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METHOD OF CONSTRUCTION OF SPECIALIZED ALGORITHMS FOR SOLVING DIFFERENTIAL EQUATIONS

Considered the problem of optimization of the process of constructing specialized algorithms for the given form nonlinear ordinary differential equations systems' numerical solution. The problem is reduced to the determination and use of approximating representations of the numerical integration errors by the chosen basic method.

Key words: *differential equations, specialized algorithms, system dynamics, integration errors.*

Introduction. Methods for the numerical solution of the Cauchy problem for systems of ordinary differential equations

$$\dot{y} = f(y,u,t), \tag{1}$$

where: $y \in R^n$, $\mathrm{u} \in \mathrm{R}^m$, $t \in R^1$, $f : R^n \times R^m \times R^1 \to R^n$, usually have the property of universality — their design rules are constructed without taking into account the properties of the system being solved. To a certain extent, the specifics of the system being solved are taken into account in such methods as exponential and implicit, in which approximate values of the unknown quantities are found by solving nonlinear algebraic equations using the Jacobi matrix. However, the use of such methods in modern modeling and control systems, including computer facilities and operating in real time, is problematic due to the large computational costs at the integration step and the limitations on its choice.

The desire to improve the accuracy and speed properties of numerical methods for modeling the dynamics of controlled objects leads to the idea of constructing specialized algorithms designed to solve a given concrete system of equations.

An example of the idea realization with respect to the solution of linear equations with constant coefficients is the calculation rule [1] $y(t+h) = y(t) + \Phi(h)y(t)$, in which the numerical values of the parameters, that is, the elements of the matrix $\Phi(h) = \int_0^h e^{A\tau} d\tau$ depend on the matrix A of the system being solved. Another example of implementation is the introduction of additional terms in the numerical integration formulas that compensate for phase distortions in the integration of sinusoidal signals [2]. In [3], we consider the construction of specialized methods for solving the equations of rotation on the basis of the Taylor series expansion and the application of the Picard method with allowance for the form of the right-hand sides.

The need to develop specialized algorithms arises when multiple numerical analysis of a system of type (1) is required, and the use of universal methods for some reason (usually associated with the ultimate goal of numerical analysis) is not feasible or impossible. This situation is possible when building control systems or modeling complex dynamic objects.

Approach to the construction of specialized algorithms for solving differential equations. We formulate the problem of constructing specialized algorithms for solving a given system (1) as a problem close to identification in the broad sense. The essence of the construction of algorithms consists in finding the representations of the remainders of approximate calculation expressions

$$r_j(h) = y_j(t+h) - z_j(t+h),\ j = \overline{1,n} \tag{2}$$

as

$$r_j(h) \approx \sum_{k=1}^{N_j} p_{kj} g_{kj}(\dot{z}(t), z(t), u(t) = \varphi_j(p_j, z, \dot{z}, u), \tag{3}$$

where $z(t+h)$ is the approximate value of the unknown vector function $y(t+h)$ obtained by some numerical (basic) method; $g_{kj}(\bullet)$ — some functions; p_{kj} — options, $j = \overline{1...n}$.

The calculated rule of a specialized algorithm is given by

$$z(t+h) = F(z(t), f) + \Psi(p, z(t), f, u), \tag{4}$$

where $F(z(t), f)$ is the vector-function obtained according to the calculation rule of the base method; $\Psi(p, z(t), f, u)$ is a vector-function, the components of which are functions of (3).

When solving the problem of constructing specialized algorithms for analyzing dynamics, the following questions arise: the theoretical justification for the existence of a solution; selection of the base method; definition or organization of search for the type, number and arguments of g_{kj} functions, evaluation of parameter values of p_{kj}.

To solve the first problem it is sufficient to justify the existence of a Ψ function that provides a certain order of accuracy of the method. In this case, the basic condition is the differentiability of the residues (2) the required number of times, the form of the right-hand side of (3) is theoretically determined by expanding $r_j(h)$ in a power series in h at the point $h = 0$.

The use of real-time systems leads to the need for an optimal choice of a numerical algorithm for analyzing dynamics using criteria that takes into account the accuracy and velocity properties [4]. In this regard, the basic method should be chosen as optimal or close to it.

The solution of the last two questions pertaining to the identification (structural and parametric) proper can be carried out using the methods of the corresponding theory, in particular approaches and methods of the theory of experiment planning, identification methods, estimation theory, and selection of empirical relationships [5].

Let us consider a statistical approach to solving problems of synthesis of specialized methods for analyzing dynamics using the correlation analysis of the significance of functions $g_{kj}(\bullet)$ from (3) (which we will call hypotheses), the method of least squares for estimating parameters p_{kj}, and the ideas of sequential complication of the type of hypotheses. The set of values of the remainders (2) is ensured by a set of initial conditions of the system (1) uniformly distributed in the range of admissible values.

Let r be the vector N of the residuals $r(h)$ (the index j is omitted for simplicity of the record), determined by the results of the solution of the given system (1) by the base $z(t + h)$ and the reference methods $(t + h)$. As a reference, a method is used whose error is much less than the error of the basic method (achieved due to a high order of accuracy and a small step of integration $h_{\in} < h$). Let, further, there are M hypotheses $g_k(\bullet)$, $k = \overline{1, M}$ and N values corresponding for each of them, forming the $N \times M$ matrix G. At the initial stage of synthesis, sets of hypotheses are formed from the simplest functions, for example, $z(t)$, $\dot{z}(t)$, u. Sets of analyzed hypotheses can contain the quantities determined in the process of solving the system (1) by the base method (for example, $z(t + h)$), individual fragments of the right parts or their approximate derivatives, and also functions that for any reason seem significant.

In the presence of r and G, one can obtain estimates of the first, second, and mixed initial moments of the residues and hypotheses, using the usual statistical formulas.

Applying the symbol of the scalar product of vectors, we obtain for the first initial moments (e-N-vector with unit components, the g_k k-column of the matrix G)

$$\eta = \frac{1}{N}(r, e),\ \lambda_k = \frac{1}{N}(g_k, e),\ k = (\overline{1 \ldots M});$$

for the second and mixed initial moments

$$\rho = \frac{1}{N}(r, r),\ \xi_k = \frac{1}{N}(g_k, r),\ \varkappa_{kl} = \frac{1}{N}(g_k, g_l),\ k, l = (\overline{1 \ldots M}).$$

The correlation coefficients of the residues and hypotheses are determined by the formula

$$R_k = \frac{\xi_k - \eta\lambda_k}{\sqrt{(\varkappa_{kk} - \lambda_k^2)(\varkappa_{ll} - \lambda_l^2)}},\ k, l = (\overline{1 \ldots M}), \qquad (5)$$

and the correlation coefficients of hypotheses — according to the formula

$$RG_{kl} = \frac{\varkappa_{kl} - \lambda_k \lambda_l}{\sqrt{(\varkappa_{kk} - \lambda_k^2)(\varkappa_{ll} - \lambda_l^2)}},\ k,l = \overline{(1...M)}. \qquad (6)$$

Suppose, further, that q hypotheses $g_k(\bullet), k = \overline{(1...M)}$ are chosen from *M* hypotheses and it is required to estimate the values of the parameters p_{k_i} in approximation (3), which are the best in the mean middle square sense. If the hypotheses $g_{k_i}(\bullet), i = \overline{1,q}$ are linearly independent, which can always be ensured, the desired estimates of the parameters p_{k_i} are determined from the system of algebraic equations

$$\varkappa_q p_q = \xi_q, \qquad (7)$$

where p_q is the vector of the required parameters; ξ_q — *q*-vector with components; ξ_{k_i}; $\varkappa_q$ — $(q \times q)$ is the matrix (q is the matrix submatrix $\varkappa$) containing only the k_i elements of rows and columns of the matrix $\varkappa$.

Let r_q — *N*-vector of new residues be $r_q = r - \sum_{i=1}^{q} p_{k_i} g_{k_i}$. Then its second initial moment $\rho_q = \rho - (\rho_q, \xi_q)$ will be minimal for the chosen set of hypotheses $g_{k_i}(\bullet), i = \overline{1,q},$ and its correlation coefficients with the remaining *M-q* hypotheses $g_k(\bullet), k \neq k_i, i = \overline{1,q}$ are determined by the expression

$$R_k^q = \frac{\xi_k - \lambda_k \eta_q - \sum_{i=1}^{q} p_{k_i} \varkappa_{ik}}{\sqrt{(\varkappa_{kk} - \lambda_k^2)(\rho_q - \eta_q^2)}}, \quad k = k_i,$$

where $\eta_q = \eta - \sum_{i=1}^{q} p_{k_i} \lambda_{k_i}$ is the estimate of the mathematical expectation of a new residue.

From the foregoing we can see how the process of sequential increase in the number of selected hypotheses can be represented, determine their significance and parameters of approximating expressions, and also the quality of the approximation.

1. The choice of the *q* + 1*th* hypothesis from the number *M* — *q* of the initial set of hypotheses (*q* = 0, 1, 2, ...), according to the condition of the most correlated with the residues.
2. Formation of the submatrix $\varkappa_{q+1}$ and the vector ξ_{q+1} of the system (7).
3. Determination of least squares estimates for the *q* + 1 parameter for selected hypotheses by solving a system of linear algebraic equations (7).
4. Evaluation of the statistical characteristics of new remnants (the first and second initial moments, the correlation coefficients with the re-

maining hypotheses, etc.) and the decision to continue the approximation process (transition to paragraph 1).

Implementation of paragraphs 1–4 in the cycle ends, if $\max\left|R_k^q\right| < R_0$ or $q = M$ (R_0 — the threshold values of correlation coefficients of hypotheses that are considered significant).

As a result of the first stage of the construction of a specialized algorithm for solving differential equations, we obtain an approximation of the residues in the form (3) q_0 by hypotheses $(1 \le q_0 \le M)$ from the initial set (the case when $q_0 = 0$ indicates an unsuccessful initial set of hypotheses and is not considered here).

The main idea of the second stage of the synthesis is to construct approximate expressions of the form (3) for some parameters of the obtained approximation of the residues (2) (to avoid complicating the indices, we assume that the approximation contains the first to the hypotheses of the initial set).

Suppose that *N* equalities of the system

$$r = \sum_{k=1}^{\bar{q}} p_k g_k + \sum_{l=\bar{q}+1}^{q_0} p_l g_l \qquad (8)$$

are carried out by changing the values of the parameters $p_k, k = \overline{(1,q)}$. By solving each equation of system (8) with respect to the P_k by method of least squares, we obtain P_k vector $\overline{p_k}$, the components of which are determined by the expression

$$p_{k_i} = \frac{g_{ki}\left(r_i - \sum_{i=\bar{q}+1}^{q_0} p_{li} g_{li}\right)}{\sum_{k=1}^{\bar{q}} g_{ki}^2}, \quad k = \overline{1,q}, \quad i = \overline{1,N}.$$

Further, the problem of approximation of parameters $p_k, k = \overline{(1,q)}$ is solved analogously to the problem of residual approximation. After substituting the approximations of the parameters $p_k, k = \overline{(1,q)}$ in (8), new, more complex hypotheses appear for the residues in the approximating expression, which together with $g_{k_i}(\bullet)$, $k = \overline{\bar{q}+1, q_0}$ are used in the formation of the next set of hypotheses for the further search for residual approximation.

The process of synthesis of methods for solving differential equations ends either when the required characteristics are achieved or in the case when $q_0 = 0$ for all approximated quantities.

The numerical experiments carried out to develop the method of constructing specialized algorithms have confirmed the possibility of obtaining methods with improved accuracy properties. Simple hypotheses such as state variables, their derivatives and their products were used. The values of the standard deviation errors of the specialized algorithm were 5-10 times less (for different state variables) of the corresponding values of the base method (Runge-Kutta of the second order). Additional costs for calculating residual approximations did not exceed 5% of the cost of the basic method.

It should be noted that, from a methodological point of view, the problem of constructing specialized methods considered above is a task of accurately adjusting the basic method for solving a given system of differential equations and therefore can be assigned to problems of precision clustering of dynamics problems solving, which leads to a set of approximating expressions for residues, used under certain conditions.

Conclusions. Thus, in the article, a method for constructing algorithms for numerical solving nonlinear ordinary differential equations is proposed, and a methodology for its implementation is provided that ensure the adaptation of the computational process to the specific features of the particular problem being solved. The results are aimed for realization of the computer-integrated systems' functioning processes in conditions of resource limitations.

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Розглянуто задачу оптимізації процесу побудови спеціалізованих алгоритмів чисельного розв'язання систем нелінійних звичайних диференціальних рівнянь заданого виду. Задача зводиться до визначення і використання апроксимуючих уявлень помилок чисельного інтегрування обраним базовим методом.

Ключові слова: *диференціальні рівняння, спеціалізовані алгоритми, динаміка систем, помилки інтегрування.*

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