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On Kinetic Formulation of First-order Hyperbolic Quasilinear Systems

Evgeniy Yu. Panov

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Abstract. We give kinetic formulation of measure valued and strong measure valued solutions to the Cauchy problem for a first-order quasilinear equation. For the corresponding kinetic equation the class of existence and uniqueness to the Cauchy problem is extracted. This class consists of so-called entropy solutions, which correspond to strong measure valued solutions of the original problem. In the last section we generalized these results to the case of symmetric generally nonconservative multidimensional systems and introduce the notion of a strong measure valued solution, based only on the kinetic approach under consideration.

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1. Introduction

In the half-space $(t, x) \in \Pi = (0, +\infty) \times \mathbb{R}^m$ consider firstly a scalar conservation law

$$u_t + \mathrm{div}_x \varphi(u) = 0, \tag{1.1}$$

$u = u(t, x)$, $\varphi(u) = (\varphi_1(u), \ldots, \varphi_m(u)) \in (C^1(\mathbb{R}))^m$.

Nonlocal theory of generalized entropy solutions (briefly — g.e.s.) to the Cauchy problem for the equation (1.1) with initial condition

$$u(0, x) = u_0(x) \in L^\infty(\mathbb{R}^m) \tag{1.2}$$

was constructed by S. N. Kruzhkov in paper [1]. Remind the definition of g.e.s.

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Definition 1.1. *A bounded measurable function $u = u(t,x)$ is called a g.e.s. to the Cauchy problem* (1.1), (1.2) *if*

a) $\forall\, k \in \mathbb{R}$

$$|u-k|_t + \mathrm{div}_x[(\varphi(u)-\varphi(k))\,\mathrm{sign}(u-k)] \le 0 \qquad (1.3)$$

in the sense of distributions on Π (*in* $\mathcal{D}'(\Pi)$)*;*

b) $\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0$ *in* $L^1_{loc}(\mathbb{R}^m)$,

i. e. there exists a set of null Lebesgue measure $\mathcal{E} \subset (0,+\infty)$ *such that for* $t \notin \mathcal{E}$ $u(t,\cdot) \in L^1_{loc}(\mathbb{R}^m)$ *and* $u(t,\cdot) \underset{t\to 0, t\notin\mathcal{E}}{\to} u_0$ *in* $L^1_{loc}(\mathbb{R}^m)$.

Condition (1.3) means that for any nonnegative test function $g = g(t,x) \in C_0^\infty(\Pi)$ $\int_\Pi \left[|u-k|g_t + \sum_{i=1}^m (\varphi_i(u)-\varphi_i(k))\,\mathrm{sign}(u-k)g_{x_i}\right] dt\,dx \ge 0.$ Taking in (1.3) $k = \pm\|u\|_\infty$ we derive that $u_t + \mathrm{div}_x\varphi(u) = 0$ and a g.e.s. $u(t,x)$ satisfies the equation (1.1) in the distributional sense. As is known (see [1]), there always exists a unique g.e.s. to the problem (1.1), (1.2).

Kinetic formulation of conservation laws was proposed in papers [2, 3] and was further developed in [4]. A function $u = u(t,x)$ was shown to be a g.e.s. to the problem (1.1), (1.2) if and only if the function $f = f(t,x,v) = \chi_{u(t,x)}(v)$, where for $u,v \in \mathbb{R}$ $\chi_u(v) = \theta(u-v) - \theta(-v)$, $\theta(\lambda) = \begin{cases} 1, & \lambda > 0 \\ 0, & \lambda \le 0 \end{cases}$ is the Heaviside function, is a generalized solution (g.s.) to the corresponding problem for a kinetic equation

$$\frac{\partial}{\partial t} f + (\varphi'(v), \nabla_x f) = \frac{\partial}{\partial v}\mu, \qquad (1.4)$$

(here $(\cdot,\cdot)$ denotes scalar product in $\mathbb{R}^m$) with some nonnegative locally finite measure $\mu = \mu(t,x,v)$ on $\Pi \times \mathbb{R}$ that has compact support with respect to the variable v, and with initial condition

$$f(0,x,v) = f_0(x,v), \qquad (1.5)$$

where $f_0(x,v) = \chi_{u_0(x)}(v)$. One should understand the initial condition (1.5) in a similar way to the condition b) in Definition 1.1, that is,

$$\operatorname*{ess\,lim}_{t\to 0} f(t,\cdot,\cdot) = f_0 \ \text{ in } L^1_{loc}(\mathbb{R}^{m+1}). \qquad (1.6)$$

In papers [5, 6] the results above were extended to the more general case of measure valued solutions.

We recall (see [7, 8]) that a measure valued function on a measurable subset Ω of some Euclid space is a weakly measurable map $x \mapsto \nu_x$ of

Ω into the space $\mathrm{Prob}_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$. The weak measurability of ν_x means that $\forall\, p(\lambda) \in C(\mathbb{R})$ the function $x \mapsto \int p(\lambda)d\nu_x(\lambda)$ on Ω is measurable. We say that a measure valued function ν_x is *bounded* if there exists $M > 0$ such that $\mathrm{supp}\,\nu_x \subset [-M, M]$ for almost all $x \in \Omega$. We shall denote by $\|\nu_x\|_\infty$ the smallest of such M. Finally, we say that the measure valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(\lambda - u)$ is the Dirac measure at $u \in \mathbb{R}$, are regular. We identify these measure valued functions and the corresponding functions $u(x)$, so that there is a natural embedding $L^\infty(\Omega) \subset MV(\Omega)$, where $MV(\Omega)$ is the set of bounded measure valued functions on Ω. In similar way the vectorial measure valued functions $x \to \nu_x \in \mathrm{Prob}_0(\mathbb{R}^n)$ are defined.

We now consider the measure valued solutions (briefly — m.s.) $\nu_{t,x} \in MV(\Pi)$ of the Cauchy problem for (1.1) with initial condition

$$\nu_{0,x} = \nu_x^0 \in MV(\mathbb{R}^m). \tag{1.7}$$

Definition 1.2 (see [8]). *A bounded measure valued function $\nu_{t,x}$ is called a m.s. of the problem* (1.1), (1.7) *if*

a) $\forall\, k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\int |\lambda - k|\, d\nu_{t,x}(\lambda) + \mathrm{div}_x \int (\varphi(\lambda) - \varphi(k))\,\mathrm{sign}(\lambda - k)\, d\nu_{t,x}(\lambda) \le 0 \tag{1.8}$$

in $\mathcal{D}'(\Pi)$*;*

b) $\forall\, p(\lambda) \in C(\mathbb{R}) \quad \operatorname*{ess\,lim}_{t\to 0} \int p(\lambda)\, d\nu_{t,x}(\lambda) = \int p(\lambda)\, d\nu_x^0(\lambda)$

in $L^1_{loc}(\mathbb{R}^m)$*.*

Remark that Definition 1.2 is consistent with Definition 1.1 in that $u(t,x)$ is a g.e.s. of the problem (1.1), (1.2) if and only if this function, understood as the regular measure valued function $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$, is a m.s. of the problem (1.1), (1.7) with regular initial function $\nu_x^0(\lambda) = \delta(\lambda - u_0(x))$.

We can treat a measure valued function $\nu_{t,x}$ as the random field $u(t,x)$ where the value of $u(t,x)$ for fixed $(t,x) \in \Pi$ is a random variable with distribution $\nu_{t,x}$. Then condition (1.8) shows that $u(t,x)$ satisfies condition (1.3) "on the average", that is,

$$E(|u - k|)_t + \mathrm{div}_x E((\varphi(u) - \varphi(k))\,\mathrm{sign}(u - k)) \le 0 \quad \text{in } \mathcal{D}'(\Pi)$$

(here E denotes expectation). Setting $k = \pm M$, where $M = \|\nu_{t,x}\|_\infty$, we conclude that $u(t,x)$ satisfies (1.1) "on the average". Some results on relations between measure valued and statistic solutions of the problem (1.1), (1.7) can be found in [9].

As is known (see [10, 11]), there always exists a m.s. of the Cauchy problem (1.1), (1.7), but it is nether unique if the initial function ν_x^0 is not regular. In [10] (see [11] for more details) the notion of a *strong measure valued solution* (briefly — s.m.s.) was introduced and existence and uniqueness of s.m.s. to the problem (1.1), (1.7) were also established.

Definition 1.3 (see [10, 11]). *A bounded measure valued function* $\nu_{t,x}$ *is called a s.m.s. of the problem* (1.1), (1.7) *if for all* $\lambda \in (0,1)$ *the function* $u(t,x,\lambda) = \inf\{\, v \mid \nu_{t,x}((v,+\infty)) \le \lambda \,\}$ *is a g.e.s. of the problem* (1.1), (1.2) *with initial function* $u_0(x,\lambda) = \inf\{\, v \mid \nu_x^0((v,+\infty)) \le \lambda \,\}$.

For a regular m.s. $\nu_{t,x}(\lambda) = \delta(\lambda - u(t,x))$ we have $u(t,x,\lambda) = u(t,x)$ and $\nu_{t,x}$ is a s.m.s. Note (see [11]) that for fixed $(t,x) \in \Pi$ the measure $\nu_{t,x}$ is the image of the Lebesgue measure $d\lambda$ on $(0,1)$ under the map $\lambda \mapsto u(t,x,\lambda)$: $\nu_{t,x} = u(t,x,\cdot)^* d\lambda$ (similarly, $\nu_x^0 = u_0(x,\cdot)^* d\lambda$). Therefore,

$$\forall\, k \in \mathbb{R} \quad \int |\lambda - k|\, d\nu_{t,x}(\lambda) = \int_0^1 |u(t,x,\lambda) - k|\, d\lambda,$$

$$\int (\varphi(\lambda) - \varphi(k))\operatorname{sign}(\lambda - k)\, d\nu_{t,x}(\lambda) = \int_0^1 (\varphi(u(t,x,\lambda)) - \varphi(k))\operatorname{sign}(u(t,x,\lambda) - k)\, d\lambda$$

and inequality (1.8) can be written in the form

$$\int_0^1 \big[|u(t,x,\lambda) - k|_t + \operatorname{div}_x((\varphi(u(t,x,\lambda)) - \varphi(k))\operatorname{sign}(u(t,x,\lambda) - k))\big]\, d\lambda \le 0. \tag{1.9}$$

Thus, Definition 1.3 contains strengthening of the condition (1.8), namely the integrand in (1.9) is required to be nonpositive (in $\mathcal{D}'(\Pi)$) for all $\lambda \in (0,1)$. In particular any s.m.s. is a m.s. in the sense of Definition 2 as well. The unique solvability of the problem (1.1), (1.2) implies easily existence and uniqueness of s.m.s. of the problem (1.1), (1.7) (see details in [11]).

2. Kinetic formulation of m.s.

Denote by F_M the space of distribution functions $f(v) = F(\nu)(v) = \nu((v,+\infty))$ of measures ν with support in the segment $[-M,M]$. Functions $f(v) \in F_M$ don't increase and are continuous from the right with respect to $v \in \mathbb{R}$; $f(v) = 0$ for $v \geq M$, $f(v) = 1$ for $v < -M$. Denote also by $F_c = \bigcup_{M>0} F_M$ the space of distribution functions of measures having compact supports on $\mathbb{R}$.

If $\nu_x \in MV(\Omega)$, Ω is a measurable domain in some Euclid space, then $f(x, v) = F(\nu_x)(v) \in F_M$ for a.e. $x \in \Omega$, $M = \|\nu_x\|_\infty$ and by weak measurability of the map $x \mapsto \nu_x$ the function $f(x, v) = F(\nu_x)(v)$ is measurable on the set of variables (x, v). Let $F_M(\Omega)$ be the class of functions with above properties. Inclusion $f(x, v) \in F_M(\Omega)$ is equivalent to existence of a measure valued function $\nu_x \in MV(\Omega)$ such that $\|\nu_x\|_\infty \leq M$ and $f(x, v) = \nu_x((v, +\infty))$.

As was shown in [5], a distribution function $f(t, x, v)$ of a m.s. $\nu_{t,x} \in MV(\Pi)$ can be described as a solution of the corresponding problem (1.4), (1.5) for the kinetic equation. In [5] we also presented kinetic formulation of s.m.s.

Theorem 2.1 ([5]). *A bounded measure valued function $\nu_{t,x}$ is a m.s. of the problem* (1.1), (1.7) *if and only if the corresponding distribution function $f(t, x, v) = \nu_{t,x}((v, +\infty))$ is a g.s. of the problem* (1.4), (1.5) *with initial function $f_0(x, v) = \nu_x^0((v, +\infty))$. Besides, $\nu_{t,x}$ is a s.m.s. of the problem* (1.1), (1.7) *if and only if for each nondecreasing function $s(u) \in C([0, 1])$ the function $s(f(t, x, v))$ is a g.s. to the problem* (1.4), (1.5) *with initial data $s(f_0(x, v))$.* (*It is understood that the measure μ in the right-hand side of* (1.4) *depends on the function s*).

It turns out that the class of strong m.s. correspond to the following important class of solutions of the kinetic problem (1.4), (1.5).

Definition 2.1. *A function $f(t, x, v)$ is called an entropy solution (e.s. for short) of the problem* (1.4), (1.5) *if $f(t, x, v) \in F_M(\Pi)$ for some $M > 0$, for any function $g(v) \in F_c$*

$$\frac{\partial}{\partial t} \int (f(t, x, v) - g(v))^2 dv + \operatorname{div}_x \int \varphi'(v)(f(t, x, v) - g(v))^2 dv \leq 0 \quad (2.1)$$

in $\mathcal{D}'(\Pi)$, and the initial condition (1.6) *is satisfied.*

Remark that Definition 2.1 does not include the measure μ, which plays the role of a free parameter in the equation (1.4). In this sense, the notion of e.s. solution of the kinetic problem seems to be more natural in comparison with the notion of g.s. We prove the following result

Theorem 2.2. *There exists an unique e.s. $f(t, x, v)$ of the problem* (1.4), (1.5). *Besides, $f(t, x, v) = \nu_{t,x}((v, +\infty))$, where $\nu_{t,x}$ is the corresponding strong m.s. of the problem* (1.1), (1.7).

Proof. Let $f_0(x,v) = \nu_x^0((v,+\infty))$ and $\nu_{t,x} \in MV(\Pi)$ be the unique s.m.s. of the problem (1.1), (1.7), $f(t,x,v) = \nu_{t,x}((v,+\infty))$ be the corresponding distribution function. Suppose that $M > \|\nu_{t,x}\|_\infty$ and $g(v) \in F_M \cap C^1(\mathbb{R})$. It is clear that the integrands in (2.1) vanish for $|v| > M$. Then, by Theorem 2.1 for $s(u) = u, u^2$

$$\frac{\partial}{\partial t} f + (\varphi'(v), \nabla_x f) = \frac{\partial}{\partial v}\mu_1, \tag{2.2}$$

$$\frac{\partial}{\partial t} f^2 + (\varphi'(v), \nabla_x f^2) = \frac{\partial}{\partial v}\mu_2 \ \text{ in } \mathcal{D}'(\Pi), \tag{2.3}$$

where $\mu_1,\ \mu_2$ are nonnegative locally finite measures on $\Pi \times (-M, M)$. Let $h(t,x) \in C_0^\infty(\Pi)$, $h \geq 0$. Then using the equalities (2.2), (2.3), we obtain

$$\int\limits_{\Pi\times(-M,M)} (f(t,x,v) - g(v))^2 \{h_t(t,x) + (\varphi'(v), \nabla_x h)\}\, dt\, dx\, dv$$

$$= \int\limits_{\Pi\times(-M,M)} (f(t,x,v))^2 \{h_t(t,x) + (\varphi'(v), \nabla_x h)\}\, dt\, dx\, dv$$

$$- 2\int\limits_{\Pi\times(-M,M)} f(t,x,v) g(v) \{h_t(t,x) + (\varphi'(v), \nabla_x h)\}\, dt\, dx\, dv$$

$$= \int\limits_{\Pi\times(-M,M)} h_v(t,x)\, d\mu_2(t,x,v) - 2\int\limits_{\Pi\times(-M,M)} h(t,x) g'(v)\, d\mu_1(t,x,v)$$

$$= -2\int\limits_{\Pi\times(-M,M)} h(t,x) g'(v)\, d\mu_1(t,x,v) \geq 0.$$

Using approximation of an arbitrary function $g(v) \in F_M$ by a sequence of smooth functions, we derive that $\forall g(v) \in F_M\ \forall\, h \in C_0^\infty(\Pi)$, $h \geq 0$

$$\int\limits_{\Pi\times\mathbb{R}} (f(t,x,v) - g(v))^2 \{h_t(t,x) + (\varphi'(v), \nabla_x h)\}\, dt\, dx\, dv \geq 0.$$

Here M takes an arbitrary great enough value and we obtain that the relation (2.1) holds. The initial condition (1.6) is also fulfilled by Theorem 2.1. Thus, $f(t,x,v)$ is a e.s. of the problem (1.4), (1.5). By construction $f(t,x,v)$ satisfies the last assertion of the theorem.

Uniqueness of e.s. is proved by using of the Kruzhkov's method of doubling variables, see the more general case of systems below (Theorem 3.1 and Remark 3.2). □

Remark 2.1. As it follows from the proofs of Theorem 2.2 and Theorem 2.1, $f(t,x,v)$ is an e.s. of the problem (1.4), (1.5) if and only if $f(t,x,v)$ and $f^2(t,x,v)$ are g.s. of this problem with the corresponding initial functions. Therefore, in addition to Theorem 2.1, we can conclude that *the sufficient condition for a bounded measure valued function* $\nu_{t,x}$ *to be a strong m.s. is that the only two functions* $s(f(t,x,v))$, *with* $s(u)=u,u^2$, *are g.s. of the problem* (1.4), (1.5).

Remark 2.2. In the paper [6] one approximation scheme of relaxation type was proposed. Namely, the following problem was considered

$$f_t + (\varphi'(v), \nabla_x f) = r(g-f), \quad f(0,x,v) = f_0(x,v) \in F_M(\mathbb{R}^m), \quad (2.4)$$

$f = f(t,x,v) \in L^\infty(\Pi \times \mathbb{R})$, $r = \text{const} > 0$, the nonlinear operator $f \mapsto Ff = g$ is defined as follows

$$g = g(t,x,v) = \begin{cases} 1, & v \le -M, \\ 0, & v \ge M, \\ l(f^*(t,x;v)), & v \in (-M,M), \end{cases}$$

where $f^*(t,x;v)$ is a nonincreasing rearrangement (see [12]), of the function $v \mapsto f(t,x,v)$ with respect to $v \in (-M,M)$; $l(f)=\max(\min(f,1),0)$ is a cut off function.

As was proved in [6], there exists an unique g.s. $f_r(t,x,v)$ of the problem (2.4), and some subsequence of the sequence f_r converges strongly to a distribution function of some m.s. of (1.1), (1.7). Now we can revise this result:

Theorem 2.3. *The sequence* $f_r(t,x,v)$ *converges in* $L^1_{loc}(\Pi \times \mathbb{R})$ *to the e.s.* $f(t,x,v) \in F_M(\Pi)$ *of the problem* (1.4), (1.5).

3. The case of hyperbolic systems

We consider the following first-order symmetric quasilinear system

$$B(u)u_t + \sum_{i=1}^m C_i(u)u_{x_i} = 0, \quad (3.1)$$
$$u = u(t,x) \in \mathbb{R}^n, \ (t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}^m,$$

with symmetric matrices $B(u) > 0$ (i.e. $B(u)$ is positively definite) and $C_i(u)$, $i=1,\dots,m$. Remark that a general (nonstrictly) hyperbolic $n \times n$ system

$$u_t + \sum_{i=1}^m A_i(u)u_{x_i} = 0 \quad (3.2)$$

could be reduced to the symmetric form (3.1) in the cases when $m = 1$ and when $n \le 2$ (and only in these cases, see [13]). The system (3.1) is equivalent to the hyperbolic system (3.2) with the matrices $A_i(u) = B^{-1}(u)C_i(u)$.

Suppose that the matrices $B(u)$ and $C_i(u)$ depend continuously on $u \in \mathbb{R}^n$ and the matrices $A_i(u)$ have bounded eigenvalues λ_{ij}, $j = 1, \dots, n$. This and symmetry of matrices $A_i(u)$ with respect to a scalar multiplication $(f, g) \to (B(u)f, g)$ imply that for some constant $C > 0$ $\forall\, u, f \in \mathbb{R}^n$

$$(B(u)A_i(u)f, A_i(u)f) \le C^2(B(u)f, f). \tag{3.3}$$

Let X be a Hilbert space of vector fields $f(u) = (f_1(u), \dots, f_n(u))$, $u \in \mathbb{R}^n$ with finite value of $\|f\|^2 = \int_{\mathbb{R}^n} (B(u)f(u), f(u))\, du$ (here $(\cdot,\cdot)$ is a scalar product in $\mathbb{R}^n$), and the scalar product $(f, g) = \int_{\mathbb{R}^n} (B(u)f(u), g(u))\, du$. Remark that the correspondence $f \mapsto A_i(u)f(u)$ yields a bounded symmetric linear operator $\tilde{A}_i$ on X for each $i = 1, \dots, m$, and $\|\tilde{A}_i\| \le C$.

Fix some convex closed set $H \subset X$, the bounded closed subsets of which are compact. By analogy with the scalar case we study the Cauchy problem for the kinetic system corresponded to (3.1), (3.2)

$$f_t + \sum_{i=1}^{m} A_i(u) f_{x_i} = 0, \tag{3.4}$$

where the unknown vector $f = f(t, x, u) \in \mathbb{R}^n$ belongs to the set $L^2_{loc}(\Pi, H)$ (which is a convex closed subset of the space $L^2_{loc}(\Pi, X)$ determined by condition: $f(t, x, \cdot) \in H$ for a.e. $(t, x) \in \Pi$), with initial condition

$$f(0, x, u) = f_0(x, u) \in L^2_{loc}(\mathbb{R}^m, H). \tag{3.5}$$

Definition 3.1. *A function $f = f(t, x, u) \in L^2_{loc}(\Pi, H)$ is called an entropy solution (e.s.) of the problem* (3.4), (3.5) *if* $\forall\, g = g(u) \in H$

$$\frac{\partial}{\partial t}\|f(t,x,\cdot) - g\|^2 + \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\big(\tilde{A}_i(f(t,x,\cdot) - g), f(t,x,\cdot) - g\big) \le 0 \quad \text{in } \mathcal{D}'(\Pi); \tag{3.6}$$

$$\operatorname*{ess\,lim}_{t\to 0} \|f(t, x, \cdot) - f_0(x, \cdot)\| = 0 \quad \text{in } L^2_{loc}(\mathbb{R}^m). \tag{3.7}$$

Theorem 3.1. *The e.s. of the problem* (3.4), (3.5) *is unique.*

Sketch of the proof. We make use of the Kruzhkov's method of doubling variables developed in [1]. Let $f = f(t, x, u)$, $\bar{f} = \bar{f}(t, x, u)$ be two e.s. of the problem (3.4), (3.5). Then, by (3.6) with $g(u) = \bar{f}(\tau, y, u)$, $(\tau, y) \in \Pi$

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)-\bar f(\tau,y,\cdot)\|^2 + \sum_{i=1}^m \frac{\partial}{\partial x_i}\big(\tilde A_i(f(t,x,\cdot)-\bar f(\tau,y,\cdot)), f(t,x,\cdot)-\bar f(\tau,y,\cdot)\big) \le 0 \quad \text{in } \mathcal{D}'(\Pi\times\Pi).$$

In the same way, changing places of variables (t,x) and (τ,y), and also solutions f and $\bar f$, we obtain that

$$\frac{\partial}{\partial \tau}\|f(t,x,\cdot)-\bar f(\tau,y,\cdot)\|^2 + \sum_{i=1}^m \frac{\partial}{\partial y_i}\big(\tilde A_i(f(t,x,\cdot)-\bar f(\tau,y,\cdot)), f(t,x,\cdot)-\bar f(\tau,y,\cdot)\big) \le 0 \quad \text{in } \mathcal{D}'(\Pi\times\Pi).$$

Putting these relations together we derive that in $\mathcal{D}'(\Pi\times\Pi)$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \tau}\right)\|f(t,x,\cdot)-\bar f(\tau,y,\cdot)\|^2 + \sum_{i=1}^m \left(\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}\right) \times \big(\tilde A_i(f(t,x,\cdot)-\bar f(\tau,y,\cdot)), f(t,x,\cdot)-\bar f(\tau,y,\cdot)\big) \le 0. \quad (3.8)$$

Applying the inequality (3.8) to the test function $p = p(t,x;\tau,y) = g(t,x)\rho_\nu(t-\tau,x-y)$ where $g(t,x)\in C_0^\infty(\Pi)$, $g\ge 0$, and the sequence $\rho_\nu \in C_0^\infty(\mathbb{R}^{m+1})$, $\nu\in\mathbb{N}$ approximates the δ-function, we obtain, after passing to the limit as $\nu\to\infty$, that

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)-\bar f(t,x,\cdot)\|^2 + \sum_{i=1}^m \frac{\partial}{\partial x_i}\big(\tilde A_i(f(t,x,\cdot)-\bar f(t,x,\cdot)), f(t,x,\cdot)-\bar f(t,x,\cdot)\big) \le 0 \quad \text{in } \mathcal{D}'(\Pi).$$

Applying the obtained relation to test functions approximated the indicator function of the subcharacteristical cone $K_R = \{\ (t,x)\in\Pi \mid |x| < R-Nt\ \}$, $N=\sqrt{m}C$, C is the constant from (3.3), we derive that for a.e. $t>0\ \forall r>0 \int_{|x|<r}\|f(t,x,\cdot)-\bar f(t,x,\cdot)\|^2dx = 0$, i.e. $f(t,x,u)=\bar f(t,x,u)$ a.e. on $\Pi\times\mathbb{R}$, and the e.s. $f(t,x,u)$ is unique. The proof is complete. □

To prove existence of e.s. we make use of the following approximation scheme of relaxation type. Namely, let $P_H: X\mapsto H$ be the projection map so that $\|f-P_H f\| = \min\limits_{h\in H}\|f-h\|$. Because X is a Hilbert space and H is its convex closed subset the projection $P_H f$ is well-defined, and the contraction property holds:

$$\|P_H f - P_H g\| \le \|f-g\| \ \forall\, f,g\in X. \tag{3.9}$$

Fix $r>0$ and consider the problem

$$f_t + \sum_{i=1}^m A_i(u) f_{x_i} = -r(f - P_H f), \tag{3.10}$$

$$f(0,x,u) = f_0(x,u) \in L^2_{loc}(\mathbb{R}^m, X). \tag{3.11}$$

We shall consider g.s. $f(t,x,u) \in L^2_{loc}(\Pi, X)$ of the problem (3.10), (3.11), which satisfy the equation (3.10) in $\mathcal{D}'(\Pi\times\mathbb{R})$ and the initial condition (3.11) in the sense of relation (3.7). Now we are ready to establish the following

Theorem 3.2. *There exists a unique g.s. of the problem* (3.10), (3.11).

Proof. For the proof fix some $R>0$ and define the cone $K_R = \{\ (t,x)\in \Pi \ \big|\ |x| < R - Nt\ \}$, and the Banach space $L\subset L^2_{loc}(K_R, X)$ consisting of vectors $f = f(t,x,u)$ with finite norm

$$\|f\| = \operatorname*{ess\,sup}_{t>0}\left(\int\limits_{|x|\le R-Nt} \|f(t,x,\cdot)\|^2 dx\right)^{1/2}.$$

Let $g = g(t,x,u)\in L$ and $f = f(t,x,u) = \Phi(g) \in L^2_{loc}(K_R, X)$ be the unique g.s. to the Cauchy problem for the linear system $f_t + \sum\limits_{i=1}^m A_i(u) f_{x_i} = -r(f - P_H g)$, with the fixed initial data (3.11). If $g_1, g_2\in L$, $f_1 = \Phi(g_1)$, $f_2 = \Phi(g_2)$ then, as it is easily verified, for a.e. $t>0$

$$\int\limits_{|x|\le R-Nt} \|f_1(t,x,\cdot) - f_2(t,x,\cdot)\|^2\, dx$$

$$\le r\int\limits_0^t e^{-r(t-\tau)} \int\limits_{|x|\le R-N\tau} \|P_H g_1(\tau,x,\cdot) - P_H g_2(\tau,x,\cdot)\|^2\, dx\, d\tau,$$

which implies that the map Φ is a contraction with coefficient $1-e^{-rR/N}$. By Banach theorem there exist the unique fixed point $f\in L$ of the map Φ, i.e. problem (3.10), (3.11) has the unique g.s. $f = f_R$ in the cone

K_R. By uniqueness solutions f_{R_1} and f_{R_2} coincide in their common region. Therefore, we can define the function $f(t,x,u) \in L^2_{loc}(\Pi, X)$ setting $f(t,x,u) = f_R(t,x,u)$ where $R > |x| + Nt$ is arbitrary. It is clear that $f(t,x,u)$ is a g.s. of the problem (3.10), (3.11). Uniqueness of g.s. $f(t,x,u)$ follows from its uniqueness in any cone K_R. □

Our next task is to prove convergence of solutions $f = f_r(t,x,u)$ of (3.10), (3.11) as the parameter $r \to \infty$. To this end we need some a priory estimates for g.s. of the problem (3.10), (3.11), which are uniform with respect to the parameter $r > 0$. Firstly, we have the following

Theorem 3.3. *Let $f_1(t,x,u)$, $f_2(t,x,u)$ be g.s. of the problem* (3.10), (3.11) *with initial data $f_{01}(x,u), f_{02}(x,u)$ respectively. Then*

1) *in $\mathcal{D}'(\Pi)$*

$$\begin{aligned}&\big(\|f_1(t,x,\cdot) - f_2(t,x,\cdot)\|^2\big)_t \\ &\quad + \sum_{i=1}^m \big(\tilde{A}_i(f_1(t,x,\cdot) - f_2(t,x,\cdot)), f_1(t,x,\cdot) - f_2(t,x,\cdot)\big)_{x_i} \le 0; \quad (3.12)\end{aligned}$$

2) *for almost each $t > 0$ $\forall\, R > 0$*

$$\int\limits_{|x|\le R} \|f_1(t,x,\cdot) - f_2(t,x,\cdot)\|^2\, dx \le \int\limits_{|x|\le R+Nt} \|f_{01}(x,\cdot) - f_{02}(x,\cdot)\|^2\, dx. \qquad (3.13)$$

Proof. To prove (3.12) we multiply the relation

$$\begin{aligned}&\big(f_1(t,x,\cdot) - f_2(t,x,\cdot)\big)_t + \sum_{i=1}^m \big[\tilde{A}_i\big(f_1(t,x,\cdot) - f_2(t,x,\cdot)\big)\big]_{x_i} \\ &\qquad = -r\big[\big(f_1(t,x,\cdot) - f_2(t,x,\cdot)\big) - \big(P_H f_1(t,x,\cdot) - P_H f_2(t,x,\cdot)\big)\big]\end{aligned}$$

by $f_1(t,x,\cdot) - f_2(t,x,\cdot)$ scalarly in X and use that

$$\begin{aligned}&\big(f_1(t,x,\cdot) - f_2(t,x,\cdot), P_H f_1(t,x,\cdot) - P_H f_2(t,x,\cdot)\big) \\ &\qquad\qquad \le \|f_1(t,x,\cdot) - f_2(t,x,\cdot)\|^2\end{aligned}$$

in view of (3.9).

At last, (3.13) follows from (3.12) after application to test functions, which approximate indicator functions of subcharacteristical cones. □

Applying relation (3.13) to pairs $f_1 = f(t,x,\cdot)$, and $f_2 = f(t+\Delta t, x+\Delta x, \cdot)$ we obtain estimates of continuity modulus of solutions $f = f_r$ of the problem (3.10), (3.11) in $L^2_{loc}(\Pi, X)$, which do not depend on parameter r. This, together with compactness of bounded subsets in H, allows us to establish strong convergence of the sequence $g_r = P_H f_r(t,x,\cdot)$ to some function $f = f(t,x,\cdot) \in L^2_{loc}(\Pi, H)$. Besides, $\forall\, h \in H$

$$-\frac{\partial}{\partial t}\|f_r(t,x,\cdot)-h\|^2 - \sum_{i=1}^m \frac{\partial}{\partial x_i}\big(\tilde{A}_i(f_r(t,x,\cdot)-h), f_r(t,x,\cdot)-h\big)$$
$$= 2r\big(f_r(t,x,\cdot)-g_r(t,x,\cdot), f_r(t,x,\cdot)-h\big) \ge 2r\|f_r(t,x,\cdot)-g_r(t,x,\cdot)\|^2.$$

(the last inequality easily follows from properties of the projection P_H).

The relation above implies that for any nonnegative function $\rho(t,x) \in C_0^\infty(\Pi)$

$$\int_\Pi \|f_r(t,x,\cdot)-g_r(t,x,\cdot)\|^2 \rho(t,x)\,dtdx \le c(\rho)/(2r) \underset{r\to\infty}{\longrightarrow} 0,$$

i.e. $f_r(t,x,u) - g_r(t,x,u) \to 0$ in $L^2_{loc}(\Pi, X)$ as $r \to \infty$. Therefore, $f_r \to f$ as well.

Passing to the limit as $r \to \infty$ in the inequality (3.12) with $f_1 = f_r$, $f_2 \equiv g \in H$ we conclude that the limit function $f = f(t,x,u)$ satisfies condition (3.6). The initial condition (3.5) is also easily verified. Thus, $f(t,x,u)$ is an e.s. of (3.4), (3.5). In view of uniqueness of e.s., the limit function $f(t,x,u)$ doesn't depend on the choice of a convergent subsequence. Therefore, the original sequence $f_r(t,x,u)$ converges to $f(t,x,u)$ in $L^2_{loc}(\Pi, X)$. We proved the following

Theorem 3.4. *Let $f_r = f_r(t,x,u)$ be a g.s. of the problem* (3.10), (3.11), $r \in \mathbb{N}$. *Then the sequence f_r converges in $L^2_{loc}(\Pi, X)$ to the unique e.s. $f(t,x,u)$ of the problem* (3.4), (3.5).

In the important particular case when H is a cone the condition (3.6) could be rewritten in different form.

Theorem 3.5. *Let H be a closed convex cone in X. Then* (3.6) *is equivalent to conditions*

$$\forall\, h \in H \ \frac{\partial}{\partial t}\big(f(t,x,\cdot), h\big) + \sum_{i=1}^m \frac{\partial}{\partial x_i}\big(\tilde{A}_i f(t,x,\cdot), h\big) \ge 0 \quad \text{in } \mathcal{D}'(\Pi) \quad \text{and} \tag{3.14}$$

$$\frac{\partial}{\partial t}\|f(t,x,\cdot)\|^2 + \sum_{i=1}^m \frac{\partial}{\partial x_i}\big(\tilde{A}_i f(t,x,\cdot), f(t,x,\cdot)\big) = 0 \quad \text{in } \mathcal{D}'(\Pi). \tag{3.15}$$

Corollary 3.1. *Let H be a closed linear subspace of X. Then condition* (3.6) *is equivalent to the identity:* $\forall\, h \in H$

$$\frac{\partial}{\partial t}\big(f(t,x,\cdot),h\big) + \frac{\partial}{\partial x}\big(\tilde{A}f(t,x,\cdot),h\big) = 0 \quad in\ \mathcal{D}'(\Pi). \tag{3.16}$$

Remark 3.1. In the case when H is a closed linear subspace existence and uniqueness of e.s. to the problem (3.4), (3.5) can be proved without the condition of strong compactness of bounded closed subsets of H (which holds only for finite-dimensional subspaces). Indeed, we do not use this condition in the proof of the uniqueness, and the existence can be established on the base of weak convergence of the sequence $f_r = f_r(t,x,u)$ of g.s. to (3.10), (3.11). Obviously, the limit function $f(t,x,u)$ must satisfy condition (3.16), which is equivalent to (3.6) by Corollary above.

Remark 3.2. In the case of a scalar equation (1.1) $u_t + \sum_{i=1}^{m} a_i(u)u_{x_i} = 0$, $a_i(u) = \varphi_i'(u)$ we set $H = \{\ f(u) - \theta(-u) \mid f \in F_M\ \}$, where $M > 0$, $\theta(\lambda)$ is the Heaviside function. We see, with using of the Helly theorem, that $H \subset L^2(\mathbb{R})$ is a compact convex set. Therefore, there exists an unique e.s. $f(t,x,u) - \theta(-u)$ of the corresponding kinetic equation $f_t + \sum_{i=1}^{m} a_i(u)f_{x_i} = 0$ with initial data $f(0,x,u) = f_0(x,u) - \theta(-u)$, $f_0(x,\cdot) \in F_M$. It is clear that $f(t,x,u)$ coincides with e.s. in the sense of Definition 2.1. As follows from Theorem 2.2 this solution is also e.s. with respect to the wider class $\tilde{H} = \{\ \lambda f \mid f \in H, \lambda \geq 0\ \}$ that is the conic envelope of H.

Consider some closed convex set $H \subset X$ bounded closed subset of which are compact, and suppose in addition that $\operatorname{div} f(u) = l + \nu\ \forall f(u) \in H$ where $\nu \in \operatorname{Prob}_0(\mathbb{R}^n)$ is a finite Borel probability measure on $\mathbb{R}^n$, $l \in \mathcal{D}'(\mathbb{R}^n)$ is some fixed distribution. Denote

$$M = \{\operatorname{div} f(u) - l \mid f(u) \in H\} \subset \operatorname{Prob}_0(\mathbb{R}^n).$$

Let $\nu_x^0 = \operatorname{div} f_0(x,u) - l$ and $\nu_{t,x} = \operatorname{div} f(t,x,u) - l$ where $f(t,x,u) \in L^2_{loc}(\Pi, H)$ is a unique e.s. of the problem (3.4), (3.5) with initial data $f_0(x,u)$. By analogy with the scalar case $n = 1$ we call the measure valued function $\nu_{t,x}$ a *strong measure valued solution* to the original problem (3.1) with the measure valued initial data ν_x^0. Of course, this notion depends on choice of "kinetic class" H, which is not uniquely determined by the family of measures M. We can restrict ourselves to the class of potential vectors $f(u) = \nabla\varphi(u)$. Then (for $l = 0$) the vector $f(u)$ is uniquely defined by its divergence since the function $\varphi(u)$ must be an unique g.s. to the Poisson equation $\Delta\varphi = \nu$ and therefore

$$f(u) = \nabla\varphi(u) = \int \nabla\mathcal{E}(u-v)d\nu(v),$$

where $\mathcal{E}(u)$ is the fundamental solution of the Laplace equation. The natural restriction to the class of measures $\nu \in M$ is the condition $f(u) \in X$.

To demonstrate connection strong measure valued solutions with measure valued solutions in DiPerna sense (see [8]) we consider the case when the matrices $A_i(u)$ are symmetric and the matrix $B(u)$ is the identity matrix. By analogy with a scalar equation we consider wider conical kinetic class H, which consists of potential vectors $f(u) \in X$, $f(u) = \nabla p(u)$ in $\mathcal{D}'(\mathbb{R}^n)$ such that the distribution $\mathrm{div} f = \Delta p$ is non-negative finite Borel measure in $\mathbb{R}^n$ (not necessary — probability). Assume as usual that H is a convex closed cone, bounded closed subsets of which are compact. Then for any initial data $f_0(x,u) \in L^2_{loc}(\mathbb{R}, H)$ there exists the unique e.s. $f = f(t,x,u) \in L^2_{loc}(\Pi, H)$ of the problem (3.4), (3.5). We consider measure valued (in wider sense) functions $\nu^0_x = \mathrm{div}_u f_0(x,u)$ and $\nu_{t,x} = \mathrm{div}_u f(t,x,u)$. Let continuous functions $p(u)$, $q(u) = (q_1(u), \dots, q_m(u))$ be such that $\nabla p(u) \in H$,

$$\mathrm{div}(\nabla q_i(u) - A_i(u)\nabla p(u)) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^n) \tag{3.17}$$

and $p, q \to 0$ as $u \to \infty$. We call the function $p(u)$ an *entropy* of the system (3.1) and the function $q(u)$ a corresponding *entropy flux* . Remark that $p(u)$ is a subharmonic function, which generalizes the usual convexity requirement. The condition (3.17) is a weaker form of the relation $\nabla q_i(u) = A_i(u)\nabla p(u)$ postulated for classical entropy pairs (p,q) in the Lax sense. In contrast to the Lax relation (which is overdetermined for $n > 2$), the flux vector q is uniquely defined by (3.17) for any general enough entropy $p(u)$.

By Theorem 3.5 the relation (3.14) is fulfilled with $h = \nabla p$. Integrating by parts we conclude that

$$(f(t,x,\cdot), \nabla p) = -\int p(u)\, d\nu_{t,x}(u);$$

$$(\tilde{A}_i f(t,x,\cdot), \nabla p) = (f(t,x,\cdot), \nabla q_i) = -\int q_i(u)\, d\nu_{t,x}(u)$$

and by (3.14) we derive the DiPerna entropy condition

$$\frac{\partial}{\partial t}\int p(u)\, d\nu_{t,x}(u) + \sum_{i=1}^m \frac{\partial}{\partial x_i}\int q_i(u)\, d\nu_{t,x}(u) \le 0 \ \text{ in } \mathcal{D}'(\Pi).$$

Remark in conclusion that some results of this paper including the case of systems with one space variable were published with detailed proofs in the preprint [14].

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Contact information

E. Yu. Panov
Mathematical Analysis Department,
Novgorod State University,
B. St.-Peterburgskaya 41,
173003 Velikiy Novgorod,
Russia
E-Mail: pey@novsu.ac.ru