < \alpha < 1$). can be introduced as the laminar phase average length. The average length so determined is finite and is easy to calculate:

$$\bar{l} = \langle l^\alpha \rangle^{\frac{1}{\alpha}} = \left( \int_0^\infty l^\alpha P(l) dl \right)^{\frac{1}{\alpha}} = \left( \frac{(2^\alpha - 1)}{\ln 2 \cdot \sin(\pi\alpha)} \right)^{\frac{1}{\alpha}}$$

It is easy to see the divergence of that expression at $\alpha \to 1$. At other $0 < \alpha < 1$ values, the average length is finite. It is of interest that at $\alpha \ll 1$ , that average length loses its dependence on and takes the order of 1, and $1 - \alpha \equiv \delta \ll 1$ $\bar{l} \approx 1/(\pi\delta \ln 2)$.

Now let the chaotic state degree of an intermittent regime be considered. Let us start from the Lyapunov exponent. To calculate it, let the following definition be used:

$$\Lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln |f'(x_n)| \tag{5}$$

Since the derivative $f'(x_n) = 1$ in the laminar phase, all such items do not contribute to the sum (6). In other words, only the items associated with scattering in the $0 < x < 1$ contribute to that sum. Let the number of such items be designated as $\tilde{N}$. Moreover, let the map (2) be used to determine the derivative values at the scattering. Then the relationship (6) can be written for convenience as

$$\Lambda = \lim_{N\to\infty} \frac{\langle n_{ch} \rangle \tilde{N}(N)}{N} \frac{1}{\langle n_{ch} \rangle \tilde{N}(N)} \sum_{k=1}^{\tilde{N}(N)} \ln \frac{1}{p_k^2} = \lim_{N\to\infty} \frac{\langle n_{ch} \rangle \tilde{N}(N)}{N} \left( \ln \frac{1}{p^2} \right)$$

where $\langle n_{ch} \rangle$ is the chaotic phase average length.

The physical sense of this relationship is simple. It can be noticed by introducing the Lyapunov exponent $\lambda_{sc} = \lim_{N\to\infty} \frac{1}{\tilde{N}(N)} \sum_{k=1}^{\tilde{N}(N)} \ln \frac{1}{p_k^2}$ for chaotic scattering. Then the Lyapunov exponent $\Lambda$ takes the form

$$\Lambda = \lambda_{sc} \lim_{N\to\infty} \frac{\langle n_{ch} \rangle \tilde{N}(N)}{N}$$

Thus, the Lyapunov exponent differs from that for chaotic scattering by the ratio of the scattering acts $\tilde{N}(N)$ to the number of iterations $N$ in the limit $N \to \infty$. To calculate the average $\overline{\ln(1/p^2)}$ or the Lyapunov exponent for chaotic scattering using ergodicity, let us go from the time average to the ensemble average:

$$\lambda_{sc} = \overline{\ln\left(\frac{1}{p^2}\right)} = \left\langle \ln\frac{1}{p^2}\right\rangle = \int_0^\infty \ln\left(\frac{1}{p^2}\right)\frac{dp}{\ln 2\,(p+1)} \approx 2.37313$$

This Lyapunov exponent is positive, thus, the scattering is chaotic.

The quantity $N/\tilde{N}$ can be treated as the ratio of the total length of laminar phases to the number thereof. Then the limit of that quantity coincides with the laminar phase average length $\langle l\rangle$. We obtain finally

$$\Lambda = \frac{\lambda_{sc}\langle n_{ch}\rangle}{\langle l\rangle} \tag{6}$$

It follows from the expression that the Lyapunov exponent is equal to zero due to anomalous statistics of laminar phases resulting in $\langle l\rangle = \infty$. Thus, a weak chaotic regime is realized in the system being studied that is very atypical of one-dimensional systems on the whole and, in particular, of intermittent ones. This fact is confirmed by direct numerical experiments with the map (1). In those experiments, the Lyapunov exponent drops slowly as the number of iterations increases and depends heavily on the initial conditions. These effects are explained by logarithmic decrease of the quantity being studied as the number of iterations grows.

In order to demonstrate the presence of chaos in the system at a weak irregularity of its dynamics, tools such as correlation function and topologic entropy can be used. The correlation function shows a slow dropping in the numerical simulation, thus evidencing the presence of long correlations and weak chaotization of the system. That behavior of the correlation function agrees well in a certain sense with the zeroing Lyapunov exponent. It is to note that a similar behavior of statistical characteristics is known also for another intermittency type [10].

Now let the topologic entropy of the system be calculated. The topologic entropy positiveness means a complex behavior of the system. This is meant as the presence of an infinite number of various periodic and aperiodic trajectories. Such systems show an exponential sensitivity to external noises [12].

To determine the topologic entropy, let the needing series $Q_+(t)$ and $Q_-(t)$ be used in the neighborhood of the minimum $x = 1$ [13] (see also [14]). The first needing series has the form $Q_+(t) = 1-t-t^2-t^3-t^4+\ldots$. In the initial map, $x = 1$ is a minimum, therefore, the first item of the needing sequence is positive. At further iterations, $f^k(1) \to \infty$, , that is, a 2nd kind break is observed in the point $x = 1$, therefore, all other items of the needing series are negative. Accordingly, $Q_-(t) = -Q_+(t)$. The equation for topologic entropy has the form [13]

$$\sum_{k=1}^{\infty} t^k - 1 = 0$$

It is $t_0 = 1/2$. that is the root of that equation nearest to zero. Thus, topologic entropy for all the maps of that type is

$$h_{top} = -\ln(t_0) = \ln 2 \simeq 0.693$$

The topologic entropy value for the map being studied does not coincide with Lyapunov exponent that is 0 in this case. Thus, the conclusion on coincidence of the Lyapunov exponent with the topologic entropy for one-dimensional maps that is cited often (see, e.g., [4]) is invalid for the considered intermittency regime.

## 4. Conclusion

To conclude, let us discuss the main originating elements of the intermittency regime of the considered type. Such regimes can be observed for a wide class of maps. It is to note first of all that the map laminar zone may be of a more complex character. The only important circumstance is the asymptotic closeness of the function defining the map to the diagonal on the laminar segment. The relationship (6) defines the Lyapunov exponent in all such cases as well as at variations in the chaotic scattering region characteristics. The physical sense of that dependence is associated with the fact that the Lyapunov exponent is not invariant as the time variable is changed. Moreover, some specific phase transitions of the Lyapunov exponent considered as the ordering parameter can be observed as the distribution function is re-built when the chaotic scattering region changes. Such transitions are connected with transitions in the laminar phase average length from finite to infinite values.

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## Переміжний режим без хаотичних фаз

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Запропоновано просте відображення, в якому виникає переміжний режим без хаотичних фаз. Вивчено головні характеристики такого переміжного режиму. Аналітично визначено функцію розподілу довжин ламінарних фаз, показник Ляпунова та топологічну ентропію. Проведено порівняння головних характеристик з результатами комп'ютерного моделювання.