INDUCED VACUUM POLARIZATION OF SCALAR FIELD BY IMPENETRABLE MAGNETIC TUBE

V.M. GORKAVENKO,[1] YU.A. SITENKO,[2] O.B. STEPANOV[1]

[1]Taras Shevchenko National University of Kyiv, Faculty of Physics
(64, Volodymyrs'ka Str., Kyiv 01601, Ukraine)

[2]Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03143, Ukraine)

PACS 11.27.+d, 11.10.Kk, 11.15.Tk
©2010

We investigate the influence of an external magnetic field in a tube on the vacuum of a massive charged scalar field for arbitrary space-time dimensions. The tube is considered impenetrable for the scalar field and obeys the Dirichlet boundary condition on the bounding surface. It was shown that, for a particular case of the $2+1$-dimensional space-time, the induced vacuum energy of the scalar field outside the tube can be numerically calculated without regularization procedure. The dependences of the induced vacuum energy upon the distance from the tube at its various transversal radii are obtained.

1. Introduction

Since Casimir's seminal paper [1] it is known that the existence of the external boundary conditions in quantum field theory leads to a non-zero vacuum expectation value of the energy-momentum tensor (see, e.g., [2, 3]). This may have far reaching consequences; in particular, the vacuum energy-momentum tensor serves as a source of gravitation, and the so-called self-consistent cosmological models of the Universe are proposed, where matter is absent, and its role is played by vacuum quantum effects [4].

In this respect, it seems to be of interest to look for various situations where the vacuum energy-momentum tensor is calculable and finite. Let X be the base space manifold of dimension d, and Y be a submanifold of dimension less than d. The matter field is quantized under a certain boundary condition imposed at Y. In most implications of the Casimir effect, Y is chosen to be noncompact disconnected (e.g., two parallel infinite plates, as generically in [1]) or closed compact (e.g., box or sphere), see [3].

In [5–7], the Casimir effect was considered in detail in the case where Y is a noncompact connected manifold that has dimension $d-2$, i.e. it is a $d-2$-brane in the d-dimensional space (the manifold is a point at $d=2$, a line at $d=3$, and a plane at $d=4$). This brane was filled with magnetic flux lines. In this case, as it was first noted by W. Ehrenberg and R. Siday in [8], the matter field out of a brane is affected by the electromagnetic potential of the brain magnetic field also, and it is a generalization of a Bohm–Aharonov [9] singular magnetic vortex in the 3-dimensional space. The condition for the matter field to vanish at Y was imposed, and the vacuum polarization of the field was investigated in a space manifold X. So, in [5–7] was considered the Casimir–Bohm–Aharonov effect. It should be noted that, in this case, the effect of vacuum polarization was calculated analytically.

In the 3-dimensional space, a solution of the singular magnetic vortex problem can be used in astrophysics (physics of cosmic strings), as well as in condensed matter physics (Abrikosov vortex in type II superconductors). But, in both these cases, a magnetic vortex has a non-zero transverse size that is determined by the Compton wavelength of a corresponding scalar field in the phase transition epoch [10] or by the radius of a Cooper pair [11], respectively. So, it is more interesting to investigate the Casimir–Bohm–Aharonov effect in the case of a finite-size magnetic tube. In this context, it is worth to find out the dependence of vacuum effects on the ratio of the transverse size of mentioned topological defects and the Compton wavelength of the field under consideration.

We choose Y to be noncompact connected and possessing dimension $d-1$, i.e. a $d-1$-brane in the d-dimensional space. This brane envelops the part of the d-dimensional space, in which we fill a $d-2$-dimensional brane with a magnetic field (the brane is a circle with an internal point-like magnetic field at $d=2$ and a cylinder with a magnetic line inside it at $d=3$). Throughout the present paper, we restrict ourselves to the case of scalar matter.

In the next Section, a general definition of the energy density for a quantized charged scalar field is reviewed, and a starting expression for its regularized vacuum expectation value against the background of a brane is given in the general case of arbitrary space-time dimensions. Unfortunately, we failed to calculate the obtained expressions analytically. So we restrict ourselves in Section 3 by the simplest case of space-time dimension $2+1$ and obtain the vacuum polarization numerically. Finally, the results are summarized in Section 4.

2. Energy Density

The operator of a quantized charged scalar field is represented in the form

$$\Psi(x^0,\mathbf{x}) = \sum\!\!\!\!\!\!\int_\lambda \frac{1}{\sqrt{2E_\lambda}}\times$$

$$\times\left[e^{-iE_\lambda x^0}\psi_\lambda(\mathbf{x})\,a_\lambda + e^{iE_\lambda x^0}\psi_{-\lambda}(\mathbf{x})\,b_\lambda^\dagger\right]. \qquad (1)$$

Here, $a_\lambda^\dagger$ and a_λ ($b_\lambda^\dagger$ and b_λ) are the scalar particle (antiparticle) creation and annihilation operators satisfying commutation relation; λ is the set of parameters (quantum numbers) specifying the state; $E_\lambda = E_{-\lambda} > 0$ is the energy of the state; the symbol $\sum\!\!\!\!\!\!\int_\lambda$ denotes the summation over discrete and the integration (with a certain measure) over continuous values of λ; the wave functions $\psi_\lambda(\mathbf{x})$ are the solutions to the stationary equation of motion,

$$\left\{-\boldsymbol{\nabla}^2 + m^2\right\}\psi_\lambda(\mathbf{x}) = E_\lambda^2\psi(\mathbf{x}), \qquad (2)$$

$\boldsymbol{\nabla}$ is the covariant differential operator in an external (background) field.

The standard expression for the energy density has the form

$$\varepsilon = \sum\!\!\!\!\!\!\int_\lambda E_\lambda\psi_\lambda^*(\mathbf{x})\,\psi_\lambda(\mathbf{x}). \qquad (3)$$

This relation can be regarded as purely formal and, strictly speaking, meaningless: it is ill-defined, by suffering from ultraviolet divergencies. The well-defined quantity is obtained with help of a regularization procedure, especially the zeta function regularization [12–14], i.e. by inserting the inverse energy in a sufficiently high power

$$\varepsilon_{\rm reg}(s) = \sum\!\!\!\!\!\!\int_\lambda E_\lambda^{-2s}\psi_\lambda^*(\mathbf{x})\,\psi_\lambda(\mathbf{x}). \qquad (4)$$

Sums (integrals) are convergent in the case where $\mathrm{Re}\, s > d/2$. Thus, the summation (integration) is performed in this case, and then the result will be analytically continued to the case of $s=-1/2$.

We consider the static magnetic field whose covariant derivative is

$$\boldsymbol{\nabla} = \boldsymbol{\partial} - i\mathbf{V}, \qquad (5)$$

and the magnetic field strength takes the form

$$B^{j_1\cdots j_{d-2}}(\mathbf{x}) = -\varepsilon^{j_1\cdots j_d}\partial_{j_{d-1}}V_{j_d}(\mathbf{x}), \qquad (6)$$

where $\mathbf{V}(\mathbf{x})$ is the bundle connection (vector potential of the magnetic field), and $\varepsilon^{j_1\cdots j_d}$ is the totally antisymmetric tensor, $\varepsilon^{12\cdots d}=1$.

As was already mentioned in Introduction, we consider the bundle curvature (magnetic field strength) to be nonvanishing in the $d-2$-brane (i.e. point in the $d=2$ case, line in the $d=3$ case, plane in the $d=4$ case, and $d-2$-hypersurface in the $d>4$ case). Denoting the location of the $d-2$-brane by $x^1=x^2=0$, we get

$$B^{3\cdots d}(\mathbf{x}) = 2\pi\Phi\delta(x^1)\delta(x^2), \qquad (7)$$

where Φ is the total flux (in units of 2π) of the bundle curvature; then the bundle connection can be chosen in the form

$$V^1(\mathbf{x}) = -\Phi\frac{x^1}{(x^1)^2+(x^2)^2}, \quad V^2(\mathbf{x}) = \Phi\frac{x^2}{(x^1)^2+(x^2)^2},$$

$$V^j(\mathbf{x}) = 0, \quad j=\overline{3,d}. \qquad (8)$$

We require the vanishing of the scalar field on the $d-1$-dimensional brane Y and within a part of the d-dimensional region that is surrounded by Y. We define the location of a d-dimensional region forbidden for the scalar field ($\Psi(\mathbf{x})=0$ in this region) as

$$\sqrt{(x^1)^2+(x^2)^2} \le r_0, \quad -\infty \le x^k \le \infty, \quad k>2. \qquad (9)$$

Then the location of a d-dimensional region with the scalar field is

$$r_0 < \sqrt{(x^1)^2+(x^2)^2} < R_0, \quad -\infty \le x^k \le \infty, \quad k>2, \qquad (10)$$

where $R_0\to\infty$. In Eqs. (9) and (10), the parameters r_0 and R_0 define the transverse size of this regions.

As was mentioned above, the scalar field function obeys the condition $\Psi(r=r_0)=0$. A condition on the other border at $r=R_0$ is not substantial. For definiteness, it can be $\Psi(r=R_0)=0$. In this case, the complete set of solutions to Eq.(2) against background (7)-(8) within region (10) is given by the functions

$$\psi_{kn\mathbf{p}}(\mathbf{x}) = (2\pi)^{\frac{1-d}{2}} \times$$

$$\times \frac{Y_{|n-\Phi|}(kr_0)J_{|n-\Phi|}(kr) - J_{|n-\Phi|}(kr_0)Y_{|n-\Phi|}(kr)}{\left[Y^2_{|n-\Phi|}(kr_0) + J^2_{|n-\Phi|}(kr_0)\right]^{1/2}} \times$$

$$\times e^{in\varphi} e^{i\mathbf{p}\mathbf{x}_{d-2}}, \qquad (11)$$

where $0 < k < \infty$, $n \in \mathbb{Z}$, $-\infty < p^j < \infty$, $j = \overline{3,d}$; $r = \sqrt{(x^1)^2 + (x^2)^2}$, $\varphi = \arctan(x^2/x^1)$, $\mathbf{x}_{d-2} = (0, 0, x^3, \ldots x^d)$; $J_\mu(u)$ and $Y_\mu(u)$ are the Bessel functions of order μ of the first and second kinds, respectively, and $\mathbb{Z}$ is the set of integers. Since solutions (11) in the limit $R_0 \to \infty$ correspond to the continuous spectrum ($E_{kn\mathbf{p}} = \sqrt{\mathbf{p}^2 + k^2 + m^2} > m$), they obey the orthonormality condition

$$\int d^d x \; \psi^*_{kn\mathbf{p}}(\mathbf{x})\psi_{k'n'\mathbf{p}'}(\mathbf{x}) = \frac{1}{k}\delta(k - k')\delta_{nn'}\delta(\mathbf{p} - \mathbf{p}'). \qquad (12)$$

To compute the zero component of the vacuum expectation value of energy density, we have to substitute (11) in Eqs. (4). We obtain

$$\varepsilon_{\text{reg}}(s) = (2\pi)^{1-d} \int d^{d-2}p \int_0^\infty dk\, k \left(\mathbf{p}^2 + k^2 + m^2\right)^{-s} \times$$

$$\times S(kr, kr_0, \Phi), \qquad (13)$$

where

$$S(kr, kr_0, \Phi) =$$

$$= \sum_{n\in\mathbb{Z}} \frac{\left[Y_{|n-\Phi|}(kr_0)J_{|n-\Phi|}(kr) - J_{|n-\Phi|}(kr_0)Y_{|n-\Phi|}(kr)\right]^2}{Y^2_{|n-\Phi|}(kr_0) + J^2_{|n-\Phi|}(kr_0)}. \qquad (14)$$

As a result of the infinite range of summation, the last expression depends only from the fractional part of the flux

$$F = \Phi - [\![\Phi]\!], \quad (0 \le F < 1), \qquad (15)$$

where $[[u]]$ is the integer part of a quantity u (i.e. the integer which is less than or equal to u).

We can also rewrite (14) in the form

$$S(kr, kr_0, F) = S(kr, F)_{\text{z.s.}} + S(kr, kr_0, F)_{\text{corr.}}, \qquad (16)$$

where $S(kr, \Phi)_{\text{z.s.}}$ corresponds to the appropriate series in the case of vacuum polarization by a magnetic tube of zero transverse size [5–7]

$$S(kr, F)_{\text{z.s.}} = \sum_{n=0}^\infty \left[J^2_{n+F}(kr) + J^2_{n+1-F}(kr)\right] =$$

$$= \int_0^{kr} d\tau \left[J_F(\tau)J_{-1+F}(\tau) + J_{-F}(\tau)J_{1-F}(\tau)\right], \qquad (17)$$

and a correction term

$$S(kr, kr_0, F)_{\text{corr.}} = 2\sum_{n=0}^\infty \Bigg[J_{n+F}(kr_0)Y_{n+F}(kr) \times$$

$$\times \frac{J_{n+F}(kr_0)Y_{n+F}(kr) - J_{n+F}(kr)Y_{n+F}(kr_0)}{J^2_{n+F}(kr_0) + Y^2_{n+F}(kr_0)} +$$

$$+ J_{n+1-F}(kr_0)Y_{n+1-F}(kr) \times$$

$$\times \frac{J_{n+1-F}(kr_0)Y_{n+1-F}(kr) - J_{n+1-F}(kr)Y_{n+1-F}(kr_0)}{J^2_{n+1-F}(kr_0) + Y^2_{n+1-F}(kr_0)} \Bigg] -$$

$$- \sum_{n=0}^\infty \Bigg[J^2_{n+F}(kr_0) \frac{J^2_{n+F}(kr) + Y^2_{n+F}(kr)}{J^2_{n+F}(kr_0) + Y^2_{n+F}(kr_0)} +$$

$$+ J^2_{n+1-F}(kr_0) \frac{J^2_{n+1-F}(kr) + Y^2_{n+1-F}(kr)}{J^2_{n+1-F}(kr_0) + Y^2_{n+1-F}(kr_0)} \Bigg]. \qquad (18)$$

In the absence of a magnetic flux in the tube (i.e. at $F = 0$), Eq. 13 takes the form

$$\varepsilon_{\text{reg}}(s)|_{F=0} = (2\pi)^{1-d} \int d^{d-2}p \int_0^\infty dk\, k \left(\mathbf{p}^2 + k^2 + m^2\right)^{-s} \times$$

$$\times S(kr, kr_0, F = 0). \qquad (19)$$

The function $S(kr, kr_0, F = 0)$ can be also split into two parts

$$S(kr, kr_0, F = 0) = \tilde{S}(kr)_{\text{z.s.}} + \tilde{S}(kr, kr_0)_{\text{corr.}}, \qquad (20)$$

where $\tilde{S}(kr)_{\text{z.s.}}$ corresponds to the appropriate series in the case of vacuum polarization by a magnetic tube of zero transverse size [5–7]

$$\tilde{S}(kr)_{\text{z.s.}} = J_0^2(kr) + 2\sum_{n=0}^{\infty} J_n^2(kr) = 1, \tag{21}$$

and a correction term

$$\tilde{S}(kr, kr_0)_{\text{corr.}} =$$

$$= 2\Bigg[J_0(kr_0)Y_0(kr)\frac{J_0(kr_0)Y_0(kr) - J_0(kr)Y_0(kr_0)}{J_0^2(kr_0) + Y_0^2(kr_0)} +$$

$$+2\sum_{n=1}^{\infty} J_n(kr_0)Y_n(kr)\frac{J_n(kr_0)Y_n(kr) - J_n(kr)Y_n(kr_0)}{J_n^2(kr_0) + Y_n^2(kr_0)}\Bigg] -$$

$$-\Bigg[J_0^2(kr_0)\frac{J_0^2(kr) + Y_0^2(kr)}{J_0^2(kr_0) + Y_0^2(kr_0)} +$$

$$+2\sum_{n=1}^{\infty} J_n^2(kr_0)\frac{J_n^2(kr) + Y_n^2(kr)}{J_n^2(kr_0) + Y_n^2(kr_0)}\Bigg]. \tag{22}$$

Expression (19) is still a bad quantity even after the analytical continuation to the case of $s = -1/2$. We need a renormalization procedure. So, we define the renormalized vacuum energy as a variation of the vacuum energy in the case of the finite transverse size tube with a magnetic flux ($F \neq 0$) and the same tube without magnetic flux ($F = 0$):

$$\varepsilon_{\text{ren}} = \lim_{s \to -\frac{1}{2}} [\varepsilon_{\text{reg}}(s) - \varepsilon_{\text{reg}}(s)|_{F=0}] =$$

$$= (2\pi)^{1-d} \int d^{d-2}p \int_0^{\infty} dk\, k \left(\mathbf{p}^2 + k^2 + m^2\right)^{-s} \times$$

$$\times G(kr, kr_0, F), \tag{23}$$

where the function $G(kr, kr_0, F)$ is independent of the space dimension d,

$$G(kr, kr_0, F) = S(kr, kr_0, F) - S(kr, kr_0, F = 0), \tag{24}$$

and $S(kr, kr_0, F)$ is defined by (16) and (20). It should be noted that ε_{ren} is a periodic function of the flux Φ, since it depends only on F (being symmetric under $F \leftrightarrow 1 - F$).

Here, we will consider the simplest situation for a magnetic tube of finite transverse size, i.e. we consider the situation for half-integer values of the magnetic flux $F = 1/2$ (in this case, we expect the maximal effect of a vacuum polarization by analogy with [5–7]) in the $2+1$-dimensional space-time, in order to avoid the additional integration over p_{d-2} components of the momentum. In this case, relation (23) becomes

$$\varepsilon_{\text{ren}} = \lim_{s \to -\frac{1}{2}} \frac{1}{2\pi} \int_0^{\infty} dk\, k \left(k^2 + m^2\right)^{-s} G(kr, kr_0), \tag{25}$$

where we denote, for simplicity of notation, $G(kr, kr_0) = G(kr, kr_0, F = 1/2)$.

We met some difficulties in the analytical evaluation of $G(kr, kr_0)$ and, unfortunately, cannot obtain the induced energy density (25) in an analytical form. But we found that this problem can be solved numerically. To do it, we have to introduce the dimensionless variables

$$kr = z, \qquad \lambda = r_0/r, \tag{26}$$

where $z \in (0, \infty)$, $\lambda \in [0, 1]$. The case of $\lambda = 1$ corresponds to $r = r_0$, i.e. the point on the tube boundary, and the case of $\lambda = 0$ corresponds to the point at infinity $r \to \infty$ or to the case of a singular tube ($r_0 = 0$). In this variables, Eq. (25) becomes

$$r^3 \varepsilon_{\text{ren}} = \lim_{s \to -\frac{1}{2}} \frac{1}{2\pi} \int_0^{\infty} dz\, z \left(z^2 + \left(\frac{mr_0}{\lambda}\right)^2\right)^{-s} G(z, \lambda z). \tag{27}$$

3. Numerical Evaluation of Energy Density

Before performing a numerical analysis, let us point out some analytical properties of the integrand: straight on the brane ($\lambda = 1$), the function is zero

$$G(z, z) = 0; \tag{28}$$

at small values of z,

$$G(z, \lambda z)|_{z \to 0} = -[\ln(\lambda)/\ln(\lambda z)]^2; \tag{29}$$

at small values of λ (i.e. at a large distance from the brane or in the case of a singular tube), the function corresponds to the case of a singular tube:

$$G(z, \lambda z)|_{\lambda \to 0} = S(z, F = 1/2)_{\text{z.s.}} - \tilde{S}(z)_{\text{z.s.}}. \tag{30}$$

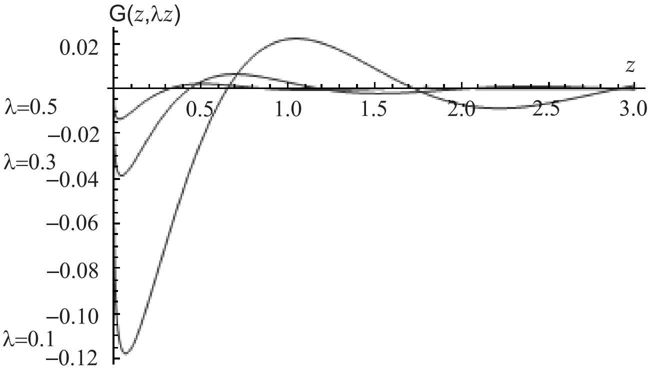

Fig. 1. Behavior of $G(z, \lambda z)$ at various values of λ.

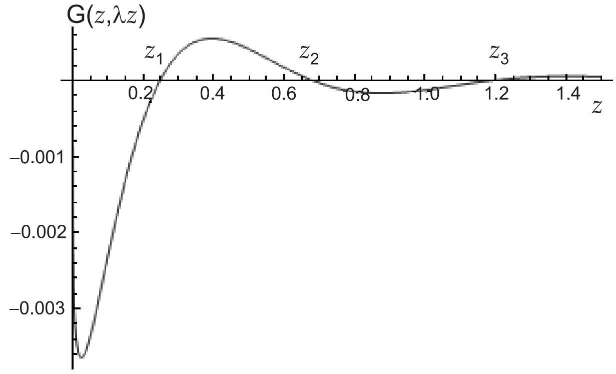

Fig. 2. Location of $G(z, \lambda z)$ roots at $\lambda = 0.7$

The numerical analysis indicates that, in the calculation of the function $G(z, \lambda z)$, one can use the series in (18) and (22) with finite limits. For the calculation of $G(z, \lambda z)$ at a point $z = z'$, it is enough to cut off the summation limits by $n = [[z' + 30]]$. In this case, the relative error is

$$\left| \frac{G(z, \lambda z)|_{n\in(0,[[z+30]])} - G(z, \lambda z)}{G(z, \lambda z)} \right| < \varepsilon(\lambda),$$

$$\varepsilon(\lambda) < 10^{-17}, \quad \lambda \in [1/10, 9/10]. \qquad (31)$$

One can make sure that the envelope of $G(z, \lambda z)$ is an exponentially decreasing function at large z, see Fig.1, except the case where $\lambda = 0$ (i.e. in the case of a singular magnetic tube). So, for the finite-size magnetic tube (27), we can immediately take $s = -1/2$ and evaluate the values of dimensionless quantity $r^3 t^{00}_{\rm ren}$ (27) for various (not very small) λ. To do it, we have to be able to perform integration in (27) with high precision. We make it in the following way.

As one can see from Fig. 2, the function $G(z, \lambda z)$ is negative from $z = 0$ to the first function root at $z = z_1$ ($z_1 \neq 0$). So, the appropriate integral in (27) is negative also. Because of the decreasing character of the envelope function, the integral from z_1 to z_3 will be positive. It is useful to define the period of the function $G(z, \lambda z)$ as an interval between two nearest roots with positive derivative. Then the full integral in (27) will be the sum of the negative integral from $z = 0$ to $z = z_1$ and the multitude of positive values of integrals over periods[1].

[1] The above description is correct in the case where $mr_0 \geq 1$. In the case of a small transverse size of the tube ($mr_0 < 1$), the integrals over some finite number of first periods may be negative. But, after it, they become and remain positive.

For small z ($z \lesssim 20$), we make a direct integration of the function $G(z, \lambda z)$ over periods using 25 digits of precision in internal computations.

For large z, we make integration for each period separately. To do it, we create a table of values of the function $G(z, \lambda z)$ for a separated period and replaced this function by a more simple function in the form

$$G_{\rm int}(z, \lambda z) = a \frac{e^{-bz}}{z^c} \frac{A_q(z^2)}{B_q(z^2)} \sin(kz + j \ln z + \phi_0), \qquad (32)$$

where the sine function ensures that the roots of $G(z, \lambda z)$ coincide with roots of $G_{\rm int}(z, \lambda z)$; $A_q(x)$ and $B_q(x)$ are q-degree polynomials, q can be $3, 4$ or 5; all unknown parameters can be found by interpolation. We allow the relative error of interpolation to be of

$$\left| \frac{G_{\rm int}(z, \lambda z) - G(z, \lambda z)}{G(z, \lambda z)} \right| < 10^{-8} \qquad (33)$$

for each period. The function $G_{\rm int}(z, \lambda z)$ can be immediately integrated with the required accuracy. In this way, we made integration up to $z \simeq 100/\lambda$ with an absolute accuracy up to 10^{-17}.

With the help of the above procedure, we obtain a table of contributions from the integration over each period, extrapolate this table to infinity, and then find the full integral in (27) as the sum of the negative integral over the first negative period(s), a multitude of positive integrals over periods up to $z \simeq 100/\lambda$, and the interpolation term. The absolute accuracy of the obtained result is 10^{-13}. It should be noted that nearly 99 % of the integral value in (27) is obtained by the direct calculation, and only nearly one percent is a contribution from the interpolation.

In contrast to the case of a singular tube [5–7], where the dimensionless energy density $r^3 \varepsilon_{\rm ren}$ under a fixed

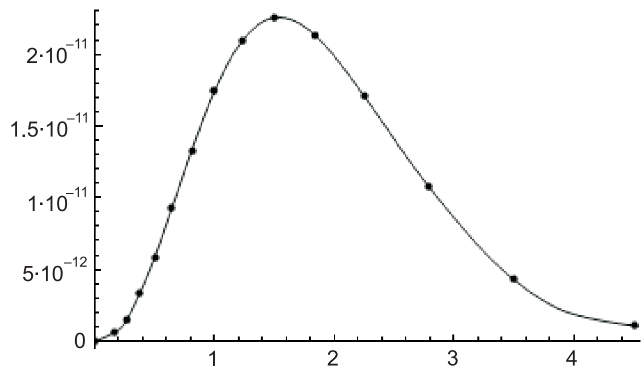

Fig. 3. The case of $mr_0 = 3/2$

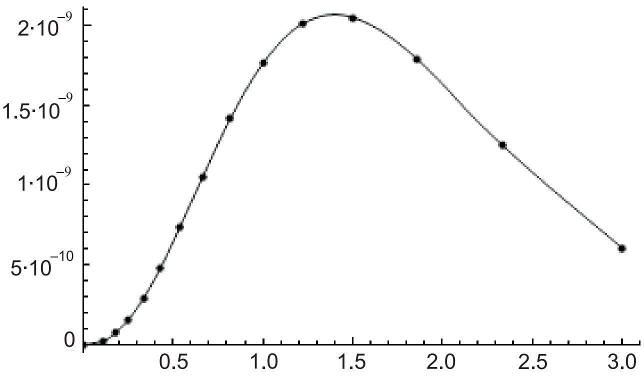

Fig. 4. The case of $mr_0 = 1$

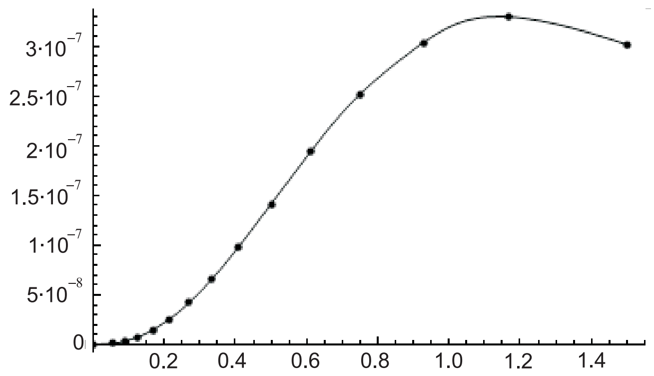

Fig. 5. The case of $mr_0 = 1/2$

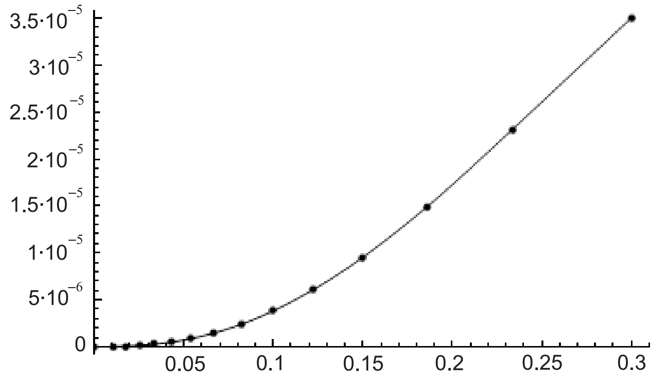

Fig. 6. The case of $mr_0 = 10^{-1}$

magnetic flux depends only on the dimensionless distance from the tube in the transverse direction (mr), the dimensionless energy density (27) depends in our case on two dimensionless parameters, namely the parameter that defines the field mass or the tube radius (mr_0) and the distance from the tube ($\lambda = r_0/r$). For comparison with the singular case, we represent the result for the energy density as a function of the distance from the tube $mr - mr_0 = mr_0(1/\lambda - 1)$ for various values of mr_0.

In this paper, we made calculations for particular cases where the tube transverse size is comparable with the Compton wavelength of a scalar field and for distances from the surface of the tube in the transverse direction ($\bar{r}=r-r_0$) up to[2] $\bar{r} = 3r_0$. The values of energy density obtained by using the above-described procedure are illustrated in Figs. 3–7. Here, the variable $m\bar{r}$ is along the x-axis, and the dimensionless quantity $r^3\varepsilon_{\rm ren}$ is presented by a solid line. The dots on the solid line correspond to points that were calculated.

[2] A further increase of the distance from a brane results in a significant increment of the computation time, because the envelope of the function $G(z, \lambda z)$ at these distances is not a so good decreasing function, as that at big λ (small distances from a brane).

4. Summary

We have shown that the vacuum of a quantized charged scalar field is polarized against the background of a $d-1$-brane with a static magnetic field inside in the flat $d+1$-dimensional space-time. We have considered a situation where the brane is impenetrable for the scalar field and obeys the Dirichlet boundary condition on the bounding surface. The vector potential of the brane induces a finite energy-momentum tensor in the vacuum; therefore, this effect may be denoted as the Casimir–Bohm–Aharonov effect. We have shown that the induced vacuum energy (23) depends periodically on the brane flux and possesses a large-distance asymptotics like that in the case of a singular magnetic tube [5–7].

It is shown that the vacuum polarization (25) in a space-time of arbitrary dimension is determined by the above-introduced function $G(z, \lambda z)$. Unfortunately, we could not perform the required integration of this function in analytical form. We restrict ourselves by the simplest case of a space-time of $2+1$ dimension (a circle with a point-like magnetic field inside) and find out that the energy density can be numerically calculated without regularization procedure. We have directly obtained the behavior of the energy density at small transverse distances from the brane for the particular case of a half-integer magnetic flux and when the transverse size of the tube is comparable with the Compton wavelength of the scalar field, see Figs. 3–7.

Qualitatively, as one can see from Figs. 3–5, the energy density ($r^3\varepsilon_{\rm ren}$) is zero at the brane (because of the Dirichlet boundary condition), increases with increase in the distance, reaches a maximum at $\bar{r}\sim m^{-1}$, decreases, and reaches zero asymptotically from above. One can see that, as the parameter mr_0 decreases, the maximum of $r^3\varepsilon_{\rm ren}$ essentially increases.

For $mr_0 \ll 1$ (see Figs. 6 and 7), we expect the similar behavior, but it needs the additional numerical confirmation. Notably we expect that, while moving away from the tube, the energy density ($r^3\varepsilon_{\rm ren}$) rises up to its maximum value at large distances from the tube[3] (see Fig. 8) and then slowly decreases to zero. At the decrease of mr_0, the maximum of $r^3\varepsilon_{\rm ren}$ will increase and move away from the tube. If the field is massless, then we expect that the energy density ($r^3\varepsilon_{\rm ren}$) rises up to the greatest of all possible constant values, which corresponds to the case of a massless field in the case of a singular topological defect [5–7].

Let us suppose the physical situation where the topological defect in the $2+1$-dimensional space-time was created under a phase transition of some scalar field with mass m_h. Then the tube radius is of the order of the Compton wavelength of the scalar field that yields a string at the corresponding phase transition. It is connected with the dimensionless transverse size of the tube by the obvious relation $mr_0 = m/m_h$. Such a statement of the problem allows us to study the dependence of the vacuum polarization on the ratio of the Compton wavelengths of the scalar field that yields a string and the scalar field under consideration. So, in the case where $m_h \lesssim m$ ($mr_0 \gtrsim 1$), the vacuum effects can be neglected. But, in the case where $m_h \gg m$ ($mr_0 \ll 1$), the vacuum effects are essential and are similar in magnitude to the case of a singular topological defect. We hope that these results can be applied also for a $3+1$-dimensional space-time, where the radius of a tube (cosmic string) is defined by an energy scale or in the grand unification time or in the electroweak phase transition time. Then the appropriate phase transition will affect only the vacuum of the field with a mass smaller than the energy scale of a phase transition.

[3] We expect it will be at a distance of the order of the Compton wavelength of the field. So, if $mr_0 = \alpha$, $\alpha \ll 1$, then the maximum of $r^3\varepsilon_{\rm ren}$ will be nearly at $r/r_0 = 1/(mr_0) = \alpha^{-1} \gg 1$.

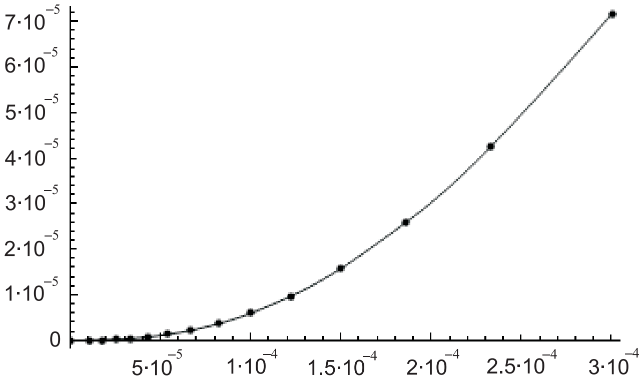

Fig. 7. The case of $mr_0 = 10^{-4}$

Comparing our results with those in the case of a singular magnetic tube [5–7], one can see the next general distinguishing characteristics. First, the induced vacuum energy in the case of an impenetrable magnetic tube is zero on the bounding surface in contrast to the case of a singular magnetic tube. Second, one can see the striking dependence of magnitudes of induced vacuum energies on the tube radius (mr_0). Third, the vacuum energy density integrated over transverse coordinates is infinite in the case of a singular magnetic tube, but it is finite for an impenetrable finite-size magnetic tube. The origin of this difference is in different topologies of the bounding surface and in different conditions on it: in the case of an impenetrable magnetic tube, the scalar field obeys the Dirichlet boundary condition; in case of a singular magnetic tube – the regularity condition[4].

It should be noted that the energy densities for the quantized scalar and spinor matters in magnetic backgrounds inside the tube of a finite transverse size in low-dimensional spaces ($d = 2, 3$) were considered in [15–19]. Since the authors of these works are concerned with the case where the region of a nonvanishing background field is overlapped with that of the nonvanishing quantized matter, their results considerably differ from ours:

[4] In the case of a singular tube with a not integer magnetic flux ($F \neq 0$), the regularity condition coincides with the Dirichlet boundary condition, see (17). But, without a magnetic flux ($F = 0$), it is not true, see (21). So, after the renormalization, the analog of the function $G(kr, kr_0, F)$ (24) for a singular tube does not satisfy the Dirichlet boundary condition.

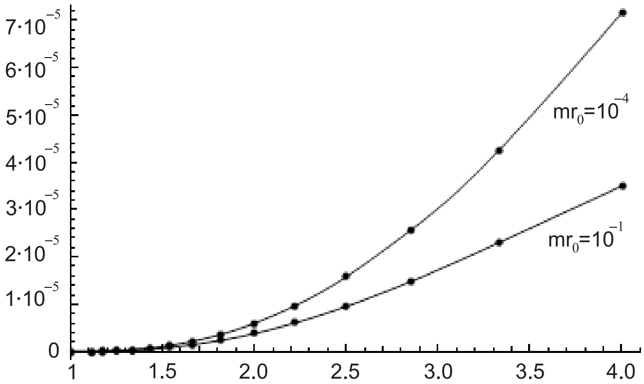

Fig. 8. Dimensionless energy density $r^3\varepsilon$ as a function of $r/r_0 = 1/\lambda$

in particular, the dependence on the flux of the background magnetic field is not periodic.

From the general features of the Bohm–Aharonov effect, it is known that, in our case (i.e. where the region with the scalar field do not overlap with the region with the magnetic field), the effects outside the brane is determined only by the fractional part of the brane flux. The interesting question for a further investigation is the dependence of the induced vacuum energy on the distribution of a magnetic field inside the tube at a fixed flux. Another question is the investigation of vacuum effects under various boundary conditions on the brane.

Yu.A. Sitenko acknowledges the support from the State Foundation for Basic Research of Ukraine under project F28.2/083 (application of the string theory and the field theory methods to the study of nonlinear phenomena in low-dimensional systems).

1. H.B.G. Casimir, Physica **19**, 846 (1953).
2. N.D. Birrel and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, Cambridge, 1982).
3. V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect and Its Applications* (Clarendon Press, Oxford, 1997).
4. A.A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
5. Yu.A. Sitenko and A.Yu. Babansky, Mod. Phys. Lett. A **13(5)**, 379 (1998).
6. Yu.A. Sitenko and V.M. Gorkavenko, Ukr. Fiz. Zh. **48**, 1286 (2003).
7. Yu.A. Sitenko and V.M. Gorkavenko, Phys. Rev. D **67**, 085015 (2003).
8. W. Ehrenberg and R. Siday, Proc. Phys. Soc. B **62**, 8 (1949).
9. Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
10. A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, 1994).
11. A.A. Abrikosov, Zh. Eksp. Teor. Fiz. **32(6)**, 1442 (1957); D.A. Kirzhnits, Pis'ma Zh. Eksp. Teor. Fiz. **15(12)**, 745 (1972).
12. A. Salam and J. Strathdee, Nucl. Phys. B **90**, 203 (1975).
13. J.S. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976).
14. S.W. Hawking, Commun. Math. Phys. **55**, 133 (1977).
15. D. Cangemi, G. Dunne, and E. D'Hoker, Phys. Rev. D **52**, 3163 (1995).
16. M.P. Fry, Phys. Rev. D **54**, 6444 (1996).
17. G. Dunne and T.M. Hall, Phys. Lett. B **419**, 322 (1998).
18. M. Bordag and K. Kristen, Phys. Rev. D **60**, 105019 (1999).
19. M. Scandurra, Phys. Rev. D **62**, 085024 (2000).

Received 15.11.09

ПОЛЯРИЗАЦІЯ ВАКУУМУ СКАЛЯРНОГО ПОЛЯ НЕПРОНИКЛИВОЮ ТРУБКОЮ З МАГНІТНИМ ПОЛЕМ

В.М. Горкавенко, Ю.О. Ситенко, О.Б. Степанов

Р е з ю м е

У роботі досліджено узагальнену на випадок простору-часу довільної розмірності задачу про вплив на вакуум зарядженого масивного скалярного поля зовнішнього магнітного поля, розташованого в трубці скінченого радіуса. Трубка є непроникливою для бозонного поля та має на поверхні граничні умови типу Діріхле. Показано, що для часткового випадку простору-часу розмірністю 2+1 індукована густина енергії вакууму ззовні трубки може бути знайдена чисельними методами без застосування процедури регуляризації. Отримано залежності індукованої густини енергії вакууму від відстані до трубки при різних значеннях її поперечного радіуса.