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HYPERBOLIC BOUNDARY VALUE PROBLEM FOR UNLIMITED PIECEWISE-HOMOGENEOUS HOLLOW CYLINDER

By means of the method of integral and hybrid integral transforms, in combination with the method of main solutions (influence functions and Green functions) the integral image of exact analytical solution of hyperbolic boundary value problem of mathematical physics for unlimited piecewise-homogeneous hollow cylinder is obtained for the first time.

Keywords: *hyperbolic equation, initial and boundary conditions, conjugate conditions, integral transforms, the main solutions.*

Introduction. The theory of hyperbolic boundary value problems for partial differential equations is an important section of the modern theory of differential equations which is intensively developing in the present time. The popularity of the problem is the consequence of the significance of its results in the development of many mathematical problems, as well as of its numerous applications in mathematical modeling of different processes and phenomenon of mechanics, physics, engineering, new technologies.

Significant results from the theory of Cauchy and boundary value problems for hyperbolic equations were obtained in the works of J. Hadamard [1], L. Gording [2], Yu. Mitropolsky, G. Khoma, M. Hromyak [3], A. Samoilenko, B. Tkach [4], M. Smirnov [5], V. Chernyatyn [6] and others.

It is well known that the complexity of a boundary-value problem significantly depends on the coefficients of equations (different types of degeneracy and features) and the geometry of domain (smoothness of the boundary, the presence of corner points, etc.) in which the problem is considered. The dependence of the properties of solutions of boundary value problems for linear, quasi-linear, and certain classes of nonlinear equations (hyperbolic, parabolic, elliptic) in homogeneous domains on the abovementioned properties of the coefficients of equations and geometry of domain are studied in detail, and functional spaces of correctness of problems in the sense of Hadamard are constructed.

However, many important applied problems of thermophysics, thermodynamics, theory of elasticity, theory of electrical circuits, theory of vibrations lead to boundary value problems for partial differential equations not only in homogeneous domains (when the coefficients of the equations are continuous), but also in inhomogeneous and piecewise homogeneous domains if the coefficients of the equations are piecewise continuous or piecewise constant [7, 8].

The method of hybrid integral transforms generated by hybrid differential operators when in each component of connectivity of piecewise homogeneous domain are treated different differential operators or differential operators look the same, but with different sets of coefficients is an effective method of constructing exact solutions for a fairly broad class of linear boundary value problems in piecewise homogeneous domains [9–12].

By means of the method of hybrid integral transforms the exact solution of hyperbolic boundary value problem of mathematical physics for unlimited piecewise homogeneous hollow cylinder is obtained in this article.

Formulation of the problem. Let's consider the problem of structure of 2π-periodic for angular variable φ solution of partial differential equations of hyperbolic type of 2nd order [13]

$$\frac{\partial^2 u_j}{\partial t^2} - \left[a_{rj}^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) + \frac{a_{\varphi j}^2}{r^2}\frac{\partial^2}{\partial \varphi^2} + a_{zj}^2 \frac{\partial^2}{\partial z^2} \right] u_j + \\ + \chi_j^2 u_j = f_j(t, r, \varphi, z); \; r \in I_j; \; j = \overline{1, n+1} \tag{1}$$

which is bounded in the set

$$D = \{(t, r, \varphi, z) : t > 0; r \in I_n^+ = \bigcup_{j=1}^{n+1} I_j \equiv \bigcup_{j=1}^{n+1} (R_{j-1}; R_j),\, R_0 > 0, R_{n+1} = R < +\infty; \\ \varphi \in [0; 2\pi); z \in (-\infty; +\infty)\}$$

with initial conditions

$$u_j \big|_{t=0} = g_j^1(r, \varphi, z); \quad \frac{\partial u_j}{\partial t}\Big|_{t=0} = g_j^2(r, \varphi, z); \quad r \in I_j; \quad j = \overline{1, n+1}, \tag{2}$$

boundary conditions

$$\left(\alpha_{11}^0 \frac{\partial}{\partial r} + \beta_{11}^0 \right) u_1 \Big|_{r=R_0} = g_0(t, \varphi, z); \quad \left(\alpha_{22}^{n+1} \frac{\partial}{\partial r} + \beta_{22}^{n+1} \right) u_{n+1} \Big|_{r=R} = g(t, \varphi, z); \tag{3}$$

$$\frac{\partial^s u_j}{\partial z^s}\Big|_{z=-\infty} = 0; \quad \frac{\partial^s u_j}{\partial z^s}\Big|_{z=+\infty} = 0; \quad s = 0, 1; \tag{4}$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^k \frac{\partial}{\partial r} + \beta_{j1}^k \right) u_k - \left(\alpha_{j2}^k \frac{\partial}{\partial r} + \beta_{j2}^k \right) u_{k+1} \right] \Big|_{r=R_k} = 0; \\ j = 1, 2; \; k = \overline{1, n}. \tag{5}$$

Here a_{rj}, $a_{\varphi j,}$ a_{zj}, χ_j, α_{js}^k, β_{js}^k — some not negative constants;

$$\alpha_{11}^0 \le 0;\ \beta_{11}^0 \ge 0;\ \left|\alpha_{11}^0\right| + \beta_{11}^0 \ne 0;\ \alpha_{22}^{n+1} \ge 0;\ \beta_{22}^{n+1} \ge 0;\ \alpha_{22}^{n+1} + \beta_{22}^{n+1} \ne 0;$$

$$c_{jk} = \alpha_{2j}^k \beta_{1j}^k - \alpha_{1j}^k \beta_{2j}^k \ne 0;\ \ c_{1k} \cdot c_{2k} > 0;$$

$$f(t,r,\varphi,z) = \left\{ f_1(t,r,\varphi,z), f_2(t,r,\varphi,z), \ldots, f_{n+1}(t,r,\varphi,z) \right\}; \ddot{x}$$

$$g^1(r,\varphi,z) = \left\{ g_1^1(r,\varphi,z), g_2^1(r,\varphi,z), \ldots, g_{n+1}^1(r,\varphi,z) \right\};$$

$$g^2(r,\varphi,z) = \left\{ g_1^2(r,\varphi,z), g_2^2(r,\varphi,z), \ldots, g_{n+1}^2(r,\varphi,z) \right\};$$

$g_0(t,\varphi,z)$; $g(t,\varphi,z)$ are known bounded continuous functions;

$u(t,r,\varphi,z) = \left\{ u_1(t,r,\varphi,z), u_2(t,r,\varphi,z), \ldots, u_{n+1}(t,r,\varphi,z) \right\}$ is the desired function.

The main part. Let's assume that the solution of the problem (1)–(5) exists and defined and the unknown functions satisfy the conditions of applicability of integral transformations (6)–(8) [14–16].

Let's apply the integral Fourier transform on Cartesian axis $(-\infty;+\infty)$ relative to variable z to the problem (1)–(5) [14]:

$$F\left[g(z)\right] = \int_{-\infty}^{+\infty} g(z)\exp(-i\sigma z)dz \equiv \tilde{g}(\sigma),\ \ i = \sqrt{-1}\ , \tag{6}$$

$$F^{-1}\left[\tilde{g}(\sigma)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{g}(\sigma)\exp(i\sigma z)d\sigma \equiv g(z), \tag{7}$$

$$F\left[\frac{d^2 g}{dz^2}\right] = -\sigma^2 F[g(z)] \equiv -\sigma^2 \tilde{g}(\sigma). \tag{8}$$

The integral operator F due to the formula (6) as a result of identity (8) three-dimensional initial boundary value problem of conjugation (1)–(5) puts in accordance the task of constructing solution which is limited in the set $D' = \left\{(t,r,\varphi); t > 0; r \in I_n^+; \varphi \in [0;2\pi)\right\}$ and is 2π-periodical of angular variable φ of differential equations

$$\begin{gathered}\frac{\partial^2 \tilde{u}_j}{\partial t^2} - \left[a_{rj}^2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) + \frac{a_{\phi j}^2}{r^2}\frac{\partial^2}{\partial \varphi^2} \right]\tilde{u}_j + \left(a_{zj}^2 \sigma^2 + \chi_j^2 \right)\tilde{u}_j = \\ = \tilde{f}_j(t,r,\varphi,\sigma);\ r \in I_j;\ j = \overline{1,n+1}\end{gathered} \tag{9}$$

with initial conditions

$$\tilde{u}_j\Big|_{t=0} = \tilde{g}_j^1(r,\varphi,\sigma);\ \ \frac{\partial \tilde{u}_j}{\partial t}\bigg|_{t=0} = \tilde{g}_j^2(r,\varphi,\sigma);\ \ r \in I_j;\ j = \overline{1,n+1}; \tag{10}$$

boundary conditions

$$\left(\alpha_{11}^{0}\frac{\partial}{\partial r}+\beta_{11}^{0}\right)\tilde{u}_1\bigg|_{r=R_0}=\tilde{g}_0(t,\varphi,\sigma);\ \left(\alpha_{22}^{n+1}\frac{\partial}{\partial r}+\beta_{22}^{n+1}\right)\tilde{u}_{n+1}\bigg|_{r=R}=\tilde{g}(t,\varphi,\sigma);\quad (11)$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^{k}\frac{\partial}{\partial r}+\beta_{j1}^{k}\right)\tilde{u}_k-\left(\alpha_{j2}^{k}\frac{\partial}{\partial r}+\beta_{j2}^{k}\right)\tilde{u}_{k+1}\right]\bigg|_{r=R_k}=0;$$
$$j=1,2;\ k=\overline{1,n}. \quad (12)$$

Let's apply finite integral Fourier transform relative to the variable φ to the problem (9)–(12) [15]:

$$F_m\left[g(\varphi)\right]=\int_0^{2\pi} g(\varphi)\exp(-im\phi)d\varphi\equiv g_m, \quad (13)$$

$$F_m^{-1}[g_m]=\frac{\mathrm{Re}}{2\pi}\sum_{m=0}^{\infty}\varepsilon_m g_m\exp(im\varphi)\equiv g(\varphi), \quad (14)$$

$$F_m\left[\frac{d^2 g}{d\varphi^2}\right]=-m^2 F_m[g(\varphi)]\equiv -m^2 g_{m,} \quad (15)$$

here $\mathrm{Re}(\cdots)$ — the real part of the expression $(\cdots)$ relative to the variable φ; $\varepsilon_0=1,\ \varepsilon_k=2;\ k=1,2,3\ldots$

The integral operator F_m due to the formula (13) as a result of identity (15) two-dimensional initial boundary value problem of conjugation (9)–(12) puts in accordance the task of constructing solution which is limited in the set $D''=\left\{(t,r);t>0;r\in I_n^{+}\right\}$ of differential equations

$$\frac{\partial^2\tilde{u}_{jm}}{\partial t^2}-a_{rj}^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\nu_{jm}^2}{r^2}\right)\tilde{u}_{jm}+\left(a_{zj}^2\sigma^2+\chi_j^2\right)\tilde{u}_{jm}=$$
$$=\tilde{f}_{jm}(t,r,\sigma);\ r\in I_j;\ j=\overline{1,n+1}; \quad (16)$$

with initial conditions

$$\tilde{u}_{jm}\Big|_{t=0}=\tilde{g}_{jm}^{1}(r,\sigma);\ \frac{\partial\tilde{u}_{jm}}{\partial t}\bigg|_{t=0}=\tilde{g}_{jm}^{2}(r,\sigma);\ r\in I_j;\ j=\overline{1,n+1}, \quad (17)$$

boundary conditions

$$\left(\alpha_{11}^{0}\frac{\partial}{\partial r}+\beta_{11}^{0}\right)\tilde{u}_{1m}\bigg|_{r=R_0}=\tilde{g}_{0m}(t,\sigma);\ \left(\alpha_{22}^{n+1}\frac{\partial}{\partial r}+\beta_{22}^{n+1}\right)\tilde{u}_{n+1,m}\bigg|_{r=R}=\tilde{g}_m(t,\sigma);\quad (18)$$

and conjugate conditions

$$\left[\left(\alpha_{j1}^{k}\frac{\partial}{\partial r}+\beta_{j1}^{k}\right)\tilde{u}_{km}-\left(\alpha_{j2}^{k}\frac{\partial}{\partial r}+\beta_{j2}^{k}\right)\tilde{u}_{k+1,m}\right]\Bigg|_{r=R_k}=0; \tag{19}$$

$$j=1,2;\ k=\overline{1,n}.$$

Let's apply finite hybrid integral Hankel transform of 2nd kind relative to the variable r in piecewise homogeneous segment I_n^+ of n conjugation points to the problem (16)–(19) [16]:

$$H_{sn}\left[f(r)\right]=\int_{R_0}^{R} f(r)V(r,\lambda_s)\sigma(r)rdr\equiv\tilde{f}(\lambda_s), \tag{20}$$

$$H_{sn}^{-1}\left[\tilde{f}(\lambda_s)\right]=\sum_{s=1}^{\infty}\tilde{f}(\lambda_s)\frac{V(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|u^2}\equiv f(r), \tag{21}$$

$$\begin{aligned}H_{sn}\left[B_{(m)}[f(r)]\right]=-\lambda_s^2\tilde{f}(\lambda_s)-\sum_{k=1}^{n+1}\gamma_k^2\int_{R_{k-1}}^{R_k} f(r)V_k(r,\lambda_s)\sigma_k rdr-\\ -a_1^2R_0\sigma_1\left(\alpha_{11}^0\right)^{-1}V_1(R_0,\lambda_s)\left(\alpha_{11}^0\frac{df}{dr}+\beta_{11}^0 f\right)\Bigg|_{r=R_0}+\\ +a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}\left(\alpha_{22}^{n+1}\frac{df}{dr}+\beta_{22}^{n+1}f\right)\Bigg|_{r=R}.\end{aligned} \tag{22}$$

Spectral function $V(r,\lambda_s)$, weight function $\sigma(r)$ and hybrid Bessel differential operator $B_{(m)}=\sum_{k=1}^{n+1}a_{rk}^2\theta(r-R_{k-1})\theta(R_k-r)B_{\nu_{km}}$, written in [16], take part in formulas (20)–(22).

Here $B_{\nu_{km}}=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}-\dfrac{\nu_{km}^2}{r^2}$ is Bessel differential operator, $\theta(x)$ is the Heaviside step function.

Let's write the system (16) and the initial conditions (17) in matrix form

$$\begin{bmatrix}\left(\dfrac{\partial^2}{\partial t^2}-a_{r1}^2B_{\nu_{1m}}+q_1^2(\sigma)\right)\tilde{u}_{1m}\\ \left(\dfrac{\partial^2}{\partial t^2}-a_{r2}^2B_{\nu_{2m}}+q_2^2(\sigma)\right)\tilde{u}_{2m}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \left(\dfrac{\partial^2}{\partial t^2}-a_{r,n+1}^2B_{\nu_{n+1,m}}+q_{n+1}^2(\sigma)\right)\tilde{u}_{n+1,m}\end{bmatrix}=\begin{bmatrix}\tilde{f}_{1m}(t,r,\sigma)\\ \tilde{f}_{2m}(t,r,\sigma)\\ \cdots\cdots\\ \tilde{f}_{n+1,m}(t,r,\sigma)\end{bmatrix}, \tag{23}$$

$$\left.\begin{bmatrix} \tilde{u}_{1m}(t,r,\sigma) \\ \tilde{u}_{2m}(t,r,\sigma) \\ \cdots\cdots\cdots \\ \tilde{u}_{n+1,m}(t,r,\sigma) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \tilde{g}^{1}_{1m}(r,\sigma) \\ \tilde{g}^{1}_{2m}(r,\sigma) \\ \cdots\cdots\cdots \\ \tilde{g}^{1}_{n+1,m}(r,\sigma) \end{bmatrix};$$

$$\frac{\partial}{\partial t}\left.\begin{bmatrix} \tilde{u}_{1m}(t,r,\sigma) \\ \tilde{u}_{2m}(t,r,\sigma) \\ \cdots\cdots\cdots \\ \tilde{u}_{n+1,m}(t,r,\sigma) \end{bmatrix}\right|_{t=0} = \begin{bmatrix} \tilde{g}^{2}_{1m}(r,\sigma) \\ \tilde{g}^{2}_{2m}(r,\sigma) \\ \cdots\cdots\cdots \\ \tilde{g}^{2}_{n+1,m}(r,\sigma) \end{bmatrix}, \tag{24}$$

here $q_j^2(\sigma) = a_{zj}^2\sigma^2 + \chi_j^2$; $j = \overline{1, n+1}$.

The integral operator H_{sn} is represented as an operator matrix-row due to the rule (20):

$$H_{sn}\left[\cdots\right] = \left[\int_{R_0}^{R_1} \cdots V_1(r,\lambda_s)\sigma_1 r dr \ \int_{R_1}^{R_2} \cdots V_2(r,\lambda_s)\sigma_2 r dr \right. \quad . \tag{25}$$
$$\left. \int_{R_{n-1}}^{R_n} \cdots V_n(r,\lambda_s)\sigma_n r dr \ \int_{R_n}^{R} \cdots V_{n+1}(r,\lambda_s)\sigma_{n+1} r dr \right].$$

Let's apply the operator matrix-row (25) to the problem (23), (24) according to the matrix multiplication rule. As a result of the identity (22), we get a Cauchy problem

$$\sum_{j=1}^{n+1}\left(\frac{d^2}{dt^2} + \lambda^2 + \gamma_j^2 + q_j^2(\sigma)\right)\tilde{\tilde{u}}_{jm}(t,\lambda,\sigma) = \sum_{j=1}^{n+1}\tilde{f}_{jm}(t,\lambda,\sigma) - a_1^2 R_0\sigma_1 \times \tag{26}$$
$$\times\left(\alpha_{11}^0\right)^{-1} V_1(R_0,\lambda_s)\tilde{g}_{0m}(t,\sigma) + a_{n+1}^2 R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1} V_{n+1}(R,\lambda_s)\tilde{g}_m(t,\sigma),$$

$$\left.\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma)\right|_{t=0} = \sum_{j=1}^{n+1}\tilde{\tilde{g}}^{1}_{jm}(\lambda_s,\sigma);$$
$$\left.\frac{d}{dt}\sum_{j=1}^{n+1}\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma)\right|_{t=0} = \sum_{j=1}^{n+1}\tilde{\tilde{g}}^{2}_{jm}(\lambda_s,\sigma), \tag{27}$$

here $\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma) = \int_{R_{j-1}}^{R_j} \tilde{u}_{jm}(t,r,\sigma)V_j(r,\lambda_s)\sigma_j r dr$; $j = \overline{1, n+1}$,

$$\tilde{\tilde{f}}_{jm}(t,\lambda_s,\sigma) = \int_{R_{j-1}}^{R_j} \tilde{f}_{jm}(t,r,\sigma)V_j(r,\lambda_s)\sigma_j r dr, \quad j = \overline{1, n+1},$$

$$\tilde{\tilde{g}}_{jm}^{k}(\lambda_s,\sigma)=\int\limits_{R_{j-1}}^{R_j}\tilde{g}_{jm}^{k}(r,\sigma)V_j(r,\lambda_s)\sigma_j r dr;\quad k=1,2;\ \ j=\overline{1,n+1}.$$

Let's suppose that $\max\left\{q_1^2, q_2^2, ..., q_{n+1}^2\right\}=q_1^2$ and put everywhere $\gamma_j^2=q_1^2-q_j^2;\ j=\overline{1,n+1}$. Cauchy problem (26), (27) takes the form

$$\frac{d^2\tilde{\tilde{u}}_m}{dt^2}+\Delta^2(\lambda_s,\sigma)\tilde{\tilde{u}}_m=\tilde{\tilde{f}}_m(t,\lambda_s,\sigma)-a_1^2R_0\sigma_1\left(\alpha_{11}^0\right)^{-1}V_1(R_0,\lambda_s)\tilde{g}_{0m}(t,\sigma)+ \tag{28}$$
$$+a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(t,\sigma),$$

$$\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)\Big|_{t=0}=\tilde{\tilde{g}}_m^1(\lambda_s,\sigma);\ \ \frac{d\tilde{\tilde{u}}_m}{dt}\Bigg|_{t=0}=\tilde{\tilde{g}}_m^2(\lambda_s,\sigma), \tag{29}$$

where $\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)=\sum\limits_{j=1}^{n+1}\tilde{\tilde{u}}_{jm}(t,\lambda_s,\sigma);\ \ \tilde{\tilde{f}}_m(t,\lambda_s,\sigma)=\sum\limits_{j=1}^{n+1}\tilde{\tilde{f}}_{jm}(t,\lambda_s,\sigma),$

$$\tilde{\tilde{g}}_m^1(\lambda_s,\sigma)=\sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jm}^1(\lambda_s,\sigma),\ \ \tilde{\tilde{g}}_m^2(\lambda_s,\sigma)=\sum_{j=1}^{n+1}\tilde{\tilde{g}}_{jm}^2(\lambda_s,\sigma),$$
$$\Delta^2(\lambda_s,\sigma)=\lambda_s^2+a_{z1}^2\sigma^2+\chi_1^2.$$

It is directly verify that the only solution of the inhomogeneous Cauchy problem (28), (29) is a function

$$\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)=\frac{\sin(\Delta(\lambda_s,\sigma)t)}{\Delta(\lambda_s,\sigma)}\tilde{\tilde{g}}_m^2(\lambda_s,\sigma)+\frac{d}{dt}\frac{\sin(\Delta(\lambda_s,\sigma)t)}{\Delta(\lambda_s,\sigma)}\tilde{\tilde{g}}_m^1(\lambda_s,\sigma)+$$
$$+\int\limits_0^t\frac{\sin(\Delta(\lambda_s,\sigma)(t-\tau))}{\Delta(\lambda_s,\sigma)}\Big[\tilde{\tilde{f}}_m(\tau,\lambda_s,\sigma)-a_1^2R_0\sigma_1\left(\alpha_{11}^0\right)^{-1}V_1(R_0,\lambda_s)\times \tag{30}$$
$$\times\tilde{g}_{0m}(t,\sigma)+a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(t,\sigma)\Big]d\tau.$$

Integral operator H_{sn}^{-1}, as inverse to H_{sn}, we represent as the operator matrix-column:

$$H_{sn}^{-1}\left[\cdots\right]=\begin{bmatrix}\sum\limits_{s=1}^{\infty}\cdots\dfrac{V_1(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}\\ \sum\limits_{s=1}^{\infty}\cdots\dfrac{V_2(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \sum\limits_{s=1}^{\infty}\cdots\dfrac{V_{n+1}(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}\end{bmatrix} \tag{31}$$

Let's apply operator matrix-column (31) to the matrix-element $\left[\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)\right]$, where the function $\tilde{\tilde{u}}_m(t,\lambda_s,\sigma)$ is defined by formula (30) due to matrices multiplication rule. As a result we get the only solution of one-dimensional hyperbolic initial boundary problem of conjugation (16)–(19):

$$\tilde{u}_{jm}(t,r,\sigma)=\sum_{s=1}^{\infty}\left[\frac{\sin(\Delta(\lambda_s,\sigma)t)}{\Delta(\lambda_s,\sigma)}\tilde{\tilde{g}}_m^2(\lambda_s,\sigma)+\frac{\partial}{\partial t}\frac{\sin(\Delta(\lambda_s,\sigma)t)}{\Delta(\lambda_s,\sigma)}\tilde{\tilde{g}}_m^1(\lambda_s,\sigma)\right]\times$$

$$\times\frac{V_j(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}+\sum_{s=1}^{\infty}\int_0^t\frac{\sin(\Delta(\lambda_s,\sigma)(t-\tau))}{\Delta(\lambda_s,\sigma)}\left[\tilde{\tilde{f}}_m(\tau,\lambda_s,\sigma)-a_1^2R_0\sigma_1\left(\alpha_{11}^0\right)^{-1}\times\right. \quad (32)$$

$$\left.\times V_1(R_0,\lambda_s)\tilde{g}_{0m}(\tau,\sigma)+a_{n+1}^2R\sigma_{n+1}\left(\alpha_{22}^{n+1}\right)^{-1}V_{n+1}(R,\lambda_s)\tilde{g}_m(\tau,\sigma)\right]d\tau\frac{V_j(r,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}.$$

If to apply consistently inverse operators F^{-1} and F_m^{-1} to functions $\tilde{u}_{jm}(t,r,\sigma),$ which are defined by formulas (32) and perform the some simple transformation, we get functions

$$u_j(t,r,\varphi,z)=\sum_{k=1}^{n+1}\int_0^t\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\int_{-\infty}^{+\infty}E_{jk}(t-\tau,r,\rho,\varphi-\alpha,z-\xi)f_k(\tau,\rho,\alpha,\xi)\times$$

$$\times\sigma_k\rho d\xi d\alpha d\rho d\tau+\frac{\partial}{\partial t}\sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\int_{-\infty}^{+\infty}E_{jk}(t,r,\rho,\varphi-\alpha,z-\xi)g_k^1(\rho,\alpha,\xi)\times$$

$$\times\sigma_k\rho d\xi d\alpha d\rho+\sum_{k=1}^{n+1}\int_{R_{k-1}}^{R_k}\int_0^{2\pi}\int_{-\infty}^{+\infty}E_{jk}(t,r,\rho,\varphi-\alpha,z-\xi)g_k^2(\rho,\alpha,\xi)\times \quad (33)$$

$$\times\sigma_k\rho d\xi d\alpha d\rho+\int_0^t\int_0^{2\pi}\int_{-\infty}^{+\infty}\left[W_{j,r}^1\left(t-\tau,r,\varphi-\alpha,z-\xi\right)g_0(\tau,\alpha,\xi)+\right.$$

$$\left.W_{j,r}^2\left(t-\tau,r,\varphi-\alpha,z-\xi\right)g(\tau,\alpha,\xi)\right]d\xi d\alpha d\tau;\ j=\overline{1,n+1},$$

Functions (33) define the only solution of hyperbolic initial boundary problem of conjugation (1)–(5).

In formulas (33) there are components

$$E_{jk}(t,r,\rho,\varphi,z)=$$

$$=\frac{1}{2\pi^2}\sum_{m=0}^{\infty}\varepsilon_m\left(\sum_{s=1}^{\infty}\int_0^{+\infty}\frac{\sin(\Delta(\lambda_s,\sigma)t)}{\Delta(\lambda_s,\sigma)}\cos(\sigma z)d\sigma\frac{V_j(r,\lambda_s)V_k(\rho,\lambda_s)}{\left\|V(r,\lambda_s)\right\|^2}\right)\cos(m\varphi)$$

of matrix of influence (function of influence), components $W_{j,r}^1(t,r,\varphi,z)=-a_1^2R_0\sigma_1(\alpha_{11}^0)^{-1}E_{j1}(t,r,R_0,\varphi,z)$ of left radial Green's matrix (left Green's function) and components $W^2_{j,r}(t,r,\varphi,z) = = a^2_{n+1} R\sigma_{n+1}\left(\alpha^{n+1}_{22}\right)^{-1} E_{j,n+1}(t,r,R,\varphi,z)$ of right radial Green's matrix (right Green's function) of considered problem.

Using a properties of functions of influence $E_{jk}(t,r,\rho,\varphi,z)$ and radial Green's functions $W^k_{j,r}(t,r,\varphi,z), (k=1,2)$ we can verify that functions $u_j(t,r,\varphi,z)$ which are defined by formulas (33) satisfy the equation (1), the initial conditions (2), the boundary conditions (3), (4) and conjugate conditions (5) in the sense of theory of generalized functions [17].

The uniqueness of the solution (33) follows from its structure (integrated image) and from uniqueness of the main solutions (functions of influence and Green's functions) of problem (1)–(5).

By methods from [17, 18] can be proved that under appropriate conditions on the initial data, formulas (33) define a limited classical solution of the hyperbolic initial boundary problem of conjugation (1)–(5).

We get the following theorem as the summary of the above results.

Theorem. If functions $f_j(t,r,\varphi,z)$, $g^1_j(r,\varphi,z)$, $g^2_j(r,\varphi,z)$ satisfy conditions:

1) are continuously differentiated twice for each variable;
2) have a limited variation for the geometric variables;
3) are absolutely summable with the variable z in $(-\infty;+\infty)$;
4) conjugate conditions are true and functions $g_0(t,\varphi,z)$, $g_R(t,\varphi,z)$ are continuously differentiated twice for each variable, have a limited variation for the geometric variables, are absolutely summable with the variable z in $(-\infty;+\infty)$, then hyperbolic initial boundary value problem (1)-(5) has the only limited classical solution, which is determined by formula (33).

Remark 1. In the case of $a_{rj} = a_{\varphi j} = a_{zj} \equiv a_j > 0$ formulas (33) define the structure of the solution of hyperbolic initial boundary value problem (1)–(5) in an infinite isotropic piecewise homogeneous hollow cylinder.

Remark 2. Parameters α^0_{11}, β^0_{11}; α^{n+1}_{22}, β^{n+1}_{22} allow to allocate the solutions of initial boundary value problems from formulas (33) in the case of boundary conditions of the 1st, 2nd and 3rd kind and their possible combinations on the radial surface $r = R_0$, $r = R$.

Remark 3. Analysis of the solution (33) is done directly from the general structure according to the analytical expression of functions $f_j(t,r,\varphi,z)$, $g^k_j(r,\varphi,z), (k=1,2)$, $g_0(t,\varphi,z)$, $g(t,\varphi,z)$.

Remark 4. In the case of $\chi_j^2 \equiv 0$ equation (1) is a classic three-dimensional inhomogeneous wave equation (the equation of fluctuations) for an orthotropic environment in cylindrical coordinates.

Remark 5. In the case of $\alpha_{11}^k = 0,\ \beta_{11}^k = 1;\ \alpha_{12}^k = 0,\ \beta_{12}^k = 1;$ $\alpha_{21}^k = E_1^k,\ \beta_{21}^k = 0;\ \alpha_{22}^k = E_2^k,\ \beta_{22}^k = 0,$ here $E_1^k,\ E_2^k$ — Young's modulus ($k = \overline{1,n}$), the conjugate conditions (5) coincide with conditions of ideal mechanical contact.

Thus, in these cases 4, 5 (at $f_j(t,r,\varphi,z) \equiv 0$) considered hyperbolic boundary value problem (1)–(5) is a mathematical model of free oscillating processes in unlimited piecewise homogeneous hollow cylinder.

Conclusions. By means of method of integral and hybrid integral transforms with the method of principal solutions (influence functions and Green's functions) integral image of exact analytical solution of hyperbolic boundary-value problem of mathematical physics in unlimited piecewise homogeneous hollow cylinder is obtained. The obtained solution is of algorithmic character, continuously depend on the parameters and data of problem and can be used in further theoretical research and in practical engineering calculations of real processes which are modeled by hyperbolic boundary-value problems of mathematical physics in piecewise homogeneous domains.

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Методом інтегральних і гібридних інтегральних перетворень у поєднанні з методом головних розв'язків (функцій впливу та функцій Гріна) вперше побудовано інтегральне зображення точного аналітичного розв'язку гіперболічної крайової задачі математичної фізики для необмеженого кусково-однорідного порожнистого циліндра.

Ключові слова: *гіперболічне рівняння, початкові та крайові умови, умови спряження, інтегральні перетворення, головні розв'язки.*

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