SPONTANEOUS MAGNETIZATION OF QUANTUM XY-CHAIN FROM FINITE CHAIN FORM-FACTORS

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PACS 75.10.Jm, 75.10.Pq, 05.50+q, 02.30Ik

Using the explicit factorized formulas for matrix elements (form-factors) of the spin operators between the eigenvectors of the Hamiltonian of a finite quantum XY-chain in a transverse field, the spontaneous magnetization for σ^{x} and σ^{y} is re-derived in a simple way.

1. Introduction

The quantum XY-chain is one of the simplest models which is rich enough from the point of view of physics and, at the same time, admits a strict mathematical analysis. The study of this model was started in [1], where it was rewritten in terms of fermionic operators by means of the Jordan–Wigner transformation. Now this relation is a standard mean to study different properties (the spectrum of Hamiltonian [1, 2], correlation functions [3–5], emptiness formation probability [6], and entanglement entropy [7–9]) of the XY-chain. Although the Hamiltonian of the model is equivalent to the Hamiltonian of a free fermion system, the spin operators σ^{x} and σ^{y} are expressed in terms of fermionic operators in a non-local way. This non-locality leads to non-zero averages $\langle\sigma^{\mathrm{x}}\rangle$ and $\langle\sigma^{\mathrm{y}}\rangle$ (spontaneous magnetization) in the ferromagnetic phase of the model in the thermodynamic limit.

In [10], we propose an alternative way to study correlation functions of the XY-model: we derive the formulas for matrix elements of the spin operators σ^{x} and σ^{y} between the eigenvectors of the Hamiltonian of a finite quantum XY-chain in a transverse field. These formulas allow one to obtain at least formal expressions for the multipoint multitime correlation functions at a finite temperature. In this paper, as an application of the formulas for form-factors, the value of spontaneous magnetization for σ^{x} and σ^{y} is re-derived in a simple way.

In Section 2 we recall the definition of a finite quantum XY-chain in a transverse field, its phase diagram, and eigenvalues of the Hamiltonian and give general comments on matrix elements of the spin operators between the eigenvectors of the Hamiltonian. Section 3 is devoted to the description of a relation between the model of quantum XY-chain and the Ising model on a 2D lattice. In Section 4, we present formulas for matrix elements (form-factors) of the spin operators σ^{x} and σ^{y} between the eigenvectors of the Hamiltonian of a finite quantum XY-chain derived in [10]. In Section 5, these formulas are rewritten for the case of an infinite-length chain. Here, we also re-derive the value of spontaneous magnetization for σ^{x} and σ^{y}.

2. The Finite Quantum XY-chain in a Transverse Field

The Hamiltonian of the XY-chain of length n in a transverse field h is [1, 2]

$$\mathcal{H} = -\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1+\varkappa}{2}\sigma_k^{\mathrm{x}}\sigma_{k+1}^{\mathrm{x}} + \frac{1-\varkappa}{2}\sigma_k^{\mathrm{y}}\sigma_{k+1}^{\mathrm{y}} + h\,\sigma_k^{\mathrm{z}}\right), \tag{1}$$

where σ_k^i are Pauli matrices, and $\varkappa$ is the anisotropy. In the case $\varkappa = 0$, we get an XX-chain (isotropic case). The value $\varkappa = 1$ corresponds to the quantum Ising chain in a transverse field. In what follows, we restrict ourselves to the case $0 < \varkappa \leq 1$ and suppose the periodic boundary condition $\sigma_k^i = \sigma_{k+n}^i$.

Now consider the values of h. Due to the relationship of the XY chain and the 2D Ising model which will be discussed below, the coupling constant h plays the role of a temperature-like variable. The value $h > 1$ corresponds to the paramagnetic (disordered) phase. The value $0 \leq h < 1$ corresponds to the ferromagnetic (ordered) phase. At $h = 1$, there is a second-order phase transition. If $0 \leq h < (1-\varkappa^2)^{1/2}$, it is an oscillatory region (because of the oscillatory behavior of the two-point correlation function). Another peculiarity related to this region is the following. At fixed $\varkappa$, $0 < \varkappa \leq 1$, in the region where $(1-\varkappa^2)^{1/2} < h < 1$, the NS-vacuum energy is lower than the R-vacuum energy (asymptotically, if $n \to \infty$, they coincide). In the region $0 \le h < (1-\varkappa^2)^{1/2}$, there are intersections at special values of h of these vacuum levels even at finite n. The number of these intersections grows with n. For a detailed analysis of the oscillatory region, see [3, 11].

In this paper, we consider the ferromagnetic phase which corresponds to $0 \le h < 1$ in order to obtain a non-zero value of spontaneous magnetization.

Using the Jordan–Wigner and Bogoliubov transformations, the Hamiltonian $\mathcal{H}$ of the XY-chain can be rewritten as the Hamiltonian of a system of free fermions and can be diagonalized [1, 2]. The relation between the energies ε and the momenta q of fermionic excitations is

$$\varepsilon(q) = \left((h-\cos q)^2 + \varkappa^2 \sin^2 q\right)^{1/2}, \qquad q \neq 0, \pi\,, \tag{2}$$

$$\varepsilon(0) = 1-h, \qquad \varepsilon(\pi) = h+1\,.$$

The Hamiltonian $\mathcal{H}$ commutes with the operator $\mathbf{V} = \sigma_1^z\sigma_2^z\cdots\sigma_n^z$. Since $\mathbf{V}^2 = 1$, the eigenvectors are separated into two sectors with respect to the eigenvalue of $\mathbf{V} = \sigma_1^z\sigma_2^z\cdots\sigma_n^z$ with specific sets of possible momenta ($\mathcal{E}$ is the energy of state, i.e. the eigenvalue of $\mathcal{H}$):

– NS-sector: $\mathbf{V} \to +1 \Rightarrow$ “half-integer” momenta

$$q \in \mathrm{NS} = \left\{\frac{2\pi}{n}(j+1/2)\right\} \;\Rightarrow\; \mathcal{E} = -\frac{1}{2}\sum_{q\in \mathrm{NS}} \pm\varepsilon(q).$$

Each $-\varepsilon(q)$ in the expression for the energy corresponds to a fermionic excitation with momentum q. The number of excitations is even.

– R-sector: $\mathbf{V} \to -1 \Rightarrow$ “integer” momenta

$$q \in \mathrm{R} = \left\{\frac{2\pi}{n}j\right\} \;\Rightarrow\; \mathcal{E} = -\frac{1}{2}\sum_{q\in \mathrm{R}} \pm\varepsilon(q). \tag{3}$$

The number of excitations is even. In the paramagnetic phase ($h > 1$), the energy of the fermionic excitation $\varepsilon(0)$ becomes negative. In this case, we define $\varepsilon(0) = h-1$ together with the swapping between the absence/presence of the excitation with zero momentum. In other words, although the analytical expressions for the energies $\mathcal{E}$ in terms of h and $\varkappa$ are the same in both phases, due to the redefinition of $\varepsilon(0)$ in the case of $h > 1$, the number of excitations in the paramagnetic/ferromagnetic phase is odd/even.

We will denote the eigenvectors by the values of the excited momenta q corresponding to $-\varepsilon(q)$ in the expression for the energy $\mathcal{E}$.

Formally, in order to calculate any correlation function for the XY-chain, it is sufficient to insert a resolution of the identity operator as a sum of projectors to eigenspaces of the Hamiltonian between spin operators. It is the so-called Lehmann expansion. Then the problem is reduced to the problem of finding the matrix elements of the spin operators σ_k^{x}, σ_k^{y}, and σ_k^{z} between eigenstates of the Hamiltonian $\mathcal{H}$.

– *Matrix elements of* σ_k^{z}*:*

The operator σ_k^{z} commutes with $\mathbf{V} = \sigma_1^z\sigma_2^z\cdots\sigma_n^z$. Therefore, the action of σ_k^{z} does not change the sector. In fact, the operator σ_k^{z} can be presented as a bilinear combination of the operators of creation and annihilation of fermionic excitations. Thus, the matrix elements of σ_k^{z} between the eigenvectors of $\mathcal{H}$ can be easily calculated (most of them are 0). We will not consider such matrix elements in this paper.

– *Matrix elements of* σ_k^{x} *and* σ_k^{y}*:*

The operators σ_k^{x} and σ_k^{y} anticommute with $\mathbf{V} = \sigma_1^z\sigma_2^z\cdots\sigma_n^z$. Therefore, their action changes the sector. The operators σ_k^{x} and σ_k^{y} cannot be presented in terms of fermionic operators in a local way. All the matrix elements of them between the eigenvectors of $\mathcal{H}$ from different sectors are nonzero!

The idea of the derivation [10] of form-factors for σ_k^{x} and σ_k^{y} of a quantum finite XY-chain was to use the relations between three models: the model of quantum XY-chain in a transverse field, the Ising model on a 2D lattice and the $N = 2$ Baxter–Bazhanov–Stroganov (BBS) model. The relation between the first and second models was observed in [12], the relation between the second and third models was found in [13]. The latter relation together with the results on the separation of variables for the BBS model allowed one to prove [14] the formulas for the matrix elements of a spin operator of the Ising model found in [15, 16]. In [10], by using these relations between the models, we transferred the formulas for the form-factors of the $N = 2$ BBS model to the model of quantum XY-chain. A summarizing overview of the results on the separation of variables of the BBS model is given in [17].

In the following sections, we describe the relation between the model of quantum XY-chain in a transverse field and the Ising model on a 2D lattice. The parameters of the models are $(h, \varkappa)$ and (K_x, K_y), respectively. Then we present the formulas for the matrix elements of the spin operators $\sigma^{\rm x}$ and $\sigma^{\rm y}$ between eigenvectors of the Hamiltonian of a finite quantum XY-chain derived in [10] and take the thermodynamic limit of these formulas. The obtained formulas allow us to re-derive the value of spontaneous magnetization for $\sigma^{\rm x}$ and $\sigma^{\rm y}$.

3. Relation between a Quantum XY-chain and the Ising Model on a Lattice

The row-to-row transfer-matrix of the two-dimensional Ising model with parameters K_x, K_y can be chosen as

$$\mathbf{t}_{\rm Is} = T_1^{1/2} T_2 T_1^{1/2}\,, \tag{4}$$

where

$$T_1 = \exp\left(\sum_{k=1}^{n} K_y^* \,\sigma_k^{\rm z}\right),\; T_2 = \exp\left(\sum_{k=1}^{n} K_x \,\sigma_k^{\rm x}\sigma_{k+1}^{\rm x}\right). \tag{5}$$

The spin configurations of the rows are chosen to be labeled by the eigenvectors of the operators $\sigma_k^{\rm x}$, and the parameter K_y^* is dual to K_y, i.e. $\tanh K_y = \exp(-2K_y^*)$.

In [12], M. Suzuki observed that if we choose K_x and K_y^* such that

$$\tanh 2K_x = \frac{\sqrt{1-\varkappa^2}}{h}\,, \qquad \cosh 2K_y^* = \frac{1}{\varkappa}\,, \tag{6}$$

then Hamiltonian (1) of a XY-chain will commute with the transfer-matrix of the 2D Ising model (4), and these two operators have a common set of eigenvectors.

The dispersion relation for the fermions of the 2D Ising model with energies $\gamma(p)$ and momenta p is

$$\cosh\gamma(p) = \frac{(t_x + t_x^{-1})(t_y + t_y^{-1})}{2(t_x^{-1} - t_x)} - \frac{t_y^{-1} - t_y}{t_x^{-1} - t_x}\cos p\,, \tag{7}$$

$t_x = \tanh K_x$, $t_y = \tanh K_y$. We also have a relation between $\varepsilon(p)$ given by (2) and $\gamma(p)$:

$$\sinh\gamma(p) = \frac{\sqrt{1-\varkappa^2}}{\varkappa\sqrt{\varkappa^2+h^2-1}}\,\varepsilon(p)\,. \tag{8}$$

Relation (8) between the energies of fermionic excitations of these two models seems to be new. The existence of such a relation is surprising, because the commutativity of Hamiltonian (1) of the XY-chain and the transfer-matrix (4) of the 2D Ising model does not imply *a priori* any relation between their eigenvalues.

4. Formula for the Matrix Elements

We use the Bugrij–Lisovyy formula [15, 16] for the matrix element of a spin operator between the eigenvectors $|\Phi_0\rangle_{\rm Is} = |q_1, q_2, \ldots, q_K\rangle_{\rm Is}^{\rm NS}$ and $|\Phi_1\rangle_{\rm Is} = |p_1, p_2, \ldots, p_L\rangle_{\rm Is}^{\rm R}$ of the transfer matrix (4) for the finite 2D Ising model

$$\Xi_{\Phi_0,\Phi_1} = |{}_{\rm Is}\langle\Phi_0|\sigma_m^{\rm x}|\Phi_1\rangle_{\rm Is}|^2 =$$

$$= \xi\,\xi_T\left(\frac{t_y - t_y^{-1}}{t_x - t_x^{-1}}\right)^{(K-L)^2/2} \prod_{\substack{1\le k\le K\\ 1\le l\le L}} \frac{\sinh^2\frac{\gamma(q_k)+\gamma(p_l)}{2}}{\sin^2\frac{q_k-p_l}{2}}\times$$

$$\times\prod_{k=1}^{K}\frac{\prod_{q\ne q_k}^{\rm NS}\sinh\frac{\gamma(q_k)+\gamma(q)}{2}}{n\prod_{p}^{\rm R}\sinh\frac{\gamma(q_k)+\gamma(p)}{2}}\;\prod_{l=1}^{L}\frac{\prod_{p\ne p_l}^{\rm R}\sinh\frac{\gamma(p_l)+\gamma(p)}{2}}{n\prod_{q}^{\rm NS}\sinh\frac{\gamma(p_l)+\gamma(q)}{2}}\times$$

$$\times\prod_{k<k'}^{K}\frac{\sin^2\frac{q_k-q_{k'}}{2}}{\sinh^2\frac{\gamma(q_k)+\gamma(q_{k'})}{2}}\;\prod_{l<l'}^{L}\frac{\sin^2\frac{p_l-p_{l'}}{2}}{\sinh^2\frac{\gamma(p_l)+\gamma(p_{l'})}{2}}\,, \tag{9}$$

$$\xi^4 = 1 - (\sinh 2K_x \sinh 2K_y)^{-2} = \frac{1-h^2}{\varkappa^2}\,, \tag{10}$$

$$\xi_T^4 = \frac{\prod_{q}^{\rm NS}\prod_{p}^{\rm R}\sinh^2\frac{\gamma(q)+\gamma(p)}{2}}{\prod_{q,q'}^{\rm NS}\sinh\frac{\gamma(q)+\gamma(q')}{2}\prod_{p,p'}^{\rm R}\sinh\frac{\gamma(p)+\gamma(p')}{2}}\,, \tag{11}$$

$$\frac{t_y - t_y^{-1}}{t_x - t_x^{-1}} = \frac{1-\varkappa^2}{\varkappa\sqrt{\varkappa^2+h^2-1}}\,, \tag{12}$$

where we used (6) to write equivalent expressions in terms of different parameters.

We use the following main result of paper [10]: the matrix elements of spin operators between the eigenvectors $|\Phi_0\rangle_{\rm XY} = |q_1, q_2, \ldots, q_K\rangle_{\rm XY}^{\rm NS}$ from the NS-sector and $|\Phi_1\rangle_{\rm XY} = |p_1, p_2, \ldots, p_L\rangle_{\rm XY}^{\rm R}$ from the R-sector of Hamiltonian (1) of the XY-chain are

$$|{}_{\rm XY}\langle\Phi_0|\sigma_m^{\rm x}|\Phi_1\rangle_{\rm XY}|^2 =$$

$$= \frac{\varkappa}{2(1+\varkappa)}\left(C_{\Phi_0,\Phi_1}^{1/2} + C_{\Phi_0,\Phi_1}^{-1/2}\right)^2\Xi_{\Phi_0,\Phi_1}\,, \tag{13}$$

$$|{}_{\rm XY}\langle\Phi_0|\sigma_m^{\rm y}|\Phi_1\rangle_{\rm XY}|^2 =$$

$$= \frac{\varkappa}{2(1-\varkappa)}\left(C_{\Phi_0,\Phi_1}^{1/2} - C_{\Phi_0,\Phi_1}^{-1/2}\right)^2\Xi_{\Phi_0,\Phi_1}\,, \tag{14}$$

where Ξ_{Φ_0,Φ_1} is given by (9) and

$$C_{\Phi_0,\Phi_1}=\frac{\prod_{p\in \mathrm{R}} e^{\gamma(p)/2}}{\prod_{q\in \mathrm{NS}} e^{\gamma(q)/2}}\frac{\prod_{k=1}^{K} e^{\gamma(q_k)}}{\prod_{l=1}^{L} e^{\gamma(p_l)}}\,. \tag{15}$$

In the case of the quantum Ising chain ($\varkappa=1$), the formula for the matrix element of the spin operator σ_k^{x} derived in [14] can be expressed in terms of the energies of excitations $\varepsilon(q)$. In the case of a general XY-chain, we were not able to find an analogous explicit formula and need to use relation (8).

5. Asymptotics of Form-Factors in the Limit of Infinite Chain and Spontaneous Magnetization

In this section, we analyze the asymptotics of different parts of form-factors in the thermodynamic limit (the length $n\to\infty$) of the XY-chain. They can be obtained from the integral representations for form-factors at finite n [16, 18]. We slightly change the method. Our derivation is based on the following formulas valid for arbitrary $|\lambda|<1$ and $|\lambda|>1$, respectively:

$$\lim_{n\to\infty}\log\frac{\prod_p^{\mathrm{R}}(\lambda-e^{\gamma(p)})}{\prod_p^{\mathrm{NS}}(\lambda-e^{\gamma(p)})}=0\,. \tag{16}$$

$$\lim_{n\to\infty}\log\frac{\prod_p^{\mathrm{NS}}(\lambda-e^{-\gamma(p)})}{\prod_p^{\mathrm{R}}(\lambda-e^{-\gamma(p)})}=0\,. \tag{17}$$

At $\lambda=0$ and $\lambda=e^{\gamma(q)}$, they give, respectively,

$$\Lambda^{-1}=\frac{1}{2}\left(\sum_q^{\mathrm{NS}}\gamma(q)-\sum_p^{\mathrm{R}}\gamma(p)\right)\to 0\,,$$

$$e^{\eta(q)}=\frac{\prod_p^{\mathrm{NS}}\left(1-e^{-\gamma(q)-\gamma(p)}\right)}{\prod_p^{\mathrm{R}}\left(1-e^{-\gamma(q)-\gamma(p)}\right)}\to 1\,.$$

In turn, these two formulas yield

$$\frac{\prod_p^{\mathrm{NS}}\sinh\frac{\gamma(q)+\gamma(p)}{2}}{\prod_p^{\mathrm{R}}\sinh\frac{\gamma(q)+\gamma(p)}{2}}\to 1\,.$$

Using it twice for ξ_T (see (11)) for fixed q and p, respectively, and taking into account that the left-hand sides of (16) and (17) vanish exponentionally in n (see the derivation below), we get

$$\frac{\prod_q^{\mathrm{NS}}\prod_p^{\mathrm{R}}\sinh\frac{\gamma(q)+\gamma(p)}{2}}{\prod_{q,q'}^{\mathrm{NS}}\sinh\frac{\gamma(q)+\gamma(q')}{2}},\quad \frac{\prod_q^{\mathrm{NS}}\prod_p^{\mathrm{R}}\sinh\frac{\gamma(q)+\gamma(p)}{2}}{\prod_{p,p'}^{\mathrm{R}}\sinh\frac{\gamma(p)+\gamma(p')}{2}}\to 1.$$

Therefore, $\xi_T\to 1$ in the thermodynamic limit.

Finally, in the limit of the infinite XY-chain, formulas (13), (14) and (9) become

$$|{}_{\mathrm{XY}}\langle\Phi_0|\sigma_m^{\mathrm{x}}|\Phi_1\rangle_{\mathrm{XY}}|^2=$$

$$=\Xi_{\Phi_0,\Phi_1}\frac{2\varkappa}{1+\varkappa}\cosh^2\frac{\sum_{k=1}^{K}\gamma(q_k)-\sum_{l=1}^{L}\gamma(p_l)}{2}\,, \tag{18}$$

$$|{}_{\mathrm{XY}}\langle\Phi_0|\sigma_m^{\mathrm{y}}|\Phi_1\rangle_{\mathrm{XY}}|^2=$$

$$=\Xi_{\Phi_0,\Phi_1}\frac{2\varkappa}{1-\varkappa}\sinh^2\frac{\sum_{k=1}^{K}\gamma(q_k)-\sum_{l=1}^{L}\gamma(p_l)}{2}\,, \tag{19}$$

$$\Xi_{\Phi_0,\Phi_1}=\xi\left(\frac{t_y-t_y^{-1}}{t_x-t_x^{-1}}\right)^{(K-L)^2/2}\prod_{\substack{1\le k\le K\\ 1\le l\le L}}\frac{\sinh^2\frac{\gamma(q_k)+\gamma(p_l)}{2}}{\sin^2\frac{q_k-p_l}{2}}\times$$

$$\times\prod_{k=1}^{K}\frac{1}{n\sinh\gamma(q_k)}\prod_{l=1}^{L}\frac{1}{n\sinh\gamma(p_l)}\times$$

$$\times\prod_{k<k'}^{K}\frac{\sin^2\frac{q_k-q_{k'}}{2}}{\sinh^2\frac{\gamma(q_k)+\gamma(q_{k'})}{2}}\prod_{l<l'}^{L}\frac{\sin^2\frac{p_l-p_{l'}}{2}}{\sinh^2\frac{\gamma(p_l)+\gamma(p_{l'})}{2}}\,. \tag{20}$$

Formulas (18) and (19) at $K=L=0$ allow us to re-obtain the formulas for the spontaneous magnetization found in [12]. Indeed, for the quantum XY-chain in the ferromagnetic phase ($0\le h<1$) and in the thermodynamic limit $n\to\infty$ (when the energies of $|\Phi_0\rangle_{\mathrm{XY}}=|\mathrm{vac}\rangle_{\mathrm{NS}}$ and $|\Phi_1\rangle_{\mathrm{XY}}=|\mathrm{vac}\rangle_{\mathrm{R}}$ asymptotically coincide, giving the degeneration of the ground state), the spontaneous magnetization is

$$\langle\sigma^{\mathrm{x,y}}\rangle_{\mathrm{XY}}={}_{\mathrm{XY}}\langle\Phi_0|\sigma^{\mathrm{x,y}}|\Phi_1\rangle_{\mathrm{XY}}\,,$$

$$\langle\sigma^{\mathrm{x}}\rangle_{\mathrm{XY}}=\sqrt{2}\left(\frac{\varkappa^2(1-h^2)}{(1+\varkappa)^4}\right)^{1/8},\qquad \langle\sigma^{\mathrm{y}}\rangle_{\mathrm{XY}}=0\,.$$

At the end of this section, we give the derivation of (16) and (17). It repeats the derivation of the asymptotics of $\eta(q)$ in [18]. We use the function

$$T^2(z)=\frac{\prod_q^{\mathrm{R}}(\cosh\bar\gamma(p)-\cos q)}{\prod_q^{\mathrm{NS}}(\cosh\bar\gamma(p)-\cos q)}\,,\qquad z=e^{-\mathrm{i}p}\,,$$

where $\bar{\gamma}(p)$ is defined as $\gamma(p)$ in (7) but with the interchange $t_x \leftrightarrow t_y$ (i.e. $\bar{\gamma}(p)$ is the energy of fermionic excitations corresponding to the evolution in the transverse direction on a 2D Ising lattice). The evaluation of the products over q gives

$$T(z) = \tanh(n\bar{\gamma}(p)/2)\,. \tag{21}$$

On the other hand, due to the relation

$$\cosh\bar{\gamma}(p) - \cos q = \frac{t_x - t_x^{-1}}{t_y - t_y^{-1}}\,(\cosh\gamma(q) - \cos p),$$

we have

$$\log T(z) = \frac{1}{2}\log\left(\frac{\prod_q^{\rm R}(z - e^{\gamma(q)})(z - e^{-\gamma(q)})}{\prod_q^{\rm NS}(z - e^{\gamma(q)})(z - e^{-\gamma(q)})}\right)\,.$$

At $|\lambda| > 1$,

$$\frac{1}{{\rm i}\pi}\oint_{|z|=1}\frac{dz\,\log T(z)}{z-\lambda} = -\frac{1}{{\rm i}\pi}\oint_{|z|=1} dz\,\log(z-\lambda)\frac{T'(z)}{T(z)} =$$

$$= \log\frac{\prod_q^{\rm NS}(\lambda - e^{-\gamma(q)})}{\prod_q^{\rm R}(\lambda - e^{-\gamma(q)})}\,,$$

where we have integrated by parts and taken the contribution of the simple poles of $T'(z)/T(z)$ into account. Similarly, this integral at $|\lambda| < 1$ is

$$\frac{1}{{\rm i}\pi}\oint_{|z|=1}\frac{dz\,\log T(z)}{z-\lambda} = \log\frac{\prod_q^{\rm R}(\lambda - e^{\gamma(q)})}{\prod_q^{\rm NS}(\lambda - e^{\gamma(q)})}\,,$$

where we also took the simple pole at $z = \lambda$ into account. Due to (21), we have $\log T(z) \to 0$ if $n \to \infty$. This proves (16) and (17) in both cases of λ.

This article is based on the talk given at the Bogolyubov Kyiv conference "Modern Problems of Theoretical and Mathematical Physics", September 15-18, 2009, Kyiv, Ukraine. The work was supported by the Program of Fundamental Research of the Physics and Astronomy Division of the NAS of Ukraine, the Ukrainian FRSF grants 28.2/083 and 29.1/028, by French-Ukrainian program "Dnipro" M17-2009 and the joint project PICS of CNRS and NAS of Ukraine.

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Received 24.09.09

СПОНТАННА НАМАГНІЧЕНІСТЬ КВАНТОВОГО XY-ЛАНЦЮЖКА З ФОРМФАКТОРІВ ДЛЯ СКІНЧЕННОГО ЛАНЦЮЖКА

М.З. Іоргов

Р е з ю м е

Використовуючи факторизовані формули для матричних елементів (формфакторів) спінових операторів між власними векторами гамільтоніана скінченного квантового XY-ланцюжка в поперечному полі, дано простий вивід формули для спонтанної намагніченості $\sigma^{\rm x}$ та $\sigma^{\rm y}$.