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Uniqueness and topological properties of number representation

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Abstract. Let b be a complex number with $|b| > 1$ and let D be a finite subset of the complex plane $\mathbb{C}$ such that $0 \in D$ and card $D \geq 2$. A number z is representable by the system (D, b) if $z = \sum\limits_{j=-\infty}^{M} a_j b^j$, where $a_j \in D$. We denote by F the set of numbers which are representable by (D, b) with $M = -1$. The set W consists of numbers that are (D, b) representable with $a_j = 0$ for all negative j. Let F_1 be a set of numbers in F that can be uniquely represented by (D, b). It is shown that: The set of all extreme points of F is a subset of F_1. If $0 \in F_1$, then W is discrete and closed. If $b \in \{z : |z| > 1\} \backslash D'$, where D' is a finite or countable set associated with D and W is discrete and closed, then $0 \in F_1$. For a real number system (D, b), F is homeomorphic to the Cantor set C iff $F \backslash F_1$ is nowhere dense subset of $\mathbb{R}$.

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1. Introduction

Suppose we have a finite set D of complex numbers, $0 \in D$, card$D \geq 2$ and a number $b \in \mathbb{C}$, $|b| > 1$. We denote by F the set of "fractions" for the system (D, b) and by W the corresponding set of integers:

$$f \in F \iff f = \sum_{j=-\infty}^{-1} a_j b^j, \tag{1.1}$$

$$w \in W \iff w = \sum_{j=0}^{M} k_j b^j, \tag{1.2}$$

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where a_j and k_j belong to D.

A "general" number q is representable by the system (D, b) iff

$$q = \sum_{j=-\infty}^{M} a_j b^j, \quad a_j \in D, \tag{1.3}$$

i.e., $g = w + f$, $w \in W$, $f \in F$. We shall write G for the set of all representable numbers, by definitions (1.1), (1.2) and (1.3)

$$G = F + W.$$

The definitions of F and W and various examples of real and complex number systems can be found in [4]. See also [3], [5], [6] for an information about representability of complex numbers by the special complex systems. Topological properties of real number representations were studied in more general situations, in [2], [9], [11].

The purpose of this work is the investigation of similarities between the uniqueness of the representations by the system (D, b) and topological properties of F, W and G.

To avoid ambiguities we recall the following definition.

Definition 1.1. *Let f be an element of the set F. The element f has a unique representation in the form* (1.1) *iff for any two series* $\sum_{j=-\infty}^{-1} k_j^{(1)} b^j$ *and* $\sum_{j=-\infty}^{-1} k_j^{(2)} b^j$, *where all* $k_j^{(1)}$ *and* $k_j^{(2)}$ *belong to* D:

$$\left(f = \sum_{j=-\infty}^{-1} k_j^{(1)} b^j = \sum_{j=-\infty}^{-1} k_j^{(2)} b^j \right) \Longrightarrow (k_j^{(1)} = k_j^{(2)})$$

for each negative integer j.

Let $F_1 = F_1(D, b)$ denote the set of numbers that can be uniquely expressed as (1.1) and let $F_2 = F \backslash F_1$ be the corresponding complementary subset of F. Similarly, we introduce sets W_1, G_1, W_2 and G_2: $w \in W_1$ iff w has a unique representation (1.2); $g \in G_1$ iff g has a unique representation (1.3); $W_2 = W \backslash W_1$ and $G_2 = G \backslash G_1$.

2. Statements of results

It should be noted that some numbers have a single representation in the one form but the same numbers may fail to have the single representation in another form. The first three propositions illuminate this phenomen.

Proposition 2.1. *Let (D, b) be a number system. Then the following three properties are equivalent:*

$$F_2 \neq \emptyset; \tag{2.1}$$

$$G_2 \neq \emptyset; \tag{2.2}$$

$$(F - F) \cap ((D - D) \backslash \{0\}) \neq \emptyset, \tag{2.3}$$

where $F - F = \{x - y : x \in F,\ y \in F\}$ and

$$(D - D)\backslash\{0\} = \{x - y : x \in D,\ y \in D,\ x \neq y\}.$$

Proposition 2.2. *Let (D, b) be a number system. Then the following two properties are equivalent:*

$$F_1 \cap G_2 \neq \emptyset; \tag{2.4}$$

$$(F_1 - F) \cap (D\backslash\{0\}) \neq \emptyset, \tag{2.5}$$

where $F_1 - F = \{x - y : x \in F_1,\ y \in F\}$ and

$$D\backslash\{0\} = \{x : x \in D,\ x \neq 0\}.$$

Example 2.1 Let (D, b) be the usual binary system: $D = \{0, 1\}$, $b = 2$. Then we have $0 \in F$, $1 \in F_1 \cap G_2$ and $1 = 1 - 0$.

Let $B_{2F} = B_{2F}(D)$ and $B_{2W} = B_{2W}(D)$ be the subsets of $\{z \in \mathbb{C} : |z| > 1\}$ defined by the next relations:

$$(b \in B_{2F}) \iff (F_2(D, b) \neq \emptyset), \tag{2.6}$$

$$(b \in B_{2W}) \iff (W_2(D, b) \neq \emptyset). \tag{2.7}$$

Proposition 2.3. *Let D be a finite set of complex numbers,* card $D \geq 2$, $0 \in D$. *Then:*

2.3.1. *B_{2W} is at most countable and nonempty;*

2.3.2. $B_{2F} \supseteq [-2, -1) \cup (1, 2]$.

Example 2.2 Let $b = 3$ and $D = \{0, 2\}$. Then F is the Cantor ternary set C. In this case, it is known that $F_2 = \emptyset$. Consequently, by Proposition 2.1, $G_2 = \emptyset$ and from $W_2 \subseteq G_2$ follows $W_2 = \emptyset$.

If (D, b) is a number system, then the convex hull of F will be denoted by $\hat{F}$. The set of all extreme points of $\hat{F}$ will be denoted by Ext $\hat{F}$. The following theorem shows that there is no number system with $F_1 = \emptyset$.

Theorem 2.1. *Let* (D, b) *be a number system. Then* $\operatorname{Ext} \hat{F}$ *is subset of* F_1*. In symbols,*

$$\operatorname{Ext}\hat{F} \subseteq F_1. \tag{2.8}$$

Corollary 2.1. *Let* (D, b) *be a complex (real) number system. Then* F_1 *is a nonempty* G_δ *subset of* $\mathbb{C}$ *(of* $\mathbb{R}$*) and* F_2 *is* F_σ *subset of* $\mathbb{C}$ *(of* $\mathbb{R}$*).*

Example 2.3 Let (D, b) be the standard decimal system: $D=\{0, 1, \ldots, 9\}$ and $b = 10$. Then we have that: $F = [0, 1]$, $\operatorname{Ext} F = \{0, 1\}$, $0 \in F_1 \cap G_1$ and $1 \in F_1 \cap G_2$.

Remark 2.1. The set of all extreme points of an arbitrary closed convex plane set is closed [1, Exercise 11.9.8]. Since F is compact, $\operatorname{Ext} \hat{F}$ is a compact subset of F.

Theorem 2.2. *Let* (D, b) *be a number system. If* $0 \in G_1$*, then* W *is closed and discrete in* $\mathbb{C}$*.*

Theorem 2.3. *Let* D *be a finite set of complex numbers,* card $D \geq 2$*,* $0 \in D$*. Suppose* $b \in \{z : |z| > 1\} \backslash B_{2W}$*. If* W *is closed and discrete in* $\mathbb{C}$*, then* $0 \in G_1(D, b)$*.*

Remark 2.2. By Proposition 2.3 the set B_{2W} is at most countable and hence Theorem 2.12 is an "almost converse" of Theorem 2.2.

Example 2.4 Let $b = 10$ and $D = \{1, 1, -9\}$. Then $b \in B_{2W}$, zero is not in $G_1(D, b)$, but W is closed and discrete.

Theorem 2.4. *Let* (D, b) *be a number system. Then the following three statements are equivalent:*

2.4.1. F *is homeomorphic to the Cantor ternary set* C*;*

2.4.2. *The small inductive dimension of* G *is zero. In symbols,* ind $G = 0$*;*

2.4.3. ind $\overline{F}_2 \leq 0$.

Corollary 2.2. *Let* (D, b) *be a real number system. Then* F *is homeomorphic to* C *iff* F_2 *is a nowhere dense subset of* $\mathbb{R}$*.*

Remark 2.3. By the definition of small inductive dimension we have $\operatorname{idn}\overline{F}_2 = -1$ iff $\overline{F}_2 = \emptyset$.

The following two propositions define more precisely some aspects of Theorem 2.4 and Corollary 2.2.

Proposition 2.4. *Let* (D, b) *be a number system. If* card$D = 2$ *and* $b \in (1, +\infty)$*, then* F *is homeomorphic to* C *iff* F_2 *is empty.*

Proposition 2.5. *If $n \in \mathbb{N}$, $b \in \mathbb{C}$ and $|b| > n \geq 3$, then there exists a finite set $D \subseteq \mathbb{C}$ such that* card $D = n$, $0 \in D$, *$F(D,b)$ is homeomorphic to C and $F_2(D,b) \neq \emptyset$.*

Our final theorem gives some survey of topological properties of number representation by systems with $F_2 = \emptyset$.

Theorem 2.5. *Let (D,b) be a number system. If $F_2(D,b) = \emptyset$, then:*

2.5.1. *F is compact, perfect, zero-dimensional, that is homeomorphic to the Cantor set C;*

2.5.2. *W is a closed, discrete and unbounded subset of $\mathbb{C}$;*

2.5.3. *G is closed, perfect and zero-dimensional subset of $\mathbb{C}$.*

Remark 2.4. For an arbitrary (D,b), we have the following: F is compact and perfect; W is unbounded; and if W is a closed subset of $\mathbb{C}$, then G is closed, too.

Vector generalizations. Many our propositions and theorems remain valid when one passes from a number system to the following many-dimensional construction: D is a finite set in $\mathbb{R}^n$, including zero, and B is $n \times n$ nonsingular matrix with a norm $\|B\| > 1$. It should also be observed that Theorem 2.1 remains valid for a positional vector system whose definition similar to Definition 2.1 from the Petkovšek's work [9].

3. Proofs

3.1. Proof of Proposition 2.1

The trivial inclusion

$$F_2 \subseteq F \cap G_2 \tag{3.1.1}$$

shows that the implication (2.1) ⇒ (2.2) is correct. Let x be an element of the set G_2. By the definition of G_2 there are two sequences $\{a_j\}$ and $\{a'_j\}$ for which

$$x = \sum_{j=-\infty}^{M} a_j b^j = \sum_{j=-\infty}^{M} a'_j b^j \tag{3.1.2}$$

holds with $\sum\limits_{j=-\infty}^{M} |a_j - a'_j| \neq 0$. Let j_0 be the greatest subscript with $|a_{j_0} - a'_{j_0}| \neq 0$. Then using (3.1.2) we obtain

$$a_{j_0} + \sum_{j=-\infty}^{j_0-1} a_j b^{j-j_0} = a'_{j_0} + \sum_{j=-\infty}^{j_0-1} a'_j b^{j-j_0}, \tag{3.1.3}$$

where $a_{j_0} \neq a'_{j_0}$. The last equality is equivalent to (2.3). So we have only to establish implication (2.3) $\Rightarrow$ (2.1). Suppose that (3.1.3) holds with $a_{j_0} \neq a'_{j_0}$. Then taking the number t as

$$t = \sum_{j=-\infty}^{j_0} a_j b^{j-j_0-1} = \sum_{j=-\infty}^{j_0} a'_j b^{j-j_0-1},$$

we have that $t \in F_2$. Hence we get $F_2 \neq \emptyset$.

3.2. Proof of Proposition 2.2

Suppose d is an element of $(F_1 - F) \cap (D \backslash \{0\})$. Then $d \in D$, $d \neq 0$ and $d = t_1 - t$, with $t_1 \in F_1$ and $t \in F$. Hence $t_1 = d + t \in F_1 \cap G_2$, and we have (2.5) $\Rightarrow$ (2.4). Now suppose f is an element of $G_2 \cap F_1$. Since $f \in F_1$ we have a unique representation

$$f = \sum_{j=-\infty}^{-1} a_j b^j, \tag{3.2.1}$$

where each $a_j \in D$. Let $\mathcal{F} = \mathcal{F}(f)$ be the family of all representations of f which are different from (3.2.1). Since $f \in G_2$, we have that $\mathcal{F} \neq \emptyset$. Let (s) be an element of $\mathcal{F}$

$$(s) = \left(f = \sum_{j=-\infty}^{M_s} a_j^{(s)} b^j \right), \tag{3.2.2}$$

and let $j_0 = j_0(s)$ be the greatest subscript for which $a_{j_0}^{(s)} \neq 0$. Since $f \in F_1$, we have $j_0(s) \geq 0$. Now to prove the implication (2.4) $\Rightarrow$ (2.5), it suffices to justify the equality

$$\min \{j_0(s) : (s) \in \mathcal{F}(f), \quad f \in F_1 \cap G_2\} = 0. \tag{3.2.3}$$

Consider any number $f_0 \in F_1 \cap G_2$ with a representation $(s) \in \mathcal{F}(f_0)$ such that

$$(s) = \left(f_0 = \sum_{j=-\infty}^{M} a_j b^j \right) \quad a_M \neq 0,$$

$$M = \min \{j_0(s) : (s) \in \mathcal{F}(f), \quad f \in F_1 \cap G_2\}.$$

In order to check that (3.2.3) holds it is sufficient to show that

$$(M > 0) \Rightarrow (b^{-1} f_0 \in G_2 \cap F_1).$$

It is clear that $f_0 \in G_2$ implies $b^{-1}f_0 \in G_2$. Suppose $f_0 \in G_2 \cap F_1$, $M > 0$ and $b^{-1}f_0 \in F_2$. By the last supposition we can find two different representations

$$\sum_{j=-\infty}^{-1} a_j^{(1)} b^j = \sum_{j=-\infty}^{-1} a_j^{(2)} b^j = b^{-1} f_0. \tag{3.2.4}$$

If $|a_{-1}^{(1)}| + |a_{-1}^{(2)}| = 0$ holds, then it follows from (3.2.4) that

$$f_0 = \sum_{j=-\infty}^{-2} a_j^{(1)} b^{j+1} = \sum_{j=-\infty}^{-2} a_j^{(2)} b^{j+1}.$$

This contradicts to $f_0 \in F_1$. Consequently, $|a_{-1}^{(1)}| + |a_{-1}^{(2)}| \neq 0$, and we have

$$f_0 = a_{-1}^{(1)} + \sum_{j=-\infty}^{-2} a_j^{(1)} b^{j+1} = a_{-1}^{(2)} + \sum_{j=-\infty}^{-2} a_j^{(2)} b^{j+1},$$

contrary to the assumption $M > 0$.

3.3. Proof of Proposition 2.3

Lemma 3.1. *Let D be a finite set of complex numbers with* card $D \geq 2$ *and $0 \in D$. Then a complex number b belongs to B_{2W} iff $|b| > 1$ and there is a polynomial $p(z) = \sum_{i=0}^{n} a_i z^i$ such that $p(b) = 0$, $n \geq 1$, $a_n \neq 0$ and $a_i \in (D - D)$ for $i = 0, 1, \ldots, n$.*

Lemma 3.2. *The polynomial $p(z) = z^3 - z + 1$ has a real root z_0 with $|z_0| > 1$.*

Lemma 3.3. *Let $D_1 \subseteq D$ be two finite sets of complex numbers, and let $0 \in D_1$,* card $D_1 \geq 2$. *Then*

$$F_2(D_1, b) \subseteq F_2(D, b),$$

for each b with $|b| > 1$.

Lemma 3.4. *Let (D, b) be a number system and let z be a nonzero complex number. Then*

$$F_i(zD, b) = zF_i(D, b).$$

for $i = 1, 2$.

The simple proofs of these lemmas are omitted.

Lemma 3.5. *Let b be a real number with $|b| > 1$ and let $D = \{0,1\}$, then $F_2(D,b)$ is nonempty if and only if*

$$b \in [-2,-1) \cup (1,2].$$

Proof. It follows from Proposition 2.1 that $F_2(D,b)$ is nonempty iff there exists a sequence $\{a_j\}_{-\infty}^{-1}$ whose elements belong to the set $\{-1,0,1\}$ and

$$1 = \sum_{j=-\infty}^{-1} a_j b^j. \tag{3.3.1}$$

Hence in the case $D = \{0,1\}$ we have the equivalence

$$(F_2(D,b) = \emptyset) \equiv (F_2(D,-b) = \emptyset). \tag{3.3.2}$$

Consequently, we shall restrict ourselve, to the case $b > 1$. If $b > 2$, then $\sum\limits_{j=-\infty}^{-1} |a_j b^j| < 1$ and equality (3.3.1) cannot holds. It therefore remains to verify that 1 is a distance between two points of $F(D,b)$ for $b \in (1,2]$. It follows directly from the early Randolph's result [10]

Theorem 3.1 (Randolph). *Let $\{a_n\}^\infty$ be a sequence with $a_n > 0$, $a_1 \geq a_2 \geq \ldots$, and $\sum_{n=1}^{\infty} a_n = 1$. For a fixed $\{a_n\}_{n=1}^{\infty}$, let S be the set of all sums of the form $\sum \varepsilon_n a_n$ where ε_n is equal 1 or 0. Then the set $S - S$ fills the unit interval $[0,1]$ iff*

$$a_n \leq \sum_{k=n+1}^{\infty} a_k.$$

We can now easily prove Proposition 2.3.

Lemma 3.1 implies that B_{2W} is at most countable, and from Lemmas 3.1 and 3.2 it follows that B_{2W} is nonempty. By Lemmas 3.5 and 3.4 we have $B_{2F} \supseteq [-2,-1) \cup (1,2]$ for each two-point set D, and using Lemma 3.3 we have (2.3.2) for the case card $D > 2$. □

3.4. Proof of Theorem 2.1

Let z_0 be an extreme point of $\hat{F}$. Since $\operatorname{Ext} \hat{F} \subseteq \partial \hat{F}$, there is a straight line l_0 which contains z_0, and one of its closed half-plane includes $\hat{F}$ [8, Theorem 3.2]. This is a so-called straight line of support of a convex set $\hat{F}$.

For the sake of simplicity, suppose l_0 and real axis are mutually perpendicular,

$$l_0 = \{z \in \mathbb{C} : \operatorname{Re} z = \operatorname{Re} z_0\}.$$

This we can always do by choosing the suitable $\Theta \in [0, 2\pi)$ and passing on to the set $e^{i\Theta}D = \{e^{i\Theta}d_1, \ldots, e^{i\Theta}d_k\}$ from the "old" set $D = \{d_1, \ldots, d_k\}$. Passing to $e^{i\Theta}D$ we obtain $e^{i\Theta}F$, $e^{i\Theta}F_1$, $e^{i\Theta}\hat{F}$ and $e^{i\Theta}\mathrm{Ext}\,\hat{F}$ from F, F_1, $\hat{F}$ and $\mathrm{Ext}\,\hat{F}$. We can assume, without loss of generality, that

$$\mathrm{Re}\ z \le \mathrm{Re}\ z_0 \tag{3.4.1}$$

for all $z \in \hat{F}$.

Consider first the case where

$$l_0 \cap \hat{F} = \{z_0\}. \tag{3.4.2}$$

For any negative integer j, define D_j by the rule:

$$(a \in D_j) \iff (a \in D \text{ and } \mathrm{Re}(ab^j) = \max_{d \in D} \mathrm{Re}(db^j)). \tag{3.4.3}$$

Since D is finite and nonempty, we have $D_j \neq \emptyset$ for each negative integer j. Let t_0 be the number with a representation

$$t_0 = \sum_{j=-\infty}^{-1} a_j b^j,$$

where $a_j \in D_j$ for $j = -1, -2, \ldots$.

We claim that $t_0 = z_0$. It is obvious that t_0 is an element of F. From the definition of extreme point we have $\mathrm{Ext}\,\hat{F} \subseteq F$ [8, Theorem 4.2]. Hence $z_0 \in F$ and, by (3.4.3) $\mathrm{Re}\ z_0 \le \mathrm{Re}\ t_0$. The reverse inequality follows from (3.4.1). Consequently, $\mathrm{Re}\ z_0 = \mathrm{Re}\ t_0$. From the last equality and (3.4.2) we have $t_0 = z_0$.

The equality $z_0 = t_0$ implies that D_j has the unique element for each negative integer j. Really, given any negative integer j_0, we fix elements $a_{j_0}^{(1)}$ and $a_{j_0}^{(2)}$ of the set D_{j_0}, then for any sequence $\{a_j\}$ such that $a_j \in D_j$ we have

$$a_{j_0}^{(1)} b^{j_0} + \sum_{\substack{j=-\infty \\ j \neq j_0}}^{-1} a_j b^j = z_0 = a_{j_0}^{(2)} b^{j_0} + \sum_{\substack{j=-\infty \\ j \neq j_0}}^{-1} a_j b^j.$$

Hence $a_{j_0}^{(1)} = a_{j_0}^{(2)}$ and D_{j_0} is an one-point set.

We can now easily show that $z_0 \in F_1$. If there are two representations

$$z_0 = \sum_{j=-\infty}^{-1} c_j b^j, \quad c_j \in D$$

and

$$z_0 = \sum_{j=-\infty}^{-1} a_j b^j, \quad a_j \in D_j,$$

then by (3.4.3) the inequality

$$\operatorname{Re}(c_j b^j) \le \operatorname{Re}(a_j b^j) \tag{3.4.4}$$

holds for each negative integer j but

$$\sum_{j=-\infty}^{-1} \operatorname{Re}(c_j b^j) = \sum_{j=-\infty}^{-1} \operatorname{Re}(a_j b^j). \tag{3.4.5}$$

The relations (3.4.4) and (3.4.5) imply the equality

$$\operatorname{Re}(c_j b^j) = \operatorname{Re}(a_j b^j)$$

for each negative integer j. Since D_j is an one-point set, we have $c_j = a_j$ for all j.

Consider now the case where

$$\exists\ z_1 \in l_0 \cap \hat{F} : z_1 \ne z_0.$$

We can restrict ourselves to the situation of the inequality $\operatorname{Im} z_1 < \operatorname{Im} z_0$. From the last inequality it follows that

$$\forall\ z \in \hat{F} \cap l_0 : \operatorname{Im} z \le \operatorname{Im} z_0. \tag{3.4.6}$$

(In the opposite case, z_0 is an interior point of the interval $[z_1, z_2]$ where z_2 is some point of $\hat{F}$. This contradicts to the inclusion $z_0 \in \operatorname{Ext} \hat{F}$.)

For any negative integer j, define D_j^o by the rule:

$$(a \in D_j^o) \iff (a \in D_j \text{ and } \operatorname{Im}(ab^j) = \max_{d \in D_j} \operatorname{Im}(db^j)) \tag{3.4.7}$$

where D_j was defined by (3.4.3). We claim that D_j^o is an one-point set. Let j be a negative integer, and let a_1, a_2 be elements of D_j^o. Then we have:

$$\operatorname{Re}(a_1 b^j) = \operatorname{Re}(a_2 b^j) = \max_{d \in D} \operatorname{Re}(db^j),$$

$$\operatorname{Im}(a_2 b^j) = \operatorname{Im}(a_2 b^j) = \max_{d \in D_j} \operatorname{Im}(db^j).$$

Hence $a_2 b^j = a_1 b^j$ holds. Since $b \ne 0$, it follows that $a_1 = a_2$.

Let us denote by a_j the unique element of D_j^o. Consider an arbitrary representation of z_0,

$$z_0 = \sum_{j=-\infty}^{-1} c_j b^j,$$

where $c_j \in D$ for each j. Now to prove that $z_0 \in F_1$ it suffices to demonstrate the equality $c_j = a_j$ for each negative integer j.

Set $t_0 := \sum\limits_{j=-\infty}^{-1} a_j b^j$ where $a_j \in D_j^o$. As it has been proved above, (3.4.1) and (3.4.3) imply Re $z_0 =$ Re t_0, and hence $t_0 \in \hat{F} \cap l_0$. It follows from the equality Re $z_0 =$ Re t_0 that

$$c_j \in D_j \tag{3.4.9}$$

for each negative integer j. The relation $t_0 \in \hat{F} \cap l_0$ and (3.4.6) imply the inequality

$$\sum_{j=-\infty}^{-1} \operatorname{Im}(a_j b^j) \leq \sum_{j=-\infty}^{-1} \operatorname{Im}(c_j b^j). \tag{3.4.10}$$

By formulal (3.4.7) and (3.4.9) we have

$$\operatorname{Im}(c_j b^j) \leq \operatorname{Im}(a_j b^j), \quad j = -1, -2, \dots .$$

From this and (3.4.10) it follows that $\operatorname{Im}(c_j b^j) = \operatorname{Im}(a_j b^j)$, and hence $c_j \in D_j^o$ for each negative integer j. Since $D_j^o = \{a_j\}$, the equality $a_j = c_j$ hold for all negative integer j.

3.5. Proof of Corollary 2.1

We may assume without loss of generality that (D, b) is a complex number system. Since a convex hull of a compact subset of $\mathbb{R}^n$ is compact [8, Theorem 2.6] and F is a compact subset of $\mathbb{C}$ [4, Proposition 2.2.23], it follows that $\hat{F}$ is compact, and by the Krein-Milman theorem we have that Ext $\hat{F} \neq \emptyset$ [8, Corollary of Theorem 4.2]. The last inequality and (2.8) imply that $F_1 \neq \emptyset$.

We turn to the proof that F_2 is a F_σ. Let us denote by D^ω the product of a countable collection of copies of the discrete space $D = \{d_1, \dots, d_k\}$. As usual, we assume that D^ω has a product (Tychonoff) topology. The classic Tychonoff theorem implies that D^ω is a compact space. All elements of D^ω can be regarded as sequences $\{a_j\}_{j=-\infty}^{-1}$ with $a_j \in D$ for each negative integer j. Define a map $\Phi : D^\omega \to F$ by the rule: if $a = \{a_j\}_{j=-\infty}^{-1} \in D^\omega$, then

$$\Phi(a) = \sum_{j=-\infty}^{-1} a_j b^j. \tag{3.5.1}$$

It is easy to see that Φ is continuous and onto.

Let j_0 be a negative integer and let $d \in D$. Then we set

$$\Pi_d^{j_0} := \{a \in D^\omega : a = (a_{-1}, a_{-2}, \ldots),\ a_{j_0} = d\}.$$

All Π_d^j are closed subsets of the compact D^ω, and hence all Π_d^j are compact.

From the definition 1.1 it follows that

$$F_2 = \bigcup_{j=-\infty}^{-1} \bigcup_{i=1}^{k-1} \bigcup_{l=i+1}^{k} (\Phi(\Pi_{d_i}^j) \cap \Phi(\Pi_{d_l}^j)) \tag{3.5.2}$$

where d_i and d_l are elements of the set $D = \{d_1, \ldots, d_k\}$. Since a continuous image of a compact set is compact, $\Phi(\Pi_d^j)$ is closed for each Π_d^j. Hence, by formula (3.5.2) F_2 is a F_σ.

The definition of F_1 implies that

$$F_1 = (\mathbb{C} \backslash F_2) \cap F. \tag{3.5.3}$$

Since for a metric spaces each closed set is G_δ, it follows that F is G_δ. The complement of an F_σ is G_δ, hence $\mathbb{C}\backslash F_2$ is G_δ. Therefore, by (3.5.3) F_1 is G_δ.

3.6. Proof of Theorem 2.2

Suppose there is either a point $t_0 \in \overline{W} \backslash W$ or a point $t_0 \in W'$ where $\overline{W}$ is the closure of W and W' is the set of all accumulation points of W. In the both cases, we can find a sequence $\{z_n\}$, $n \in \mathbb{N}$, such that:

$$\lim_{n\to\infty} z_n = t_0;\ \forall n \in \mathbb{N} : z_n \in W; \tag{3.6.1}$$

$$\forall n, m \in \mathbb{N} : (n \neq m) \Rightarrow (z_n \neq z_m).$$

For each z_n there exists a representation

$$z_n = \sum_{j=0}^{Q_n} a_j^{(n)} b^j$$

where $a_j^{(n)} \in D$ and $Q_n \geq 0$. Using conditions (3.6.1), we can find a subsequence $\{z_{n_k}\}$, $z_{n_k} = a_{Q_{n_k}}^{(n_k)} b^{Q_{n_k}} + \ldots + a_0^{(n_k)} b^0$, of the sequence $\{z_n\}$ for which

$$\forall n_k \in \mathbb{N} : a_{Q_{n_k}}^{(n_k)} \neq 0 \text{ and } Q_{n_{k+1}} > Q_{n_k}. \tag{3.6.2}$$

We may assume without loss of generality that $\{z_{n_k}\}$ and $\{z_n\}$ coincide. Conditions (3.6.1) and (3.6.2) imply that

$$\lim_{n\to\infty}(a^{(n)}_{Q_n} + a^{(n)}_{Q_{n-1}}b^{-1} + \ldots + a^{(n)}_0 b^{-Q_n}) = \lim_{n\to\infty}\frac{t_0}{b^{Q_n}} = 0. \qquad (3.6.3)$$

Since D is finite and for each $n \in \mathbb{N} : 0 \neq a^{(n)}_{Q_n} \in D$, there exists a constant infinite subsequence $\{a^{(n_k)}_{Q_{n_k}}\}$ of $\{a^{(n)}_{Q_n}\}$ such that

$$a^{(n_k)}_{Q_{n_k}} = d \qquad (3.6.4)$$

with $d \in D$, $d \neq 0$. Now we can, once again, take $n_k = n$. Put

$$\Delta_n := a^{(n)}_{Q_n-1}b^{-1} + \ldots + a^{(n)}_0 b^{-Q_n}.$$

The equalities (3.6.3) and (3.6.4) show that

$$\lim_{n\to\infty}\Delta_n = -d.$$

Since $\Delta_n \in F$ and F is compact, we have $-d \in F$. From the definition of F it follows that the nonzero number $-d$ has a representation

$$-d = \sum_{j=-\infty}^{-1} a_j b^j.$$

Hence

$$0 = d - d = d + \sum_{j=-\infty}^{-1} a_j b^j,$$

that is $0 \in G_2$.

3.7. Proof of Theorem 2.3

Suppose 0 is not in $G_1(D, b)$ but W is closed and discrete in $\mathbb{C}$. It suffices to show that these assumptions imply $b \in B_{2W}(D)$. By the supposition $0 \in G_2(D, b)$, and hence we can find a representation

$$0 = \sum_{j=-\infty}^{Q} a_j b^j \qquad (3.7.1)$$

with $\sum\limits_{j=-\infty}^{Q} |a_j| \neq 0$ and $a_j \in D$ for each j. If there is some $k < Q$ such that $a_j = 0$ for all $j < k$, then

$$0 = \sum_{j=-\infty}^{Q} a_j b^{j-k} = \sum_{j=0}^{Q-k} a_{j+k} b^j.$$

It follows, in this case, that $b \in B_{2W}(D)$ (see Lemma 3.1). Hence we can restrict ourselves to the case when

$$\forall\, j < Q \;\exists\, k < j : |a_k| \neq 0.$$

Let n be a positive integer. If $n \geq Q$, then by (3.7.1)

$$0 = b^n \sum_{j=-\infty}^{Q} a_j b^j = w_n + f_n$$

where

$$w_n := \sum_{j=-n}^{Q} a_j b^{j+n} \text{ and } f_n := \sum_{j=-\infty}^{-n-1} a_j b^{j+n}.$$

Since $f_n \in F$ for each n and F is compact, there is a convergent subsequence $\{f_{n_i}\}$ of the sequence $\{f_n\}$. The equality $w_n + f_n = 0$ implies that $\{w_{n_i}\}$ is convergent, too. Let w be the limit of $\{w_{n_i}\}$. By the assumptions W is discrete and closed. Consequently, for some i_0, we have

$$w = w_{n_{i_0}} = w_{n_{i_0+1}} = w_{n_{i_0+2}} = \dots\ .$$

This implies that $w \in W_2$. Thus $b \in B_{2W}(D)$.

3.8. Proof of Theorem 2.4

Lemma 3.6. *Let (D, b) be a number system. Then the set F is a compact perfect set.*

Proof. It is known that F is compact. It remains to show that every $f \in F$ is an accumulation point of F. By the definition of F we have

$$f = \sum_{j=-\infty}^{-1} a_j b^j$$

with $a_j \in D$. Fix two different elements d_1 and d_2 of the set D. Let i be a negative integer. Setting

$$a_j^{(i)} := \begin{cases} a_j & \text{if } j \neq i, \\ d_1 & \text{if } j = i \text{ and } a_j \neq d_1, \\ d_2 & \text{if } j = i \text{ and } a_j = d_1 \end{cases}$$

and

$$f_i := \sum_{j=-\infty}^{-1} a_j^{(i)} b^j,$$

we obtain the sequence $\{f_i\}_{i=-\infty}^{-1}$ such that

$$\lim_{i\to-\infty} f_i = f,$$

and $f_i \in F$, $f_i \neq f$, for each i. □

Lemma 3.7. *Let (D, b) be a number system. If $F_2 \neq \emptyset$, then F_2 is a dense subset of the set F.*

Proof. Let f_0 be an element of F_2. It is easy verified that $b^j f_0 \in F_2$ for each negative integer j. Let f be an element of F. By the definition of F we have

$$f = \sum_{j=-\infty}^{-1} a_j(f) b^j$$

where $a_j(f) \in D$. For each negative integer k, define f_k by the formula:

$$f_k := b^{k-1} f_0 + \sum_{j=k}^{-1} a_j(f) b^j.$$

Then $f_k \in F_2$ for each k and

$$\lim_{k\to-\infty} f_k = f.$$

□

Now the proof of Theorem 2.4 follows from the properties of zero-dimensional sets and Lemmas 3.6, 3.7, see below.

(2.4.1) $\Rightarrow$ **(2.4.2)** If F is homeomorphic to the Cantor set C, then F is closed and zero-dimensional. An union of a countable family of zero-dimensional closed sets in a separable metric space is zero-dimensional [4, Corollary 3.2.9]. Since W is countable and

$$G = \bigcup_{w\in W} (w + F),$$

the set G is zero-dimensional.

(2.4.2) $\Rightarrow$ **(2.4.3)** Suppose that $\operatorname{ind} G = 0$. Since $\overline{F}_2 \subseteq F \subseteq G$, we have $\operatorname{ind} \overline{F}_2 \leq \operatorname{ind} G = 0$ [4, Theorem 3.1.7].

(2.4.3) $\Rightarrow$ **(2.4.1)** Consider first the case where $\operatorname{ind} \overline{F}_2 = 0$. In this case, F_2 is a nonvoid set and by Lemma 3.7 we have $F = \overline{F}_2$. Using Lemma 3.6, we have that F is a compact, perfect zero-dimensional subset of the complex plane $\mathbb{C}$. Hence F is homeomorphic to the Cantor ternary set [12, Theorem 29.7 and Corollary 30.4]. Now, suppose that ind $\overline{F}_2 = -1$. By the definition of the small inductive dimension we have $F_2 = \emptyset$, i. e., each element of the set F has a unique representation (1.1). Under this condition the map $\Phi : D^\omega \to F$ (see formula 3.5.1) is one-to-one, continuous and onto. Hence Φ is a homeomorphism [12, Theorem 17.14]. Since every two totally disconnected, perfect, compact metrizable spaces are homeomrphic, it follows that the Cantor set C is homeorphic to D^ω. [12, Theorem 30.3 and Corollary 30.4]. Consequently, C is homeomorphic to F.

3.9. Proof Corollary 2.2

Let us denote by Int $\overline{F}_2$ the set of all interior points of the set $\overline{F}_2$. We must show that Int $\overline{F}_2 = \emptyset$ iff ind $\overline{F}_2 \leq 0$. This follows directly from the well-known

Theorem 3.2. [7, Theorem IV.3] *Let $\mathbb{R}^n$ be the Eucliden n–dimensional space, and let $A \subseteq \mathbb{R}^n$. Then* ind $A = n$ *iff* Int $A \neq \emptyset$.

3.10. Proof of Proposition 2.4

We may assume without loss of generality that $D = \{0, 1\}$, (see Lemma 3.4).

Lemma 3.8. [10] *If $b_1 \geq b_2 \geq b_3 \geq \ldots,\ b_n \ldots > 0$, $\sum\limits_{n=1}^{\infty} b_n = s < \infty$ and $b_n \leq \sum\limits_{i=n+1}^{\infty} b_i$, then corresponding to any number z, $0 \leq z \leq s$, there exists a sequence $\{\varepsilon_n\}$ each element of which is either 0 or 1, such that*

$$z = \sum_{n=1}^{\infty} \varepsilon_n b_n.$$

Suppose b is a point in the interval $(1, 2]$. Then from Randolph's Lemma 3.8 it follows that $[0, 1] \subseteq F$, and so F cannot be homeomophic to the Cantor set C. Observe also that by Lemma 3.5 we have $F_2 \neq \emptyset$ for $b \in (1, 2]$. If $b \in (2, \infty)$, then by Lemma 3.5 $F_2 = \emptyset$ and Theorem 2.4 shows that F is homeomorphic to C.

3.11. Proof of Proposition 2.5

For an arbitrary number system (D, b) with $D = \{d_1, \ldots, d_k\}$ we construct the corresponding iterated function system $\{f_1, \ldots, f_k\}$ where

$$f_j : \mathbb{C} \to \mathbb{C}, \quad f_j(z) = b^{-1}z + d_j, \quad j = 1, \ldots, k \quad d_j \in D.$$

It is easy to see that $F = F(D, b)$ is the invariant set for this iterated system, that is

$$F = \bigcup_{j=1}^{k} f_j(F),$$

and similarity dimension of F is

$$s(F) = \lg(k)/\lg(b)$$

(see, for example, [4, Chapter 4]). Since $\operatorname{ind} F \le s(F)$ [4, Theorem 6.2.10 and Theorem 6.3.8], it follows from Theorem 2.4 that if $\operatorname{card} D < |b|$, then F is homeomorphic to the Cantor set C.

Hence, if $n \in \mathbb{N}$, $n \ge 3$, $b \in \mathbb{C}$, $|b| > n$, $\{0, 1, b^{-1}\} \subseteq D \subseteq \mathbb{C}$, $\operatorname{card} D = n$, then $F(D, b)$ is homeomorphic to C and by Proposition 2.1 $F_2(D, b) \neq \emptyset$.

3.12. Proof of Theorem 2.5

Suppose W is closed. It is enough to show that G is closed.

Let g be an accumulation point of G. Then there is a sequence $\{g_n\}_{n=1}^{\infty}$ such that $g = \lim_{n\to\infty} g_n$ and $g_n \in G$ for each n. By the definition of G we have $g_n = f_n + w_n$ where $f_n \in F$ and $w_n \in W$. Since F is compact, there is a convergent subsequence $\{f_{n_k}\}$ of the sequence $\{f_n\}$. Set

$$f := \lim_{k\to\infty} f_{n_k}.$$

Then we have

$$g - f = \lim_{k\to\infty} w_{n_k},$$

and since W is closed, it follows that $g - f \in W$. Hence

$$g = (g - f) + f \in W + F = G.$$

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