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First and second sounds in a degenerate Bose gas

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In this work the propagation of sound waves in a degenerate quantum gas is considered. We modify the Wang Chang and Uhlenbeck method for description of the sounds in a Maxwell gas for the case of a degenerate quantum gas. Using this approach, we constructed a dispersion relation for sound waves in a condensed Bose gas at finite temperatures, and calculate the velocities of first and second sounds in the first approximation. The possibility of the theoretical investigation for sound damping is discussed.

PACS: 03.75.Kk Dynamic properties of condensates; collective and hydrodynamic excitations, superfluid flow;
67.40.Pm Transport processes, second and other sounds, and thermal counterflow; Kapitza resistance;
67.57.Jj Collective modes.

Keywords: quantum gas, Bose–Einstein condensation, first and second sounds.

1. Introduction

One of the most interesting problems in statistical mechanics is the description of the dynamics of collective modes. In the case of liquids and gases the collective modes are manifested mainly as sound waves (pressure or density oscillation).

More exotic hydrodynamic modes occur in the degenerate quantum liquids and gases. The behavior of the degenerate systems is determined by not only the effects of quantum mechanics but also by the statistical properties of the system [1]. The most popular ones are the superfluid ^{4}He, the condensed alkali Bose gases, superconductors, etc. One of the most spectacular features exhibited by superfluid ^{4}He is the existence of two hydrodynamic modes, so-called the first and second sounds. Theoretical description of the sounds in superfluid Bose liquid is based on the Landau two-fluid model. According to this model, superfluid liquid is a mixture of two independent, interpenetrating “fluids” or components, the “superfluid” and “normal” components, each associated with its own current and flow energy. When the normal and superfluid components are in local equilibrium, the two sound modes can be distinguished: the first sound consists of an in-phase oscillation of the superfluid and the normal fluid component, while the second sound consists of an out-of-phase oscillation of the superfluid and normal fluid components. Unfortunately, a strong interaction between helium atoms disguises clear distinction between the two components.

The experimental discovery in 1995 of Bose–Einstein condensation in dilute low-temperature trapped atomic clouds revealed the quantum phenomena in a qualitatively new regime. One of the most interesting features, exhibited by cold gas clouds of weakly interacting atoms, is that in the Bose–Einstein condensate (BEC). When essentially all atoms occupy the same quantum state, and the condensate may be described very well in terms of a mean-field theory. This is in significant contrast to liquid ^{4}He, for which a mean-field approach is inapplicable due to the strong correlations induced by the interaction between the atoms. Because the interatomic interaction is much weaker than in liquid ^{4}He, the superfluid in the gaseous BEC corresponds directly to the Bose-condensed atoms, and the normal component is the thermal system of noncondensed atoms [2].

The first and second sounds in the weakly interacting condensed Bose gas exhibit different properties than those in a Bose liquid [3]. In superfluid helium the first sound is mainly a density wave, while the second sound is almost a pure temperature wave. In contrast, in a condensed Bose gas, the first sound mode is mainly an oscillation of the density of the thermal cloud (the normal component) and the second sound is essentially an oscillation of the density of the condensate (the superfluid component). Thus, in a condensed Bose gas the both modes, first and second sounds, can be excited by a density perturbation.

There are some approaches to the theoretical investigation of the sound waves in gases and fluids in literature. The most commonly used ones are based on the solution of corresponding linearized hydrodynamic equations [4]. Theoretical investigation of the sound modes in a uniform condensed Bose gas in the framework of two-fluid hydrodynamics was proposed by Griffin and Zaremba [5]. In this paper, using a linearized equations of two-fluid hydrodynamics, the expressions for the velocities of first and second sounds have been obtained for the model of a uniform weakly interacting condensed Bose gas. This approach does not describe the sound damping processes, since has been used the equations of an ideal hydrodynamics. To describe the damping effects in sound waves we may use the Navier–Stokes hydrodynamics. In this respect, the more consistent and convenient is the approach that has been developed by Wang Chang and Uhlenbeck (WCU) [6] for the Maxwell gas. In the work of WCU the propagation of sound in monatomic gases has been studied using the method of collision integral eigenfunctions. In such approach a calculation has been provided at the level of linearized Boltzmann kinetic equation.

In the present paper, we develop a theoretical description of the propagation of sound waves in a degenerate quantum gas using a WCU approach. For this purpose, the WCU method has been modified for the case of weakly interacting condensed Bose gas. At very low temperatures, when almost all atoms are in the condensate, the dynamics of a condensed Bose gas can be described by the time-dependent Gross–Pitaevskii equation for the macroscopic wave function $\Phi(\mathbf{r},t)$, associated with the Bose condensate. At higher temperatures, when an appreciable fraction of atoms is excited out of the condensate, the dynamics of the trapped gas involves the condensate and the noncondensate degrees of freedom. The theoretical description of the trapped degenerated Bose gas at nonzero temperatures is based on the coupled equations of motion for both the condensate and noncondensate. The first one is the generalized Gross–Pitaevskii equation, and the second one is the corresponding quantum Boltzmann kinetic equation for the Wigner distribution function of noncondensate atoms $f(\mathbf{p},\mathbf{r},t)$ [7,8].

Using WCU technique, we described the weakly nonequilibrium processes in a Bose gas at the presence of a condensate. In particular, we constructed a coupled systems of equations for dynamics of the condensate and noncondensate degrees of freedom, derived the dispersion relation for sound waves in the form of determinant for the corresponding system of algebraic equations. This determinant can be evaluated using the method of successive approximations. This approach can be quite convenient and fruitful tool for the theoretical description of weakly nonequilibrium processes [9].

2. Linearized dynamics for the condensed Bose gas

2.1. Dynamics of the trapped Bose gas at finite temperatures

The coupled dynamics of condensate and noncondensate in the trapped Bose gas at finite temperatures can be described by generalized Gross–Pitaevskii equation for condensate density $n_c(\mathbf{r},t) = |\Phi(\mathbf{r},t)|^2$ and velocity $\mathbf{v}_c(\mathbf{r},t)$, as well as the quantum Boltzmann kinetic equation for Wigner distribution function of the thermal cloud atoms $f(\mathbf{p},\mathbf{r},t)$ (see for details [7,8]):

$$\frac{\partial n_c}{\partial t} + \nabla(n_c\mathbf{v}_c) = R[f],$$

$$m\left(\frac{\partial}{\partial t} + \mathbf{v}_c\nabla\right)\mathbf{v}_c = -\nabla\mu_c, \tag{1}$$

$$\frac{\partial f(\mathbf{p},\mathbf{r},t)}{\partial t} + \frac{\mathbf{P}}{m}\cdot\nabla f(\mathbf{p},\mathbf{r},t) - \nabla V_{\text{eff}}(\mathbf{r},t)\cdot\nabla_{\mathbf{p}} f(\mathbf{p},\mathbf{r},t) =$$

$$= C_{22}[f] + C_{12}[f,\Phi],$$

where total effective potential $V_{\text{eff}} = V_{\text{ext}}(\mathbf{r}) + 2gn(\mathbf{r},t)$. Here V_{ext} is the parabolic trap potential (the axial trap spring constant is so small that the condensate along the z axis can be treated as effectively uniform in such propagation studies). For simplicity, we consider the space homogeneous case $V_{\text{ext}} = 0$.

The total density $n(\mathbf{r},t)$ of the gas can be written down as the sum of the condensate and noncondensate densities

$$n(\mathbf{r},t) = n_c(\mathbf{r},t) + \tilde{n}(\mathbf{r},t),$$

where the density of particles out of condensate is determined in the form

$$\tilde{n}(\mathbf{r},t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} f(\mathbf{p},\mathbf{r},t).$$

The chemical potential of condensate μ_c is defined in the following way [7,8]:

$$\mu_c = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}} + gn_c + 2g\tilde{n}.$$

C_{22} and C_{12} are the terms of the collision integral (proportional to second order of the weak interaction constant g)

$$C_{12}[f,\Phi] = \frac{2g^2 n_c}{(2\pi)^2\hbar^4}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3\delta\left(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3\right)\times$$

$$\times\delta(\tilde{\varepsilon}_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3})[\delta(\mathbf{p}-\mathbf{p}_1) - \delta(\mathbf{p}-\mathbf{p}_2) - \delta(\mathbf{p}-\mathbf{p}_3)]\times$$

$$\times[(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)], \tag{2}$$

and

$$C_{22}[f]=\frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\times$$
$$\times\delta\left(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4}\right)[(1+f)(1+f_2)f_3f_4-$$
$$-ff_2(1+f_3)(1+f_4)]. \quad (3)$$

Here $f=f(\mathbf{p},\mathbf{r},t)$, and $f_i=f(\mathbf{p}_i,\mathbf{r},t)$ is the distribution function for ith particle,

$$\tilde{\varepsilon}_c=\mu_c+\frac{mv_c^2}{2},\quad \tilde{\varepsilon}_{p_i}=\frac{p_i^2}{2m}+V_{\text{eff}},$$

where p_i is the moment for ith particle.

The source term $R[f]$ associated with C_{12} collisions is defined to be $v_c=|\mathbf{v}_c|$,

$$R[f]=-\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}C_{12}[f].$$

We note, that for the case of absence of a trap the similar equations was derived much earlier by authors [10–12].

At the local equilibrium state $C_{22}[f_0]+C_{12}[f_0,\Phi_0]=0$. The local equilibrium distribution function is

$$f_0(\mathbf{p},\mathbf{r},t)=\frac{1}{e^{\beta\left(\mathbf{p}^2/2m+V^0_{\text{eff}}-\tilde{\mu}\right)}-1}.$$

2.2. Linearizing procedure

Let us consider a small deviation from local equilibrium state with the nonperturbed density n_{c0}:

$$n_c(\mathbf{r},t)=n_{c0}+\delta n_c(\mathbf{r},t),\quad \mathbf{v}_c(\mathbf{r},t)=\delta\mathbf{v}_c(\mathbf{r},t), \quad (4)$$

where δn_c is perturbation of the particle density. Inserting (4) into first two equations of the system of equations (1), we find linearized Gross–Pitaevskii equation taking into account the presence of a thermal cloud

$$\frac{\partial\delta n_c}{\partial t}+n_{c0}\nabla\delta\mathbf{v}_c=\delta R[f],$$
$$m\frac{\partial\delta\mathbf{v}_c}{\partial t}=-\nabla\delta\mu_c, \quad (5)$$

where

$$\delta R[f]=-\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\delta C_{12}[f],\qquad \delta\mu_c=g\delta n_c+2g\delta\tilde{n}.$$

Excluding the superfluid velocity from the system (5), we obtain that

$$\left(m\frac{\partial^2}{\partial t^2}-gn_{c0}\nabla^2\right)\delta n_c=2gn_{c0}\nabla^2\delta\tilde{n}-\frac{\partial\delta R[f]}{\partial t}.$$

In the case of the diffusion equilibrium between a condensate and the corresponding “noncondensate cloud” in the Bose gas, the value of $\delta R[f]$ becomes zero [8]. Inasmuch as a sound propagation is much slower process than the establishment of the local and diffusion equilibria, we can neglect the item, containing $\delta R[f]$, in the linearized equation for the condensate.

Therefore, the linearized equation for the condensate has the following form:

$$\left(m\frac{\partial^2}{\partial t^2}-gn_{c0}\nabla^2\right)\delta n_c=2gn_{c0}\nabla^2\delta\tilde{n}. \quad (6)$$

If atoms, that are out of the condensate, are neglected here, than the above equation becomes well known Stringari wave equation [13] for spatially uniform condensate.

Next we consider the deviation from the local equilibrium distribution function f_0. Let us write the first correction term to f_0 in the form

$$\delta f=f_0(1+f_0)h. \quad (7)$$

Using the third equation of the system (1) and (7) we obtain linearized Boltzmann kinetic equation (on details see Appendix 5)

$$f_0(1+f_0)\left(\frac{\partial h}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla\left(h-2\beta g(\delta n_c+\delta\tilde{n})\right)\right)=L[h]. \quad (8)$$

Here

$$L[h]=\frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\times$$
$$\times\delta\left(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4}\right)f_0f_{02}\times$$
$$\times(1+f_{03})(1+f_{04})[h_4+h_3-h_2-h].$$

Let us introduce the dimensionless variables

$$\mathbf{c}=\sqrt{\frac{\beta}{2m}}\,\mathbf{p},\quad \tau=\sqrt{\frac{2}{\beta m}}\,t,\quad \alpha=\beta(\mu_{0c}-V^0_{\text{eff}}),$$

then

$$\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\cdots=\frac{1}{\pi^{3/2}\Lambda^3}\int d\mathbf{c}\ldots,$$

where $\Lambda=\sqrt{2\pi\hbar^2\beta/m}$ is the thermal de Broglie wavelength. Local equilibrium distribution function has the following form. Then the

$$f_0=\frac{1}{e^{c^2-\alpha}-1}.$$

In the new variables equations (6) and (8) have such form:

$$\left(\frac{\partial^2}{\partial\tau^2}-\frac{\beta gn_{c0}}{2}\nabla^2\right)\delta n_c=\beta gn_{c0}\nabla^2\delta\tilde{n}, \quad (9)$$

$$(1+f_0)\left(\frac{\partial h}{\partial \tau}+\mathbf{c}\cdot\nabla(h-2\beta g(\delta n_c+\delta\tilde{n}))\right)=\tilde{L}[h] \quad (10)$$

with the linearized collision integral

$$\tilde{L}[h]=\frac{8g^2m^2}{\pi^5\hbar^7}\left(\frac{m}{\beta}\right)^{3/2}\int d\mathbf{c}_2\int d\mathbf{c}_3\int d\mathbf{c}_4\delta\left(\mathbf{c}+\mathbf{c}_2-\mathbf{c}_3-\mathbf{c}_4\right)\times$$

$$\times\delta\left(c^2+c_2^2-c_3^2-c_4^2\right)f_{02}(1+f_{03})(1+f_{04})\times$$

$$\times[h_4+h_3-h_2-h].$$

Removing δn_c from (9) and (10), after some algebra we obtain a closed equation like below

$$(1+f_0)\left(\frac{\partial}{\partial\tau}\hat{A}h+\mathbf{c}\cdot\nabla[\hat{A}h-2\beta g(\beta g n_{c0}\nabla^2\delta\tilde{n}+\hat{A}\delta\tilde{n})]\right)=\tilde{L}[\hat{A}h], \quad (11)$$

where $\hat{A}\equiv\left(\frac{\partial^2}{\partial\tau^2}-\frac{\beta g n_{c0}}{2}\nabla^2\right)$.

Equation (11) describes small deviation from a local equilibrium of the distribution function for atoms in a thermal cloud at the presence of condensate. It generalizes a linearized Boltzmann kinetic equation for Maxwell gas to the case of weakly interacting condensed Bose gas.

3. Propagation of the sound waves in trapped Bose gas

3.1. Dispersion relation

Here we consider the solution of the Eq. (11) in the form of plane wave, that propagates along the *z* axis with frequency ω_0 and wave number k_0. Thus,

$$h(\mathbf{c},\mathbf{r},\tau)=h_0(\mathbf{c})\mathrm{e}^{i(kz-\omega_0\tau)},$$

$$\delta\tilde{n}(\mathbf{c},\mathbf{r},\tau)=\delta\tilde{n}_0(\mathbf{c})\mathrm{e}^{i(kz-\omega_0\tau)}.$$

Inserting these formulas into (11), we obtain algebraical equation for the coefficient $h_0(\mathbf{c})$

$$i(1+f_0)(c_z k[h_0(\mathbf{c})+2\beta g(1-\beta g n_{c0}A^{-1}k^2)\delta\tilde{n}_0(\mathbf{c})]-$$

$$-\omega_0 h_0(\mathbf{c}))=\tilde{L}[h_0(\mathbf{c})], \quad (12)$$

where $A=-\omega_0^2+\frac{\beta g n_{c0}k^2}{2}$.

Following to the Wang Chang–Uhlenbeck method [6], we express the $h_0(\mathbf{c})$ in the terms of the eigenfunctions $\psi_l(\mathbf{c})$ for the linearized collision integral $\tilde{L}[\psi(\mathbf{c})]$

$$h_0(\mathbf{c})=\sum_l a_l\psi_l(\mathbf{c}). \quad (13)$$

Some standard mathematical manipulations yields with (12) and (13) the system algebraic equations for the expansion coefficients a_l:

$$\sum_l a_l[ik(M_{ml}+2\beta g(1-\beta g n_{c0}A^{-1}k^2)N_{ml})+$$

$$+\lambda_l Q_{ml}-i\omega_0\delta_{ml}]=0, \quad (14)$$

where the matrix elements are given by

$$M_{ml}=\int d\mathbf{c}f_0(\mathbf{c})(1+f_0(\mathbf{c}))\psi_m(\mathbf{c})c_z\psi_l(\mathbf{c}),$$

$$N_{ml}=\frac{1}{\pi^{3/2}\Lambda^3}\int d\mathbf{c}f_0(\mathbf{c})(1+f_0(\mathbf{c}))\psi_m(\mathbf{c})c_z\times$$

$$\times\int d\mathbf{c}'f_0(\mathbf{c}')(1+f_0(\mathbf{c}')\psi_l(\mathbf{c}'),$$

$$Q_{ml}=\int d\mathbf{c}f_0(\mathbf{c})\psi_m(\mathbf{c})\psi_l(\mathbf{c}),$$

and λ are eigenvalues.

Nontrivial solution of (14) corresponds to zero value of the determinant

$$\det\left\|ik(M_{ml}+2\beta g(1-\beta g n_{c0}A^{-1}k^2)N_{ml})-i\omega_0\delta_{ml}+\lambda_l Q_{ml}\right\|=0. \quad (15)$$

This equation defines a sound wave dispersion relation.

In the first approximation (without dissipation), keeping in the determinant (15) only first three columns and three rows that correspond to $\lambda=0$ ($\psi_1\sim 1, \psi_2\sim c_z, \psi_2\sim c_z$, $\psi_3\sim\mathbf{c}^2$), we find

$$\omega_0[\omega_0^2-k^2(M_{12}^2+M_{23}^2+2\beta g(1-\beta g n_{c0}A^{-1}k^2)N_{12}M_{12})]=0. \quad (16)$$

The trivial solution of this equation $\omega_0=0$ describes the heat relaxation mode.

Using expressions for the matrix elements (see Eq. (B.4)) in the Eq. (16), we obtain

$$\omega_0^2-k^2\left(v_0^2+\beta g(1-\beta g n_{c0}A^{-1}k^2)\tilde{n}_0\right)=0, \quad (17)$$

where $v_0\sqrt{2/\beta m}=\sqrt{(5/3)P/\tilde{\rho}_0}$, and v_0 is a dimensionless sound velocity for ideal gas. P and $\tilde{\rho}_0$ are pressure and mass distribution of the thermal cloud, respectively.

3.2. First and second sounds

In this subsection we find sound velocities. In terms of Eq. (17) for the dimensionless sound velocity $u_0=\omega_0/k$, we obtain

$$u_0^4-u_0^2\left(v_0^2+\beta g\tilde{n}_0+\frac{\beta g n_{c0}}{2}\right)+\frac{\beta g n_{c0}}{2}\left(v_0^2-\beta g\tilde{n}_0\right)=0\,.$$

This biquadratic equation has the next solutions indexed by "$\pm$" sign:

$$u_{0\pm}^2=\frac{1}{2}\left[v_0^2+\beta g\tilde{n}_0+\frac{\beta g n_{c0}}{2}\pm\right.$$

$$\pm\sqrt{\left(v_0^2+\beta g\tilde{n}_0+\frac{\beta g n_{c0}}{2}\right)^2-2\beta g\left(v_0^2-\beta g\tilde{n}_0\right)}\Bigg].$$

The power series expansion for the square root with respect to g gives

$$u_{0+}^2=v_0^2+\beta g\tilde{n}_0+\frac{\beta g n_{c0}}{2}\zeta,\quad u_{0-}^2=\frac{\beta g n_{c0}}{2}-\frac{\beta g n_{c0}}{2}\zeta,$$

where $\zeta=\dfrac{2\beta g\tilde{n}_0}{v_0^2}$.

Turning back to dimensional quantities, finds

$$u_1^2=\frac{2}{\beta m}u_{0+}^2=\frac{5}{3}\frac{P}{m\tilde{n}_0}+\frac{2g\tilde{n}_0}{m}+\frac{g n_{c0}}{m}\zeta, \qquad (18)$$

$$u_2^2=\frac{2}{\beta m}u_{0-}^2=\frac{g n_{c0}}{m}-\frac{g n_{c0}}{m}\zeta. \qquad (19)$$

Thus, in the dilute condensed Bose gas two sound modes can propagate with velocities u_1 and u_2. In the case when condensate is absent $(n_{c0}=0)$, the velocity u_2 is equal to zero, and u_1 reduces to the velocity of sound for weakly interacting gas. Acoustic modes u_1 and u_2 correspond to the first and second sounds. Expressions for u_1 and u_2 coincide with accordingly velocities that were obtained by Griffin and Zaremba [5], using the method of linearized hydrodynamic equations, and are in good agreement with experimental data in [14].

4. Conclusion

Thus, in this work it was conducted the theoretical investigation of the acoustic modes in trapped condensed Bose gas at the nonzero temperatures, when a thermal cloud of excited atoms is present along with the condensate. The calculation is based on the system of the linearized Gross–Pitaevskii and the kinetic Boltzmann equations.

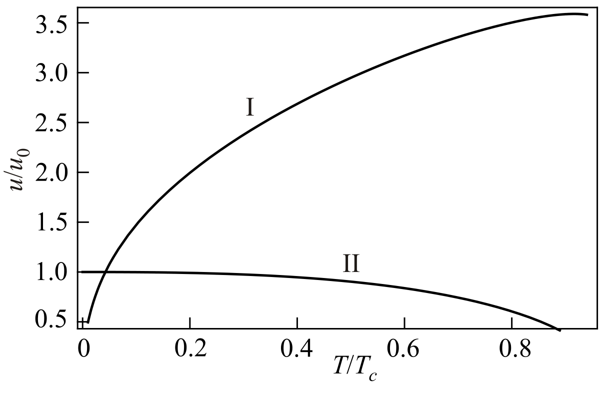

Fig. 1. First (I) and second (II) sounds velocities (normalized by $u_0=\sqrt{gn_0/m}$) plotted as a function of the reduced temperature T/T_c for a sample of ^{23}Na atoms. Scattering length a = 2.75 nm, total number of ^{23}Na atoms $N=10^9$ [3]. In this situation, one has a high density cloud of $n_0\sim10^{20}$ m^{-3}.

The use of the system of eigenfunctions for the introduced linearized collision integral (13) allowed to obtain the dispersion relation for the acoustic waves in the form of infinite determinant. Such structure of the dispersion relation permits the application of the successive approximations method. Confining the approximations to three rows and columns in the determinant, that correspond to zero eigenvalues, we derive the dispersion relation for "sounds" in the ideal case, that is without damping effects. The obtained cubic equation has three solutions. Two of them are nontrivial, corresponding the first and second sounds, and the trivial one, that describes the heat relaxation mode. These results are in agreement with the corresponding expressions, generated by other theoretical methods, and with experimental data.

The application of the method of successive approximations for the dispersive relation in the form of the infinite determinant also allows to describe damping effects, that can prove the effectiveness and flexibility of the proposed here approach. To perform the latter it is necessary to take into account more items (rows and columns) in the expansion of the determinant in series with the wave vector as a parameter. Then the main problem consists in computing the nontrivial eigenvalues and the corresponding eigenfunctions for the linearized collision integral.

Appendix A: Derivation of the linearized Boltzmann quantum kinetic equation

We proceed to the linearized Boltzmann quantum kinetic equation. Let $f=f_0+\delta f$, here f_0 is definable from the constraint $C_{22}[f_0]+C_{12}[f_0]=0$. Such condition can be satisfied by the Bose–Einstein distribution function

$$f_0(\mathbf{p},\mathbf{r},t)=\frac{1}{e^{\beta(p^2/2m+V_{\text{eff}}^0-\tilde{\mu})}-1}. \qquad (A.1)$$

Notice, that the distribution function (A.1) satisfies the condition $C_{22}[f_0]=0$ for any value of chemical potential and the condition $C_{12}[f_0]=0$ in case of $\tilde{\mu}=\mu_c$, by other words, when a diffusion equilibrium occurs between the condensate and thermal cloud.

Linearizing the collision integrals C_{22} and C_{12}, we use the following expressions:

$$\delta C[\delta f,f_0]=C[f_0+\delta f]-C[f_0]=$$

$$=\sum_{i=1}^{4}\int d\mathbf{p}_i\ \delta f(\mathbf{p}_i,\mathbf{r},t)\frac{\delta C[f]}{\delta f_i}\Big|_{f_i=f_{0i}}. \qquad (A.2)$$

Since the collision integrals C_{22} and C_{12}, defined by the expressions (3) and (2), have such structure like the following:

$$C[f_1]=\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\ P(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)F[\{f(\mathbf{p}_k,\mathbf{r},t)\}],$$

then

$$\delta C[\delta f_1, f_0] = \sum_{i=1}^{4}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\; P(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)\times$$

$$\times\delta f(\mathbf{p}_i,\mathbf{r},t)\frac{\partial F[\{f(\mathbf{p}_k,\mathbf{r},t)\}]}{\partial f_i}\bigg|_{f_i=f_{0i}}. \quad \text{(A.3)}$$

For the collision integral C_{22} the introduced function $F[\{f(\mathbf{p}_k,\mathbf{r},t)\}]$ can be expressed in the following manner:

$$F[\{f(\mathbf{p}_k,\mathbf{r},t)\}] = (1+f_1)(1+f_2)f_3f_4 - f_1f_2(1+f_3)(1+f_4).$$

After a rather simple calculation, it can be derived that

$$\sum_{i=1}^{4}\delta f(\mathbf{p}_i,\mathbf{r},t)\frac{\partial F[\{f(\mathbf{p}_k,\mathbf{r},t)\}]}{\partial f_i}\bigg|_{f_i=f_{0i}} =$$

$$= (1+f_{01})(1+f_{02})f_{03}f_{04}\sum_{i=1}^{4}s_i\frac{\delta f(\mathbf{p}_i,\mathbf{r},t)}{f_{0i}(1+f_{0i})}, \quad \text{(A.4)}$$

where $s_1 = s_2 = -1$, $s_3 = s_4 = 1$.

Setting $\delta f(\mathbf{p}_i,\mathbf{r},t) = f_{0i}(1+f_{0i})h_i$, where h is a small correlation to the distribution function, we obtain the linearized collision integral C_{22}, as it is below

$$L_{22}[h] = \frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\delta\left(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4\right)\times$$

$$\times\delta\left(\tilde{\varepsilon}_p+\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3}-\tilde{\varepsilon}_{p_4}\right)f_0f_{02}(1+f_{03})(1+f_{04})\times$$

$$\times[h_4+h_3-h_2-h]. \quad \text{(A.5)}$$

One similarly derives the linearized collision integral C_{12}:

$$L_{12}[h] = \frac{2g^2n_{c0}}{(2\pi)^2\hbar^4}\int d\mathbf{p}_1\int d\mathbf{p}_2\int d\mathbf{p}_3\delta\left(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3\right)\times$$

$$\times\delta(\mu_{c0}+\tilde{\varepsilon}_{p_1}-\tilde{\varepsilon}_{p_2}-\tilde{\varepsilon}_{p_3})[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_2)-$$

$$-\delta(\mathbf{p}-\mathbf{p}_3)]f_{01}(1+f_{02})(1+f_{03})[h_3+h_2-h_1]. \quad \text{(A.6)}$$

As it was shown in [8], when a diffusion equilibrium takes place, the quantity L_{12} can be neglected. Considering the case of diffusion balance during sound-wave propagation, we disregard the value of L_{12}.

Now we linearize the left part of kinetic equation. We have

$$\frac{\partial f(\mathbf{p},\mathbf{r},t)}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla f(\mathbf{p},\mathbf{r},t)-\nabla V_{\text{eff}}(\mathbf{r},t)\cdot\nabla_{\mathbf{p}}f(\mathbf{p},\mathbf{r},t) =$$

$$=\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla-\nabla V_{\text{eff}}(\mathbf{r},t)\cdot\nabla_{\mathbf{p}}\right)f_0(\mathbf{p},\mathbf{r},t)+$$

$$+\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla\right)\delta f(\mathbf{p},\mathbf{r},t)-2g\nabla(\delta n_c+\delta\tilde{n})\cdot\nabla_{\mathbf{p}}f_0(\mathbf{p},\mathbf{r},t) =$$

$$= f_0(1+f_0)\left(\frac{\partial h}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla(h-2g\beta(\delta n_c+\delta\tilde{n}))\right). \quad \text{(A.7)}$$

Here, it is taken into account that, in accordance with the definition of f_0, the following expression is true

$$\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla-\nabla V_{\text{eff}}(\mathbf{r},t)\cdot\nabla_{\mathbf{p}}\right)f_0(\mathbf{p},\mathbf{r},t) = 0,$$

$$\nabla_{\mathbf{p}}f_0 = f_0(1+f_0)\beta\frac{\mathbf{p}}{m}.$$

Therefore, the linearized Boltzmann kinetic equation obtains the final form:

$$f_0(1+f_0)\left(\frac{\partial h}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla(h-2\beta g(\delta n_c+\delta\tilde{n}))\right) = L_{22}[h]. \quad \text{(A.8)}$$

Appendix B: Eigensystems of $L[h]$

Because of the structure of the collision integral, the three eigenvalues, corresponding to the eigenfunctions 1, c_z, $\mathbf{c}^2$, equal zero. This is due to conservation laws for particle number, momentum, and energy, that are fulfilled during particle collisions. Farther discussion here will be restricted to the case of eigenfunctions with zero eigenvalues. Because the three eigenfunctions correspond to the same eigenvalue $\lambda = 0$, their linear combination also is an eigenfunction with the eigenvalue $\lambda = 0$. Below, we build linear combinations of the eigenfunctions, constructing an orthonormal basis with the weight $f_0(1+f_0)$.

Let $\phi_1 = 1$, $\phi_2 = c_z$, $\phi_3 = \mathbf{c}^2$. Normalized eigenfunctions are denoted as ψ_l and defined by the ratio

$$\psi_l = \frac{\phi_l}{\|\phi_l\|},$$

where

$$\|\phi_l\|^2 = \frac{1}{\pi^{3/2}\Lambda^3}\int d\mathbf{c} f_0(1+f_0)|\phi_l|^2.$$

Then we have that

$$\|\phi_1\|^2 = \frac{1}{\pi^{3/2}\Lambda^3}\int d\mathbf{c} f_0(1+f_0)|\phi_1|^2 = \frac{1}{\pi^{3/2}\Lambda^3}\int d\mathbf{c} f_0(1+f_0).$$

Because of

$$f_0(1+f_0) = \frac{df_0}{d\alpha} = \frac{d}{d\alpha}\frac{1}{e^{c^2-\alpha}-1},$$

then

$$\|\phi_1\|^2 = \frac{1}{\pi^{3/2}\Lambda^3}4\pi\frac{d}{d\alpha}\int_0^\infty dc\frac{c^2}{e^{c^2-\alpha}-1} =$$

$$= \frac{2}{\pi^{1/2}\Lambda^3} \frac{d}{d\alpha} \int_0^\infty dx \frac{x^{1/2}}{e^{x-\alpha}-1} =$$

$$= \frac{2}{\pi^{1/2}\Lambda^3} \frac{\pi^{1/2}}{2} \frac{dg_{3/2}(\alpha)}{d\alpha} = \frac{g_{1/2}(\alpha)}{\Lambda^3}.$$

Here we used the following definition of the Bose–Einstein g function in the derivation of the above expression:

$$g_{(2n+1)/2}(e^\alpha) = \frac{2^n}{(2n-1)!!\sqrt{\pi}} \int_0^\infty dx \frac{x^{(2n-1)/2}}{e^{x-\alpha}-1},$$

and the apparent ratio

$$\frac{dg_n(e^\alpha)}{d\alpha} = g_{n-1}(e^\alpha),$$

that follows from the presentation of g function as a sum

$$g_n(x) = \sum_{l=1}^\infty \frac{x^l}{l^n}.$$

Hence,

$$\psi_1 = \sqrt{\frac{\Lambda^3}{g_{1/2}(\alpha)}}. \qquad \text{(B.1)}$$

Similarly, it can be found that

$$\psi_2 = \sqrt{\frac{2\Lambda^3}{g_{3/2}(\alpha)}} c_z. \qquad \text{(B.2)}$$

Next, we check up the orthogonality of ψ_1 and ψ_2:

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{\pi^{3/2}\Lambda^3} \int d\mathbf{c} f_0 (1+f_0) \psi_1 \psi_2 =$$

$$= \frac{2}{\pi^{3/2} g_{1/2}(\alpha) g_{3/2}(\alpha)} \underbrace{\int_0^\pi d\theta \sin\theta \cos\theta}_{0} \frac{d}{d\alpha} \int_0^\infty dc \frac{c^3}{e^{c^2-\alpha}-1} = 0.$$

To orthogonalize the third "base-function" to the first and the second one, it is necessary to make choice of the following:

$$\tilde{\phi}_3 = \phi_3 - \langle \phi_3 | \psi_1 \rangle \psi_1 = c^2 - \frac{3}{2} \frac{g_{3/2}(\alpha)}{g_{1/2}(\alpha)}.$$

Then the third orthonormal eigenfunction gets the shape

$$\psi_3 = \sqrt{\frac{4\Lambda^3 g_{1/2}(\alpha)}{3(5g_{5/2}(\alpha)g_{1/2}(\alpha) - 3g_{3/2}^2(\alpha))}} \left(c^2 - \frac{3}{2} \frac{g_{3/2}(\alpha)}{g_{1/2}(\alpha)} \right). \qquad \text{(B.3)}$$

Using explicit expressions (B.1)–(B.3) for the eigenfunctions, we calculate matrix elements (15). As it can be shown,

$$M_{mm} = 0, \quad M_{12} = M_{21} = \sqrt{\frac{g_{3/2}(\alpha)}{2g_{1/2}(\alpha)}},$$

$$M_{23} = M_{32} = \sqrt{\frac{5g_{5/2}(\alpha)}{6g_{3/2}(\alpha)} - \frac{g_{3/2}(\alpha)}{2g_{1/2}(\alpha)}},$$

$$M_{13} = M_{31} = 0, \quad N_{ml} = N_{12}\delta_{1m}\delta_{2l},$$

$$N_{12} = \frac{1}{\Lambda^3} \sqrt{\frac{g_{1/2}(\alpha)g_{3/2}(\alpha)}{2}}. \qquad \text{(B.4)}$$

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