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Phonons and rotons of trapped atoms in gravitational field

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The excitations of trapped atoms with a Bose-Einstein condensate in a trap are determined by the conservation of common phonon and roton numbers of atomic motion, and these properties depend on the presence of gravitational field.

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Introduction

Phonon-like excitations of atoms in traps are observed experimentally. If we rotate a trap, the new form of atomic motion — rotons — may be expected. In ^{4}He the same maxon–roton excitations are followed by vortexes. They are defined as a singularity points of a velocity field rot $V(r) = 0$, connected to the condition for an amplitude ψ to become zero $\psi(r) = 0$ [1,2]. Here we are going to look for the properties of atomic Bose-Einstein condensate (BEC) in traps in the rotational (potential) stage of non-singular excitations. We show, that roton properties of a gas in a trap follows from the conservation of a common number of photons and rotons, found in our model. The additional potential energy of trapped atoms is promoted by coming of atoms outside of BEC via the increasing of phonon and roton energy due to the gravitational field, and these contributions are estimated below.

Phonons and rotons

The energies of translations $\mathcal{H}_{\rm ph}$ and orbital rotations $\mathcal{H}_{\rm rot}$ of atoms can be separated if the origin of coordinates is taken as the point of «equilibrium», such as the centres of a circle (2D) or a sphere (3D). Each of N atoms in a trap contributes to phonon motion, provided by an interaction between them, and to rotations as well. So we represent the Hamiltonian of N atoms as the sum of translational $\mathcal{H}_{\rm ph}$, rotational $\mathcal{H}_{\rm rot}$ energies and the rotation-translation interaction $\mathcal{H}_{\rm ph-rot}$

$$\mathcal{H}_{\rm ph} = \sum_k^{k_{\max}} \omega(k)\, \psi_k^+ \psi_k ,$$

$$\mathcal{H}_{\rm rot} = \sum_{i=1}^{N} \left(\frac{\Delta_i}{2m} + \Omega L_z^i \right) + U(r),$$

$$\mathcal{H}_{\rm ph-rot} = \frac{1}{\sqrt{N}} \sum_k^{k_{\max}} \Gamma_k \sum_i^N (L_+^i \psi_k + \psi_k^+ L_-^i) , \tag{1}$$

$$[\psi_k, \psi_{k'}^+] = \delta_{kk'},\ [L_+^i, L_-^j] = 2L_z \delta_{ij},\ [L_-^i, L_z^j] = L_-^i \delta_{ij},$$

$$[L_+^i, L_z^j] = -L_+^i \delta_{ij}.$$

Here Ω is a rotational energy; $k_{\max}$ is an upper level of a trap with a potential U; L are the operators of an orbital momentum; Γ_k is phonon–roton interaction; $\omega(k)$ is the energy of phonon with the momentum k of an atom in a rectangular trap, expressed by the well known equation [3]

$$ka = \pi n - 2 \arcsin \frac{\hbar k}{\sqrt{2mU_0}} ,$$

where U_0 is a potential of a trap and a is a trap width. Really, phonons and rotons transform to each other, and a connection between them is given by the integral of motion M

$$[\mathcal{H}, M] = 0,\quad M = \sum_k^{k_{\max}} \psi_k^+ \psi_k + \sum_{i=1}^{N} L_z^i ,$$

$$[\mathcal{H}, K] = 0,\quad K = \sum_{i=1}^{N} L_i^2 , \tag{2}$$

$$\mathcal{H} = \mathcal{H}_{\rm rot} + \mathcal{H}_{\rm ph} + \mathcal{H}_{\rm ph-rot}$$

of a system (1), represented by Hamiltonian $\mathcal{H}$. It is possible to use the mapping of rotational variables to a couple of Fermi operators for each of $i = 1, 2 \ldots N$ momentum operators by the use of substitution

$$2L_z = a^+a - b^+b, \;\; L_+ = a^+b, \;\; L_- = b^+a,$$
$$\{a,a\} = \{a,b\} = \{b,b\} = 0,$$
$$\{a^+,a\} = \{b^+,b\} = 1, \{a,b^+\} = 0$$

(we omit the i numeration and show a very simple one-particle $\mathcal{H} \to h$ version of a model in the next few formulas) . The latter definitions lead to the relations

$$[a^+a + b^+b, a^+a - b^+b] = 0, \quad [a^+a + b^+b, a^+b] = 0,$$

so that the initial h may be written in boson-fermion variables with $L^2 \to a^+a + b^+b$ for each atom in our mapping

$$h = \omega\psi^+\psi + \frac{\Omega_a}{2}a^+a - \frac{\Omega_b}{2}b^+b + \Gamma(a^+b\psi + \psi^+b^+a),$$
$$[h, \psi^+\psi + \frac{1}{2}(a^+a - b^+b)] = [h, a^+a + b^+b] = 0,$$

where the energies $\Omega_{a,b}$ of fermions contain their kinetic energies. These relations are truth for a large $k_{\max}$, so that a number m_0 of traped levels satisfy an equation $m_0 >> 1$, just like it happens in experiment. If we turn back to the total Hamiltonian $\mathcal{H}$, the thermodynamical parameters μ and ν will correspond the integrals of motion K and M

$$\mathcal{H} - \mu K - \nu M = \mathcal{H}_{\text{ph-rot}} + \sum_k^{k_{\max}} \psi_k^+\psi_k[\omega(k) - \nu] +$$
$$+\sum_{i=1}^{N} [\Omega_+^a a_i^+ a_i + \Omega_-^b b_i^+ b_i] \, ,$$
$$\Omega_\pm^{a,b} = \Omega_{a,b} - \mu_\pm, \;\; \mu_\pm = \mu \pm \nu/2 \, ,$$
$$\{a_i, a_j^+\} = \{b_i^+, b_j\} = \delta_{ij}$$

for the partition function of a system. The latter may be represented as the path integral for a large Gibbs distribution

$$Q = \text{Sp}\, \exp\, [-\beta\, (\mathcal{H} - \mu K - \nu M)] =$$
$$= \int \prod_k D\, \psi_k^* D\psi_k \prod_{i=1}^{N} Da_i^* Da_i Db_i^* Db_i \;\; \exp S,$$
$$S = \int_0^\beta L\, dt$$

over all the boson ψ, ψ^* (and fermion a, a^*, b, b^*) trajectories, satisfying the periodical (and antiperiodical) boundary conditions on $[0,\beta]$ for each k (and i). The Lagrange function L represents all the degrees of freedom of the system (1) in an ordinary way. Integral over all fermionic (a,b) fields in Q may be calculated exactly as Det $(\delta^2 S)$ [4,5], so we get a path integral over the variables ψ_k, ψ_k^* with an effective action S_{ef} for every k

$$Q = \int \prod_k D\, \psi_k^* D\psi_k \exp S_{\text{ef}}\,(0,\beta),$$
$$S_{\text{ef}}\,(0,\beta) = S_{\text{ph}} + N \ln \text{Det}\, R_k,$$
$$S_{\text{ph}} = \int_0^\beta \psi_k^* \left(-\frac{d}{dt} - \omega_k + \nu \right) \psi_k dt,$$
$$R_k = \begin{pmatrix} L_+ & \Gamma_k \psi_k / \sqrt{N} \\ \Gamma_k \psi_k^* / \sqrt{N} & L_- \end{pmatrix},$$
$$L_+ = -d/dt - \Omega_+^a, \;\; L_- = -d/dt - \Omega_-^b.$$

The quasiclassical equations of motion

$$\frac{\delta S_{\text{ef}}}{\delta \psi_k} = \frac{\delta S_{\text{ef}}}{\delta \psi_k^*} = 0 \qquad (3)$$

are used for the extremal trajectories $(\psi_k^0)^*$ and ψ_k^0 that describe an effective translational modes in the field of roton motion. In the $N >> 1$ limit the transformation $\psi_k \to \sqrt{N}\psi_k$ and $\psi_k^* \to \sqrt{N}\psi_k^*$ leads to the re-normalized action S_{eff}

$$S_{\text{ef}} \to S_{\text{eff}} = N(S_{\text{ph}} + \ln\, \text{Det}\, R_k|_{|\psi| \to |\sqrt{N}|\psi|}),$$

so that the equation of motion (3) in a new scale of boson trajectories looks like

$$\frac{\delta S_{\text{eff}}}{\delta \psi_k^*} = \frac{\delta}{\delta \psi_k^*}(S_t + \text{Sp} \ln R_k) =$$
$$= \left(\frac{d}{dt} + \omega_k - \nu \right) \psi_k(t) - \text{Sp} \left(R_k^{-1} \frac{\delta R_k}{\delta \psi_k^*(t)} \right) = 0, \quad (4)$$

where a formula $\ln \text{Det}\, R_k = \text{Sp} \ln R_k$ was used. As K is an integral of motion for (1), the first bond relation for a thermodynamical parameters are

$$\frac{1}{\beta} \frac{\partial S_{\text{eff}}}{\partial \mu} = K = \frac{1}{\beta} \sum_k \text{Sp} \left(R_k^{-1} \frac{\partial R_k}{\partial \mu} \right). \qquad (5)$$

In the same way the second bond relation is found as

$$\frac{1}{\beta} \frac{\partial S_{\text{eff}}}{\partial \nu} = M = \frac{1}{2\beta} \left[\sum_k \text{Sp} \left(R_k^{-1} \frac{\partial R_k}{\partial \nu} \right) + \int_0^\beta |\psi_k|^2 dt \right]. \qquad (6)$$

The calculation of Sp in (4)–(6) is carried out following [4,5] both in the matrix and Path Integral sense

$$\mathrm{Sp}\ln R_k = \mathrm{Sp}\ln\left\{\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}\left[1 + \begin{pmatrix} L_+^{-1} & 0 \\ 0 & L_-^{-1} \end{pmatrix}\begin{pmatrix} 0 & \Gamma_k\psi_k \\ \Gamma_k\psi_k^* & 0 \end{pmatrix}\right]\right\} =$$

$$= \mathrm{Sp}\left\{\ln\begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} - \sum_{m=1}^{\infty}\frac{1}{2m}\left[\begin{pmatrix} L_+^{-1} & 0 \\ 0 & L_-^{-1} \end{pmatrix}\begin{pmatrix} 0 & \Gamma_k\psi_k \\ \Gamma_k\psi_k^* & 0 \end{pmatrix}\right]^{2m}\right\} =$$

$$= \frac{1}{2}\mathrm{Sp}\ln T,\;\; T = (L_+ - \Gamma_k^2\psi_k L_-^{-1}\psi_k^*)(L_- - \Gamma_k^2\psi_k^* L_+^{-1}\psi_k).$$

The particular time-independent $(\psi_k^0)^* \to \psi_0^*$, $\psi_k^0 \to \psi_0$ solution of equations (4)–(6) is similar to «slow» trajectories in superfluid ^{4}He theory [1], associated with the Bose-Einstein condensation (BEC) in quantum liquid. In the same way ψ_0^*, ψ_0 trajectories correspond BEC of atoms in a trap, observed experimentally during the last years [6]. Using BEC assumption, we get the approach

$$T \to T_0 = L_+L_- - \Gamma_0^2|\psi_0|^2$$

and see the variational equation (4) in the form

$$\omega_0 = \frac{\Gamma_0^2}{2\Omega_0}\left[\tanh\frac{\beta}{2}\left(\mu_+ + \frac{\Omega_0^+}{2}\right) - \tanh\frac{\beta}{2}\left(\mu_- - \frac{\Omega_0^-}{2}\right)\right],$$

$$\Omega_0^{\pm} = (\Omega_{\pm}^2 + 4\Gamma_0^2|\psi_0|^2)^{1/2}, \tag{7}$$

while the equations for integrals of motion K, M appears as

$$K = \frac{1}{2}\sum_k\left[\tanh\frac{\beta}{2}\left(\mu_+ + \frac{\Omega_k^+}{2}\right) + \tanh\frac{\beta}{2}\left(\mu_- - \frac{\Omega_k^-}{2}\right)\right] + 2,$$

$$M = \frac{1}{2}\sum_k\left[\tanh\frac{\beta}{2}\left(\nu + \frac{\Omega_k^+}{2}\right) - \tanh\frac{\beta}{2}\left(\nu + \frac{\Omega_k^-}{2}\right)\right] +$$

$$+ 2 + |\psi_k^{\cdot}|^2.$$

These equations determine the BEC ordering in a trap via the trajectories ψ_0^*, ψ_0 and chemical potentials μ, ν in terms of energies $\omega(k)$, Ω, and Γ_k.

The right side of the equation (7) can be represented as

$$\omega_0 = \frac{\Gamma_0^2}{2\Omega_0}\frac{\sinh(A_0^+ - A_0^-)}{\cosh A_0^+\cosh A_0^-},$$

$$A_0^{\pm} = \frac{\beta}{2}\left(\mu_{\pm} \pm \frac{\Omega_0^{\pm}}{2}\right),\;\; \mu_{\pm} = \mu \pm \frac{\nu}{2}$$

and in the case of a small phonon-roton interaction $\Gamma_0 < \Omega$ the approximation

$$\Omega_0^{\pm} \simeq \Omega_{\pm} + \frac{2\Gamma_0^2|\psi_0^2|}{\Omega\pm},\quad \Omega\beta << 1$$

is fulfilled, so that $\sinh(A_0^+ - A_0^-) << 1$. Therefore, the inequality

$$\Gamma_0^2 << \omega_0\Omega_0,\;\; \Omega_0 = \sqrt{\Omega^2 + 4|\psi_0|^2\,\Gamma_0^2} \tag{8}$$

is valid. It means, that BEC trajectories exist only for low phonon-roton interaction. It is worthy to note, that for an interaction Γ between polar atoms in segnetoelectric media, expressed by the same model as (1), the condition of dipol cooperation is expressed by the inequality [4]

$$\Gamma^2 >> \omega\Omega,$$

where ω and Ω stands as the radiation and two-level system frequencies correspondingly. Just the opposite, the inequality (8) follows from the previous formulas as the condition for the equations (4)–(6) to have the BEC solution, and the condition of a large BEC density $\omega_0|\psi_0| >> \Gamma_0$ is also satisfied. The complete line of relations between the parameters of a theory looks like

$$\frac{\Gamma_0}{|\psi_0|} < \frac{\Omega}{|\psi_0^2|} < \omega_0,\; \Omega\beta << 1 \tag{9}$$

where both the BEC condition (8) and square root decomposition are taken into account. It can be seen, that the left side of (9) leads to formula (8).

Trapped atoms in gravitational field

Now we are going to look for the properties of a trapped atomic (BEC) in gravitational field. Let us consider a pure phonon Hamiltonian $\mathcal{H}_{\mathrm{ph}}^p$ (without the influence of gravitational field) for finite trap potential. In this case the ordinary commutation relations for phonon operators is broken and can be written in the form

$$[\Psi_k, \Psi_k^+] = f(\Psi_k, \Psi_k^+), \tag{10}$$

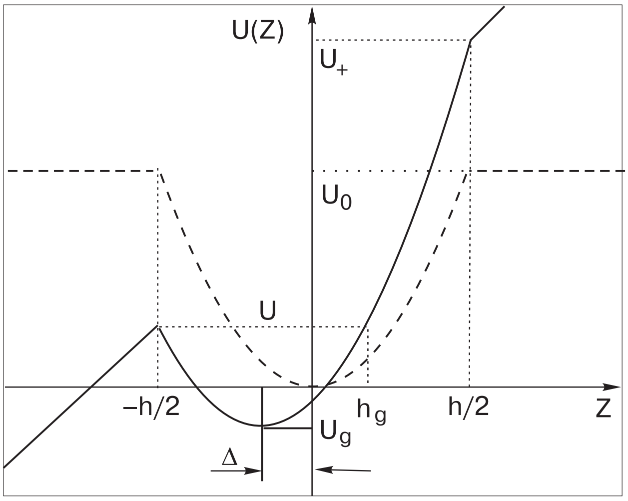

Fig. 1. The deformation of the trap potential. The dashed and solid lines for a parabolic trap potential in a free space (U_0) and in a gravitational field (U_-, U_+).

where $f(\Psi_k, \Psi_k^+)$ is a polynomial function of operators Ψ_k and Ψ_k^+. Now the Hamiltonian of gravitational field in coordinate space $\mathcal{H}_q = mgz$ can be written in initial double quantized particle operators b_k, b_k^+

$$\mathcal{H}_g = c b_k^+ b_k, \text{ where } c \text{ is constant.}$$

After diagonalizati on, $\mathcal{H}_g$ can be represented in terms of phonon operators Ψ_k, Ψ_k^+

$$\begin{cases} b_k = \alpha_k \Psi_k + \beta_k \Psi_k^+, \\ b_k^+ = \alpha_k \Psi_k^+ + \beta_k \Psi_k, \end{cases}$$

$$\mathcal{H}_g = c_1 + c_2 \Psi_k^+ \Psi_k + c_3 (\Psi_k^+ \Psi_k^+ + \Psi_k \Psi_k).$$

For the case of finite trap (10) the commutator of total Hamiltonian $\mathcal{H} = \mathcal{H}_{\text{ph}}^p + \mathcal{H}_{\text{rot}} + \mathcal{H}_{\text{ph-rot}}$ with the Hamiltonian of gravitational field $\mathcal{H}_g$ is not equal to zero.

$$[\mathcal{H}_{\text{ph}}^p + \mathcal{H}_{\text{rot}} + \mathcal{H}_{\text{ph-rot}}, \mathcal{H}_g] \neq 0.$$

The deformation of the trap potential is shown on Fig. 1. Following the paper [7] we can conclude that the gravitational field manifest itself in decreasing of upper level $k_{\max}$ and in the shift of energy levels in the trap. In particular, for ideal bose gas in the trap [7], the critical temperature and number of condensate particles are compared for the cases of gravitational field and in microgravity environment. It was shown, that the presence of gravitational field leads to the decreasing of condensate fraction and increasing of critical temperature if we keep constant the total number of particles in trap.

Conclusion

The dilute gas of interacting bosons in magnetic trap under rotation was considered as a system with Hamiltonian $\mathcal{H}$(1), (2), which consists of three terms: phonon Hamiltonian $\mathcal{H}_{\text{ph}}$, Hamiltonian of roton excitations $\mathcal{H}_{\text{rot}}$ and Hamiltonian of phonon-roton interaction $\mathcal{H}_{\text{ph-rot}}$. The partition function of the system was written as a path integral over boson and fermion trajectories with integrals of motion (2). After integration over fermions, the quasiclassical equations of motion for boson trajectory in the ground state with their integrals of motion were obtained. The conclusions on the equations (4)–(9) are as follows:

a) for a given Γ_0 the inequality (8) can be satisfied for large $|\psi_0|$, so the presence of rotons diminishes the BEC density;

b) the less amount of BEC bosons $|\psi_0^2|$ we have, the larger will be a rotational momentum of atom for a given trap.

These conclusions seems to be in accordance with the prioriting Popov [1] result, that the zero points of boson trajectory $\psi = 0$ are starting points for a vortexes in superfluid ^{4}He. As to the influence of gravity, we note here the point (c) as follows: c) the gravitational field diminishes the BEC density in a trap.

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