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TWO-PHASE STEFAN PROBLEM FOR ELLIPTIC AND PARABOLIC EQUATIONS

The Stefan problem in its classical statement is a mathematical model of the process of propagation of heat in a medium with different phase states, e.g., in a medium with liquid and solid phases. The process of propagation of heat in each phase is described by usually parabolic equation. In the presented work we assume, that a process in the solid phase is described by the parabolic equation and in the liquid phase is described by the elliptic equation. In this article we proved the existence of the global classical solution in many-dimensional space.

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1. The statement of the problem.

The Stefan problem in its classical statement is a mathematical model of the process of propagation of heat in medium with different phase states, e.g., in medium with liquid and in solid phase. As a result of melting or crystallization, the domains occupied by the liquid and solid phases undergo certain changes. This unknown interface is called a free boundary. The process of propagation of heat in each phase is described by the heat equation. If the process of propagation of heat in the liquid and solid phases is described by an elliptic equation, then the corresponding mathematical model is called the Hele-Show model. In the present work we assume that the process in the solid phase described by a parabolic equation and in the liquid phase described by an elliptic equation.In this work we prove the existence of the global classical solution.

Let $D = \{x \in \mathbb{R}^3 : \quad 0 < R_1 < |x| < R_2\}$, $D_T = D \times (0,T)$, $B_i = \{x \in \mathbb{R}^3 : |x| < R_i\}$, $i = 1, 2$, $T > 0$ is a fixed number. The problem is to find a function $u(x,t)$ and domains Ω_T, G_T, which satisfy

$$\Delta u - \frac{\partial u}{\partial t} = 0 \quad \text{in} \quad \Omega_T, \;\; \Delta u = 0 \;\text{ in }\; G_T, \tag{1.1}$$

$$\Omega_T = \{(x,t) \in D_T : 0 < u(x,t) < 1\}, \quad G_T = \{(x,t) \in D_T : u(x,t) > 1\}.$$

On the known boundary

$$u(x,t) = 0 \quad \text{on} \quad \partial B_1 \times (0,T), \;\; u(x,t) = \varphi(x,t) > 1 \quad \text{on} \quad \partial B_2 \times (0,T), \tag{1.2}$$

On the unknown (free) boundary $\gamma_T = \partial\Omega_T \cap D_T = \partial G_T \cap D_T$

$$u^+ = u^- = 1, \quad \sum_{k=1}^{3} \left(\frac{\partial u^-}{\partial x_i} - \frac{\partial u^+}{\partial x_i} \right) \cos(n, x_i) + \lambda \cos(n,t) = 0, \tag{1.3}$$

where λ is a positive constant, n is the normal to the surface γ_T directed to the side of increase of the function $u(x,t)$; $u^+(x,t), u^-(x,t)$ are the boundary values on the surface γ_T taken from the domains G_T, Ω_T respectively.
The initial conditions are

$$u(x,0) = \psi(x) \quad \text{in} \quad \overline{\Omega}_0, \;\; \psi(x) = 0 \quad \text{on} \quad \partial B_1, \quad \psi(x) = \varphi(x,0) > 1 \quad \text{on} \quad \partial B_2, \tag{1.4}$$

$$\Omega_0 = \{x \in D : 0 < \psi(x) < 1\}, \quad G_0 = \{x \in D : \psi(x) > 1\},\ \gamma_0 = \partial\Omega_0 \cap D = \partial G_0 \cap D,$$
$$\psi(x) = 1 \text{ on } \gamma_0, \ \ \Delta\psi = 0 \text{ in } G_0.$$

The function $u(x,t)$ is interpreted as the temperature of the medium, γ_T is the interface between the liquid and solid phases, $u(x,t) = 1$ is the temperature of melting.

The paper is organized as follows. In section 2 we construct a difference-differential approximation for our problem. The properties of the fundamental solutions are considered in section 3. In section 4 we prove uniform estimates and pass to the limit. Note that similar methods have been used in [1], [2].

In more simple statement the similar problem was studied in work [3].

2. Construction of the approximating problem.

Assume that problem has a classical solution. Multiply the equation (1.1) by a smooth function $\eta(x,t)$ which vanishes on ∂D_T and integrate by parts:

$$\int\limits_{D_T} [\nabla u \nabla \eta + a\chi(u) u_t \eta + \lambda \chi(u) \eta_t]\, dxdt = 0.$$

Let smooth out the function $\chi(u)$. For $\forall \varepsilon > 0$ we introduce a function $\chi_\varepsilon(\tau) \in C^\infty(\mathbb{R}^1)$:

$$\chi_\varepsilon(\tau) = 1\ \forall \tau \le 1-\varepsilon,\ \chi_\varepsilon(\tau) = 0\ \forall \tau \ge 1,\ \chi'_\varepsilon(\tau) \le 0.$$

Define the function $\{u^\varepsilon(x,t)\}$ as solutions of the following problem:

$$\Delta u^\varepsilon(x,t) - [a\chi_\varepsilon(u^\varepsilon) + \varepsilon]\frac{\partial u^\varepsilon(x,t)}{\partial t} = -\lambda \frac{\partial \chi_\varepsilon(u^\varepsilon(x,t))}{\partial t},$$
$$u^\varepsilon(x,t) = 0 \text{ on } \partial B_1 \times [0,T), \ \ u^\varepsilon(x,t) = \varphi(x,t) \text{ on } \partial B_2 \times [0,T), \tag{2.1}$$
$$u^\varepsilon(x,0) = \psi(x), \text{ in } D.$$

The obtained equation (2.1) has got smooth coefficients and is parabolic everywhere in the domain D.

Let the following conditions be satisfied:

$$\psi(x) \in C^{2+\alpha}(\overline{D}),\ \varphi(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D_T})$$

and assume that corresponding compatibility conditions at $t = 0, x \in \partial D$ hold. The solvability of this problem is evident [4]. The estimation takes place

$$\left\| u^\varepsilon(x,t) \right\|_{H^{2+\alpha,1+\alpha/2}(\overline{D_T})} \le \frac{c}{\varepsilon^\nu}, \tag{2.2}$$

the positive constant ν and c do not ε. Let us construct a system of approximating problems. We divide the cylinder D_T by the planes $t = kh, k = 1, 2, ...N, Nh = T$, integrate equation (2.1) with respect to the variable t, from $(k-1)h$ to kh and multiply by $1/h$. After simple transformations we obtain

$$\Delta u_k^\varepsilon(x) - [a\chi_\varepsilon(u^\varepsilon) + \varepsilon]\frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h} = -\lambda \frac{\chi_\varepsilon(u_k^\varepsilon(x)) - \chi_\varepsilon(u_{k-1}^\varepsilon(x))}{h} - f_k^\varepsilon, \tag{2.3}$$

where $u_k^\varepsilon(x) = u^\varepsilon(x, kh), |f_k^\varepsilon(x)| \le \frac{ch^{\alpha/2}}{\varepsilon^\nu}$. Everywhere in the further we shall assume, that $\frac{h^{\alpha/2}}{\varepsilon^\nu} \le h^\delta$, δ-some positive constant. Therefore last term in the equation (2.3) can be thrown.

The equation (2.3) can be transformed to the form

$$\Delta u_k^\varepsilon(x) - a_k^\varepsilon(x)\frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h} = 0, \tag{2.4}$$

where

$$a_k^\varepsilon(x) = \varepsilon + \int_0^1 \{a\chi_\varepsilon[u_{k-1}^\varepsilon + \tau(u_k^\varepsilon - u_{k-1}^\varepsilon)] - \lambda\chi'_\varepsilon[u_{k-1}^\varepsilon + \tau(u_k^\varepsilon - u_{k-1}^\varepsilon)]\}d\tau.$$

and let's add the transformed conditions (2.1)

$$u_k^\varepsilon(x) = 0 \text{ on } \partial B_1,\ u_k^\varepsilon(x) = \varphi(x, kh) \text{ on } \partial B_2,\ u_0^\varepsilon(x) = \psi(x) \text{ in } D. \tag{2.5}$$

If $h > 0$, $\varepsilon > 0$ are fixed, the solvability of problem (2.4) - (2.5) and smoothness of the solution are known [5]. In what follows we shall show that the linear interpolations of the functions $\{u_k(x, h, \varepsilon)\}$ with respect to t converge to a solution of the Stefan problem (1.1) - (1.4) as $\varepsilon \to 0$, $h \to 0$.

3. The fundamental solutions and its properties.

For studying of the problem (2.4), (2.5) we need in the integral representation of the solution. Let $K_R(x_0)$ be ball with center at the point x_0 and the radius R and

$$\Gamma_{n-k+1}(|x - x_0|) == \frac{ih}{2\pi a_n}\oint_{\partial L}\frac{(\sinh\sqrt{z}R)^{-1}\sinh\sqrt{z}(R - |x - x_0|)dz}{4\pi|x - x_0|(1 - \frac{zh}{a_n})(1 - \frac{zh}{a_{n-1}})(1 - \frac{zh}{a_k})}, \tag{3.1}$$

where

$$L = \{z = \xi + i\eta : Rez > -\frac{\pi^2}{R^2}, |z| < \varrho\},$$

$$\left(\frac{a_1}{h}, 0\right), \left(\frac{a_2}{h}, 0\right), \ldots \left(\frac{a_n}{h}, 0\right) \in L,\quad a_i > 0, i = 1, 2, \ldots n.$$

Property 1. Let $|x - x_0| \neq 0$, then

$$\Delta\Gamma_{n-k+1} - a_k\frac{\Gamma_{n-k+1} - \Gamma_{n-k}}{h} = 0,\ \forall k = 1\ldots(n-1),$$

$$\Delta\Gamma_1 - a_n\frac{\Gamma_1}{h} = 0,\ \Gamma_1(|x - x_0|) = \frac{\sinh\sqrt{\frac{a_n}{h}}(R - |x - x_0|)}{4\pi|x - x_0|\sinh\sqrt{\frac{a_n}{h}}}. \tag{3.2}$$

Property 2. There is the estimation

$$\int_{K_R(x_0)}\frac{\Gamma_n(|x - x_0|)}{h}dx \le \int_{K_R(x_0)}\frac{\Gamma_{n-1}(|x - x_0|)}{h}dx \le \ldots$$

$$\le \int_{K_R(x_0)}\frac{\Gamma_1(|x - x_0|)}{h}dx \le \frac{1}{a_n}\left(1 - \frac{\sqrt{\frac{a_n}{h}}R}{\sinh\sqrt{\frac{a_n}{h}}R}\right) \le 1. \tag{3.3}$$

PROPERTY 3. Let $K_\delta(x_0)$ denote the ball with its center at the point x_0 and radius δ. Then

$$\lim_{\delta\to 0}\oint\limits_{\partial K_\delta(x_0}\frac{\partial\Gamma_{m-k+1}}{\partial n}ds=\begin{cases}1, \text{ if } k=m,\\ 0, \text{ if } k\neq m,\end{cases}$$

where n is the inner normal. All reduced above property can be received by immediate calculations.

PROPERTY 4. Let $\{v_k(x)\in C^2(D)\}$ and they satisfy the following equations:

$$\Delta v_k-\frac{a_kv_k-a_{k-1}v_{k-1}}{h}=-\frac{f_k-f_{k-1}}{h},$$

then there is an integral representation

$$v_m(x_0)=\int\limits_{K_R(x_0)}\frac{(a_0v_0-f_0)\Gamma_m(|x-x_0|}{h}dx-\sum_{k=1}^m\int\limits_{\partial K_R(x_0)}v_k\frac{\partial\Gamma_{m-k+1}}{\partial n}ds+$$

$$+\sum_{k=1}^m\int\limits_{K_R(x_0)}f_k\frac{\Gamma_{m-k+1}-\Gamma_{m-k}}{h}dx. \tag{3.4}$$

This integral representation follows from the previous properties of the fundamental solutions and Green's for elliptic equations.

PROPERTY 5. The functions $\{\Gamma_{m-k+1}(|x-x_0|)-\Gamma_{m-k}(|x-x_0|)\}$ change the sign on the interval $0<|x-x_0|<R$ no more than once and

$$\left|\frac{\partial\Gamma_{k-1}(r)}{\partial r}\right|_{r=R}\leq\left|\frac{\partial\Gamma_k(r)}{\partial r}\right|_{r=R}. \tag{3.5}$$

This property follows from property1 and from principle of the maximum.

PROPERTY 6. We will denote by $r_{k,k-1}$ the points, where the functions $\Gamma_k(r)-\Gamma_{k-1}(r)$, $k=1,2...n$, are equal to zero. Then we have the inequality $r_{k,k-1}\leq r_{k+1,k}$, and

$$r_{2,1}=\sqrt{h}\frac{\ln a_{n-1}-\ln a_n}{\sqrt{a_{n-1}}-\sqrt{a_n}}+o(\sqrt{h})\ \ h\to 0, \text{ if } a_{n-1}\neq a_n, \tag{3.6}$$

$$r_{2,1}=\sqrt{h}\frac{2}{a_n}, \text{ if } a_{n-1}=a_n.$$

Proof. The functions $\Gamma_k(r)-\Gamma_{k-1}(r)$ satisfy the equation

$$\Delta(\Gamma_{k+1}(|x-x_0|)-\Gamma_k(|x-x_0|))-a_{m-k}\frac{\Gamma_{k+1}(|x-x_0|)-\Gamma_k(|x-x_0|)}{h}=$$

$$=-a_{m-k+1}\frac{\Gamma_k(|x-x_0|)-\Gamma_{k-1}(|x-x_0|)}{h}. \tag{3.7}$$

We construct an integral representation at the center of the sphere $K_{r_{k,k+1}}(x_0)$

$$\Gamma_{k+1}(0)-\Gamma_k(0)=\int\limits_{K_{r_{k,k+1}}(x_0)}a_{m-k+1}\frac{\Gamma_k(|x-x_0|)-\Gamma_{k-1}(|x-x_0|)}{h}\Gamma_1(|x-x_0|)dx=$$

$$= -\int\limits_{\partial K_{r_{k,k+1}}(x_0)} (\Gamma_{k+1}(|x-x_0|) - \Gamma_k(|x-x_0|)\frac{\partial \Gamma_1}{\partial n} ds,$$

$$\Gamma_1(|x-x_0|) = \frac{\sinh\sqrt{\frac{a_{m-k}}{h}}(r_{k+1,k} - |x-x_0|)}{4\pi|x-x_0|\sinh\sqrt{\frac{a_{m-k}}{h}} r_{k+1,k}}.$$

Taking into account that the function $\Gamma_{k+1}(r) - \Gamma_k(r)$ is equal to zero at the points $r = 0, r_{k+1,k} = 0$, we obtain

$$0 = \int\limits_{K_{r_{k,k+1}}(x_0)} a_{m-k+1}\frac{\Gamma_k(|x-x_0|) - \Gamma_{k-1}(|x-x_0|)}{h}\Gamma_1(|x-x_0|)dx.$$

If $r_{k,k-1} > r_{k+1,k}$, then this equality is impossible.

Formula (3.1) implies

$$4\pi|x-x_0|\Gamma_1(|x-x_0|) = e^{-|x-x_0|\sqrt{\frac{a_m}{h}}} + O\left(e^{-R\sqrt{\frac{a_m}{h}}}\right),$$

$$4\pi|x-x_0|\Gamma_2(|x-x_0|) = \frac{a_{m-1}}{a_{m-1}-a_m}\left(e^{-|x-x_0|\sqrt{\frac{a_m}{h}}} - e^{-|x-x_0|\sqrt{\frac{a_{m-1}}{h}}}\right) +$$
$$+O\left(e^{-R\sqrt{\frac{a_{min}}{h}}}\right), \quad \text{if} \ \ a_{m-1} \neq a_m,$$

$$4\pi|x-x_0|\Gamma_2(|x-x_0|) = \frac{|x-x_0|}{2\sqrt{a_m h}}e^{-|x-x_0|\sqrt{\frac{a_m}{h}}} + O\left(e^{-R\sqrt{\frac{a_m}{h}}}\right) \quad \text{if} \ \ a_{m-1} = a_m.$$

From here we obtain

$$r_{2,1} = \sqrt{h}\frac{\ln a_{m-1} - \ln a_m}{\sqrt{a_{m-1}} - \sqrt{a_m}} + o(h) \ \ h \to 0.$$

In particular, if $a_{m-1} = a_m$, then

$$r_{2,1} = \sqrt{h}\frac{2}{\sqrt{a_m}}.$$

The function $\Gamma_2(r) - \Gamma_1(r)$ changes the sign once. Therefore, as follows from the equations (3.7), the functions $\Gamma_k(r) - \Gamma_{k-1}(r)$ change the sign once too. It means that the inequalities (3.6) hold.

Property 7. We have following estimate

$$\left|\frac{\partial \Gamma_N}{\partial n}\right| \leq M_1\left\{\frac{1}{q^N R}\exp\{-M_2\frac{R}{\sqrt{h}}\} + \exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}\right\}, \tag{3.8}$$

where $q \geq 2$, and positive constants M_1, M_2 do not depend on N, h, R.

Proof. Let us estimate integral

$$\left.\frac{\partial \Gamma_N}{\partial n}\right|_{\partial K_R(x_0)} = \frac{-ih}{2\pi a_N}\oint\limits_L \frac{\sqrt{z}}{2\pi R\sinh(\sqrt{z}R)}\frac{dz}{(1-\frac{zh}{a_1})(1-\frac{zh}{a_2})...(1-\frac{zh}{a_N})}, \tag{3.9}$$

where ∂L is the boundary of the domain

$$L = \{z : \varrho = |z| < \frac{(1+q)\max\limits_{1\le k\le N} a_k}{h}, Rez = b_0 > -\frac{\pi^2}{R^2}, b_0 < 0, \varrho > \frac{\max\limits_{1\le k\le N} a_k}{h}\}.$$

Let us represent the integral (3.9) as a sum of two terms: I_1 and I_2, where I_1 denotes the integral along the part of the curve ∂L which is an arch of a circle, and I_2 denotes the integral along the part of the contour which lies inside the straight line $Rez = b_0$. Let us estimate the integral I_1. The estimates

$$\left|(1-\frac{zh}{a_1})(1-\frac{zh}{a_2})...(1-\frac{zh}{a_N})\right| \ge \left|\frac{|z|h}{a_{\max}} - 1\right|^N \ge q^N,$$

$$\left|\sinh(\sqrt{z}R)\right| \ge \sinh[\sqrt{|z|}R\cos(argz/2)] = \sinh[\sqrt{|z|}R\cos\varphi],$$

where $\varphi \to \frac{\pi}{4}$, if $h \to 0$, imply

$$|I_1| \le c_1\frac{1}{q^N R}\exp\{-c_2\frac{R}{\sqrt{h}}\},$$

where the constants c_1 and c_2 do not depend on h. Let us now estimate the integral I_2. As $Rez = b_0$, we obtain

$$\left|(1-\frac{zh}{a_1})(1-\frac{zh}{a_2})...(1-\frac{zh}{a_N})\right| \ge \left(1+h\frac{|b_0|}{a_{\max}}\right)^{\frac{T}{h}}.$$

Assume $b_0 = -\frac{\pi^2}{2R^2}$. Then $\left|\frac{\sqrt{|z|}R}{\sinh\sqrt{z}R}\right| \le c_4$. Thus we obtain

$$|I_2| \le c_5\exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}.$$

The constants c_3, c_4, c_5 do not depend on h.

4. Uniform estimates. Passage to the limit.

THEOREM 4.1.*Let the following conditions hold:*

$$\psi(x) \in C(\overline{D}) \cap \left(C^{2+\alpha}(\overline{\Omega_0}) \times C^{2+\alpha}(\overline{D\setminus\Omega_0})\right), \varphi(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D_T}),$$

$$\frac{\partial\varphi}{\partial t} \le 0 \ \text{ on } \partial D_T, \ \Delta\psi \le 0 \text{ in } \overline{\Omega_0}, \ \ \Delta\psi = 0 \text{ in } D\setminus\overline{\Omega_0}, \ \gamma_0 \in C^{2+\alpha},$$

$$\left(\frac{\partial\psi}{\partial n}\right)^- - \left(\frac{\partial\psi}{\partial n}\right)^+ \le 0, \ \text{ on } \ \gamma_0,$$

where $\left(\frac{\partial\psi}{\partial n}\right)^{\pm}$ *are the boundary values on the surface* γ_0 *taken from the domains* G_0, Ω_0, *respectively, and we will assume that corresponding consistency conditions hold at* $t = 0, x \in \partial D$. *Then* $\forall h > 0, \forall\varepsilon > 0$, *such that*

$$\varepsilon^{2\nu+2} \ge h^{(2+\alpha)\sigma-1}, \ \ \frac{1}{2+\alpha} < \sigma < \frac{1}{2}, \tag{4.1}$$

there exists a constant c, which does not depend on h, ε, k, such that the following estimate holds:

$$u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x) \le 0, \max_{x\in\overline{\Omega_0 \cup G_0^\varepsilon}} \left| \frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h} \right| + \|u_k^\varepsilon(x)\|_{C^{2+\alpha}(\overline{\Omega_0\cup G_0^\varepsilon})} \le c, \qquad (4.2)$$

where $G_0^\varepsilon = \{x \in D : 1 + \varepsilon < \psi(x)\}$.

Proof. Let $x_0 \in \gamma_0$. From equation (2.4) follows

$$u_1^\varepsilon(x_0) - u_0^\varepsilon(x_0) = \int\limits_{K_R(x_0)} \Delta\psi\Gamma_1(|x - x_0|)dx - \oint\limits_{K_R(x_0)} [u_1^\varepsilon(x) - u_0^\varepsilon(x)]\frac{\partial\Gamma_1}{\partial n}ds+$$

$$+ \int\limits_{K_R(x_0)} (a_1^\varepsilon(x_0) - a_1^\varepsilon(x))\frac{u_1^\varepsilon(x) - u_0^\varepsilon(x)}{h}\Gamma_1(|x - x_0|)dx + \oint\limits_{\gamma_0\cap K_R(x_0)} \left[\left(\frac{\partial\psi}{\partial n}\right)^+ - \left(\frac{\partial\psi}{\partial n}\right)^-\right]\Gamma_1 ds.$$

From here follows, that function $u_1^\varepsilon(x) - u_0^\varepsilon(x)$ cannot have a positive maximum on surface γ_0. Lack of a positive maxima in remaining points of domain $\overline{D}$ is obvious.

Let x_0 be an arbitrary point in $\overline{\Omega_0 \cup G_0^\varepsilon}$. Let us rewrite the equation (2.4) if $k = 1$ in the following form

$$\Delta[u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)] - a_k^\varepsilon(x_0)\frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h} + a_{k-1}^\varepsilon(x_0)\frac{u_{k-1}^\varepsilon(x) - u_{k-2}^\varepsilon(x)}{h} =$$

$$= -(a_k^\varepsilon(x_0) - a_k^\varepsilon(x))\frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h} + (a_{k-1}^\varepsilon(x_0) - a_{k-1}^\varepsilon(x))\frac{u_{k-1}^\varepsilon(x) - u_{k-2}^\varepsilon(x)}{h}$$

In order to obtain the estimate for the functions $\{u_k^\varepsilon - u_{k-1}^\varepsilon\}$ we will use the integral representation (Property 4). Let's notice, that $a_k^\varepsilon(x) \ge \varepsilon$. Denote by $K_R(x_0)$ the sphere with the center at the point x_0 of radius $R = \frac{2h^\sigma}{\sqrt{\varepsilon}}$. It will give

$$u_m^\varepsilon(x_0) - u_{m-1}^\varepsilon(x_0) = \int\limits_{K_R(x_0)} \Delta\psi^\varepsilon(x)\Gamma_m(|x - x_0|)dx - \sum_{k=1}^{m} \int\limits_{\partial K_R(x_0)} [u_k^\varepsilon - u_{k-1}^\varepsilon]\frac{\partial\Gamma_{m-k+1}}{\partial n}ds+$$

$$+\sum_{k=1}^{m} \int\limits_{K_R(x_0)} (a_k^\varepsilon(x_0) - a_k^\varepsilon(x))\frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h}(\Gamma_{m-k+1} - \Gamma_{m-k})dx = I_1 + I_2 + I_3.$$

Let us estimate every term. The relations (2.2) and (3.3) imply

$$|I_1| \le \frac{h}{a_m^\varepsilon(x_0)} \max_{x\in\overline{D}}(|\Delta\psi^\varepsilon(x)| \le \frac{h}{a} \max_{x\in\overline{\Omega_0}} |\Delta\psi(x)|.$$

From (2.2), (3.7) and (3.9) it follows that

$$|I_2| \le \frac{T}{h} \max_{x\in\overline{D}, 1\le k\le N} |u_k^\varepsilon - u_{k-1}|M_1 \left\{ \frac{1}{q^N R} \exp\{-M_2\frac{R}{\sqrt{h}}\} + \exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}\right\} \le$$

$$\le \frac{c_2}{\varepsilon^\nu} \exp\left\{-\frac{c_3\varepsilon}{h^{2\sigma}}\right\} \le \frac{c_2}{h^{\alpha/4}} \exp\left\{-\frac{c_3}{h^{2\sigma-\alpha/2}}\right\} = o(h),$$

where the constants c_2, c_3 do not depend on h, ε.

From (2.3) and from properties (3), (6) of the fundamental solutions we obtain

$$|I_3| \le \sum_{k=1}^{m} \int\limits_{K_R(x_0)} \frac{c|x-x_0|^\alpha}{\varepsilon^\nu} \max_{x\in\overline{D}, 1\le k\le N} |u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)| \frac{\Gamma_{m-k+1} - \Gamma_{m-k}}{h} dx \le$$

$$\le \int\limits_{K_{r_{2,1}}(x_0)} \frac{c|x-x_0|^\alpha}{\varepsilon^\nu} \max_{x\in\overline{D}, 1\le k\le N} |u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)| \frac{\Gamma_1(|x-x_0|)}{h} dx +$$

$$+ \int\limits_{K_R\setminus K_{r_{2,1}}(x_0)} \frac{c|x-x_0|^\alpha}{\varepsilon^\nu} \max_{x\in\overline{D}, 1\le k\le N} |u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)| \frac{1}{h|x-x_0|} dx \le$$

$$\le ch\left(\frac{h^\alpha}{\varepsilon^{2\nu}} + \frac{h^{(2+\alpha)\sigma-1}}{\varepsilon^{2\nu+2}}\right).$$

We differentiate the equation (2.4) with respect to one of the variables x_i and transform it to the following form:

$$\Delta u_k^{\varepsilon\prime} - \frac{b_k^\varepsilon(x_0)u_k^{\varepsilon\prime} - b_{k-1}^\varepsilon(x_0)u_{k-1}^{\varepsilon\prime}}{h} = -\frac{(b_k^\varepsilon(x_0) - b_k^\varepsilon(x))u_k^{\varepsilon\prime} - (b_{k-1}^\varepsilon(x_0) - b_{k-1}^\varepsilon(x))u_{k-1}^{\varepsilon\prime}}{h} +$$

$$+ a_k^{\varepsilon\prime}(x)\frac{u_k^\varepsilon - u_{k-1}^\varepsilon}{h},$$

where $b_k^\varepsilon(x) = a + \chi_\varepsilon'(u_k^\varepsilon)$. We use the property 4. It gives

$$u_m^{\varepsilon\prime}(x_0) = \int\limits_{K_R(x_0)} b_0^\varepsilon u_0^{\varepsilon\prime}(x)\frac{\Gamma_{m-k+1}(|x-x_0|)}{h}dx - \sum_{k=1}^{m} \int\limits_{\partial K_R(x_0)} u_k^{\varepsilon\prime}(x)\frac{\partial \Gamma_{m-k+1}}{\partial n}ds +$$

$$+ \sum_{k=1}^{m} \int\limits_{K_R(x_0)} (b_k^\varepsilon(x_0) - b_k^\varepsilon(x))u_k^{\varepsilon\prime}(x)\frac{\Gamma_{m-k+1}(|x-x_0|) - \Gamma_{m-k}(|x-x_0|)}{h}dx -$$

$$- \sum_{k=1}^{m} \int\limits_{K_R(x_0)} a_k^{\varepsilon\prime}(x)\frac{u_k^\varepsilon(x) - u_{k-1}^\varepsilon(x)}{h}\Gamma_{m-k+1}(|x-x_0|)dx. \tag{4.3}$$

From this integral representation, applying the same reasoning as above, we obtain the second part of the estimate (4.2).

THEOREM 4.2. *Let conditions of the theorem (4.1) are satisfied and*

$$\frac{\partial \psi}{\partial \rho} \ge c_0 > 0 \;\; in \;\; \overline{D},$$

where ρ is spherical radius, there exists a constant c_1, which does not depend on h, ε, k, such that the following estimate holds:

$$\frac{\partial u_k^\varepsilon}{\partial \rho} \ge c_1 > 0 \;\; \forall x \in \overline{\Omega_0 \cup G_0^\varepsilon}, \;\; k = 1, 2, ...N. \tag{4.4}$$

Proof. We differentiate the equation on a variable ρ. IT will give

$$\Delta\left(\rho\frac{\partial u_k^\varepsilon}{\partial\varrho}\right)-\frac{b_k^\varepsilon(x)\left(\rho\frac{\partial u_k^\varepsilon}{\partial\varrho}\right)-b_{k-1}^\varepsilon(x)\left(\rho\frac{\partial u_{k-1}^\varepsilon}{\partial\varrho}\right)}{h}=\frac{2}{h}\int\limits_{u_{k-1}^\varepsilon}^{u_k^\varepsilon}[\varepsilon+a\chi_\varepsilon(\tau)-\lambda\chi_\varepsilon'()\tau]d\tau.$$

After that it is necessary to take advantage of integral representation (4.3). Notice, that the principal term in the received integral representation can be estimated as follows

$$\left|\int\limits_{K_R(x_0)}b_0^\varepsilon\left(\rho\frac{\partial u_0^\varepsilon}{\partial\varrho}\right)\frac{\Gamma_{m-k+1}(|x-x_0|)}{h}dx\right|\geq c\min_{x\in\overline{D}}\left(\rho\frac{\partial\psi}{\partial\varrho}\right)+o(h^{\alpha/2}).$$

Similarly to the previous theorem it is possible to prove that all other terms in the received integral representation have limits equal to zero when $h\to 0$.

Let the function $\eta(x,t)\in C^{2,1}(\overline{D})$ be equal to zero on $(\partial D\times(0,T))\cup(D\times(t=T))$, $\eta_k(x)=\eta(x,kh)$. We multiply (2.3) by $h\eta_k(x)$, integrate it over D, and take the sum over k from 1 to N. After simple transformations we obtain

$$h\sum_{k=1}^{N}\int\limits_D\{\nabla u_k^\varepsilon\nabla\eta_k+\frac{(a\chi_\varepsilon(u_k^\varepsilon)+\varepsilon)}{h}(u_k^\varepsilon-u_{k-1}^\varepsilon)\eta_k+\lambda\chi_\varepsilon(u_k^\varepsilon)\frac{\eta_k-\eta_{k-1}}{h}\}dx=0,$$

Let us denote by $\overline{u}(x,t,h,\varepsilon)$ the piecewise linear interpolations of the functions $\{u_k^\varepsilon(x)\}$ with respect to the variable t,

$$u(x,t)=\lim_{\varepsilon,h\to 0}\overline{u}(x,t,h,\varepsilon),$$

where h,ε satisfy the conditions (4.1). The possibility of passage to the limit follows from the statements proved above. As a result we obtain

THEOREM 4.3.*Let the following conditions be satisfied:*

$$\psi(x)\in C(\overline{D})\cap\left(C^{2+\alpha}(\overline{\Omega_0})\times C^{2+\alpha}(\overline{D\setminus\Omega_0})\right),\varphi(x,t)\in H^{2+\alpha,1+\alpha/2}(\overline{D_T}),$$

$$\frac{\partial\varphi}{\partial t}\leq 0,\ \Delta\psi=0\text{ in }D\setminus\overline{\Omega_0},\ \gamma_0\in C^{2+\alpha},\ \ \min_{x\in\overline{D}}\left(\rho\frac{\partial\psi}{\partial\varrho}\right)\geq c>0,$$

$$\Delta\psi\leq 0\text{ in }\overline{\Omega_0},\ \left(\frac{\partial\psi}{\partial n}\right)^{-}-\left(\frac{\partial\psi}{\partial n}\right)^{+}\leq 0,\ \text{ on }\gamma_0,$$

and we will assume that corresponding consistency conditions hold at $t=0,x\in\partial D$. Then $\forall T>0$ there exists a solution of the problem (1.1)-(1.4) and

$$u(x,t)\in C(\overline{D_T})\cap\left(H^{2+\alpha,1+\alpha/2}(\overline{\Omega_T}\setminus\gamma_0)\times H^{2+\alpha,1+\alpha/2}(\overline{G_T}\setminus\gamma_0)\right),$$

the free boundary is given by the graph $\rho=\omega(\theta_1,\theta_2,t)$ of a function from $H^{2+\alpha,1+\alpha/2}$ class, where (ρ,θ_1,θ_2) are spherical coordinates.

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