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Phase spaces for a class of Sobolev type equations

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Abstract. The solvability of the Cauchy problem $u(0) = u_0$ of an semilinear differential operator equation $L\dot{u} = Mu + N(u)$ is under consideration. The abstract results are illustrated by the Cauchy–Dirichlet problem for degenerate reaction-diffusion equations and for Navier–Stokes equations, and by the Cauchy–Bernard problem for Oskolkov thermoconvection equations.

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1. Introduction

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, and let operators $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ (i. e. linear and continuous) and $M \in \mathcal{C}l(\mathfrak{U}; \mathfrak{F})$ (i. e. linear, closed and densely defined). We shall study the Cauchy problem

$$u(0) = u_0 \tag{1.1}$$

for the differential operator equation

$$L\dot{u} = Mu + N(u), \tag{1.2}$$

where $\ker L \neq \{0\}$, and $N : \mathrm{dom}N \subset \mathfrak{U} \to \mathfrak{F}$ is generally speaking nonlinear operator. Following [1] we shall call the Eq. (1.2) a *semilinear Sobolev type equation*, in contrast to *linear Sobolev type equation*

$$L\dot{u} = Mu. \tag{1.3}$$

The problems (1.1), (1.2) and (1.1), (1.3) are in the focus of the attention of many researches (see monographs [2]–[5] for references). In

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contrast to all these results our approach bases on the consept of the *phase space* that are understood more broadly than in application solely to Hamiltonian systems. Roughly speaking, a set $\mathfrak{P}$ is called a phase space of the Eq. (1.2) (or Eq. (1.3)) if there exists a unique solution $u = u(t)$ of problem (1.1), (1.2) (or (1.1), (1.3)) on some semiinterval $[0, T)$ for all $u_0 \in \mathfrak{P}$.

If an operator $L : \mathfrak{U} \to \mathfrak{F}$ is continuously invertible, then the Eq. (1.2) and Eq. (1.3) are reduced trivially to the equations

$$\dot{u} = Su + F(u) \tag{1.4}$$

and

$$\dot{u} = Su \tag{1.5}$$

respectively with an operator $S \in \mathcal{C}l(\mathfrak{U})$ and a nonlinear operator F : dom$F \to \mathfrak{U}$ in the right-hand side. If in addition S is a *sectorial* operator [6], then by the Solomyak–Yosida theorem the Eq. (1.5) has an analytic semigroup of resolving operators, which are represented by *Dunford integral*

$$U^t = \frac{1}{2\pi i} \int\limits_{\Gamma} (\mu I - S)^{-1} e^{\mu t} d\mu, \ t \in \mathbb{R}_+, \ U^0 = I. \tag{1.6}$$

Thereupon the problem (1.1), (1.4) is solved, if operator $F \in C^1(\mathfrak{U}_\alpha; \mathfrak{U})$, where $\mathfrak{U}_\alpha = [\mathfrak{U}_0, \mathfrak{U}_1]_\alpha$, $\alpha \in [0, 1)$, is interpolated space, $\mathfrak{U}_0 = \mathfrak{U}$, $\mathfrak{U}_1$ is dom$S$ equipped by "graphic norm" [6].

The idea of the phase space method consists in reducindg (1.2), (1.3) topt (1.4), (1.5) respectively that are given, however, not on all $\mathfrak{U}_\alpha$ (or $\mathfrak{U}$), but on some (possibly, smooth Banach) manifold imbedded in $\mathfrak{U}_\alpha$ (or $\mathfrak{U}$). In [8] the problem (1.1), (1.2) was investigated under main assumption: the point $\mu = \infty$ is a simple pole of the *L-resolvent* $(\mu L - M)^{-1}$ of operator M. In our case the point $\mu = \infty$ may not be isolated point of the L-resolvent of operator M.

The paper consists of three sections. Exept of Introduction the second section is of propaedeutic character. It contains already known results [7], that are presented in our arrangement. The main goal of this section is to show the construction of resolving semigroups of the Eq. (1.3). These semigroups are created like the semigroup (1.5).

In the third section we carry out abstract discussions, consisting in the application of the modified Lyapunov–Schmidt method to studying of the problem (1.1), (1.2). We attempt to reduce the Eq. (1.2) to the Eq. (1.3). Remark that the Cauchy problem $(\xi(0), \varphi(0)) = (0, 0)$ for equations

$$0 = \eta - \xi^2, \quad \dot{\eta} = \xi \tag{1.7}$$

has two solutions stationary $(0,0)$ or nonstationary $(t/2, t^2/4)$, but the same problem for equations

$$0 = \eta - \xi^2, \quad \dot{\eta} = \xi + 1 \tag{1.8}$$

has not solution. Since Eq. (1.7) and Eq. (1.8) are simplest examples of the Eq. (1.2) then the problem (1.1), (1.2) is not well-posed in general. This simple observation shows the necessity of the restriction of the notion of the solution to the problem (1.1), (1.2).

The fourth section contains some examples arised in applications. We apply obtained abstract results to the Cauchy–Dirichlet problem for degenerate reaction-diffusion equations and for Navier–Stokes equations, and to the Cauchy–Bernard problem for Oskolkov thermoconvection equations. The main goal of this section is to study the *morphology* (i. e. structure, lattice, organization) of the phase space of a concrete interpretation of the problem (1.1), (1.2).

In conclusion let us agree to all arguments that are carried out in real Banach spaces, but when "spectral" questions are considered, the natural complexification is introduced; all contours are oriented by "counterclockwise" motion and bound domains that lying on the "left hand" side under such motion; symbols I and $\mathbb{O}$ denote the "unique" and "null" operators respectively whose domains of definition are clear from context.

2. Relatively p-sectorial operators and degenerate analytic semigroups

Let $\mathfrak{U}$ and $\mathfrak{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$, and operator $M : \mathrm{dom} M \subset \mathfrak{U} \to \mathfrak{F}$ be linear and closed.

Definition 2.1. *Set*

$$\rho^L(M) = \{\mu \in \mathbb{C} \mid (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$$

is called a resolvent set of an operator M with respect to an operator L (or, briefly, L-resolvent set of an operator M). The set $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ is called spectrum of an operator M with respect to an operator L (or, briefly, L- spectrum of an operator M).

Remark 2.1. *When there exists an operator $L^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ L-resolvent set and L-spectrum of the operator M coincide with the resolvent set and the spectrum of the operator $L^{-1}M$ (or the operator ML^{-1}).*

Remark 2.2. *The L-resolvent set of the operator M is always open, and, consequently, the L-spectrum of the operator M is always closed.*

Definition 2.2. *Operator functions* $(\mu L - M)^{-1}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$ *are called respectively a resolvent, right resolvent, and left resolvent of an operator* M *with respect to the operator* L *(or, briefly,* L*-resolvent, right* L*-resolvent, and left* L*-resolvent of the operator* M*).*

Remark 2.3. *When there exists an operator* $L^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ *the right (left)* L*-resolvent of the operator* M *coincides with the resolvent of the operator* $L^{-1}M$ (ML^{-1})*.*

Lemma 2.1. *The* L*-resolvent, right and left* L*-resolvents of the operator* M *are continuous on* $\rho^L(M)$*.*

Theorem 2.1. *The* L*-resolvent, right and left* L*-resolvents of the operator* M *are analytic in* $\rho^L(M)$*.*

Now let an operator $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, and an operator $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$.

Definition 2.3. *An operator* M *is called* p*-sectorial with respect to an operator* L *with a number* $p \in \mathbb{N}_0$ *(or, briefly,* (L,p)*-sectorial), if*

(i) there exist constants $a \in \mathbb{R}$ *and* $\theta \in (\pi/2, \pi)$ *such that the sector*

$$S^L_{a,\theta}(M) = \{\mu \in \mathbb{C} \mid |\arg(\mu - a)| < \theta,\ \mu \neq a\} \subset \rho^L(M)\,,$$

(ii) there exists a constant $K \in \mathbb{R}_+$ *such that*

$$\max\{\|R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L^L_{(\mu,p)}(M)\|_{\mathcal{L}(F)}\} \leq K \,/ \prod_{q=0}^{p} |\mu_q - a|$$

for every $\mu_0, \mu_1, \ldots, \mu_p \in S^L_{a,\theta}(M)$*.*

Remark 2.4. *When there exists an operator* $L^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$*, the operator* M *is* $(L,0)$*-sectorial precisely when the operator* $L^{-1}M$ *is sectorial (or, which is equivalent, the operator* ML^{-1}*).*

Supposing $\rho^L \neq \emptyset$ let us introduce into consideration a pair of equations one of which is equivalent to Eq. (1.3)

$$R^L_\alpha(M)\dot{u} = (\alpha L - M)^{-1}Mu\,, \tag{2.1}$$

$$L^L_\alpha(M)\dot{f} = M(\alpha L - M)^{-1}f\,. \tag{2.2}$$

Both of these equations will be considered as concrete interpretations of the equation

$$A\dot{v} = Bv\,, \tag{2.3}$$

where operators $A, B \in \mathcal{L}(\mathfrak{V})$; $\mathfrak{V}$ is Banach space. Further vector function $v \in C^1(\mathbb{R}_+;\mathfrak{V})$ satisfying this equation will be called the *relaxed solution of Eq.* (2.3).

Definition 2.4. *A mapping $V^{\cdot} \in C^1(\mathbb{R}_+; \mathcal{L}(\mathfrak{V}))$ is called a semigroup of solving operators (or briefly, a solving semigroup) of Eq.* (2.3), *if*

(i) $V^s V^t = V^{s+t}\ \forall s, t \in \mathbb{R}_+$;

(ii) for every $v_0 \in \mathfrak{V}$ the vector function $v(t) = V^t v_0$ is the relaxed solution of Eq. (2.3).

Let us identify the semigroup with the set $\{V^t \mid t \in \mathbb{R}_+\}$. A semigroup $\{V^t \mid t \in \mathbb{R}_+\}$ will be called *analytic*, if it admits some analytic extension to a certain sector containing a ray $\mathbb{R}_+$, while retaining its properties (i), (ii), and will be called *uniformly bounded*, if

$$\exists C \in \mathbb{R} \quad \forall t \in \mathbb{R}_+ \quad \|V^t\|_{\mathcal{L}(\mathfrak{V})} \leq C.$$

Remark 2.5. *Note that the fact that a solving semigroup of Eq.* (2.3) *with an identity is not postulated.*

Theorem 2.2. *Let an operator M be (L, p)-sectorial. Then there exists an analytic and uniformly bounded solving semigroup of Eq.* (2.1) *(Eq.* (2.2)*).*

These semigroups may be represented by the integrals of *the Dunford–Taylor type*

$$U^t = \frac{1}{2\pi i} \int\limits_{\Gamma} R^L_\mu(M) e^{\mu t} d\mu,\ t \in \mathbb{R}_+, \tag{2.4}$$

$$F^t = \frac{1}{2\pi i} \int\limits_{\Gamma} L^L_\mu(M) e^{\mu t} d\mu,\ t \in \mathbb{R}_+, \tag{2.5}$$

where $\Gamma \subset S^L_{a,\theta}(M)$ is a contour such that $\arg \mu \to \pm\theta$ as $|\mu| \to \infty$, $\mu \in \Gamma$.

Remark 2.6. *The semigroup $\{U^t : t \in \mathbb{R}_+\}$ is the resolving semigroup of the Eq.* (1.3). *Let us introduce the sets*

$$\ker U^{\cdot} = \{u \in \mathfrak{U} : U^t u = 0\, \forall t \in \mathbb{R}_+\},\ \ker F^{\cdot} = \{f \in \mathfrak{F} : F^t f = 0\, \forall t \in \mathbb{R}_+\},$$

$$\mathrm{im} U^{\cdot} = \{u \in \mathfrak{U} : \lim_{t \to 0+} U^t u = u\}, \quad \mathrm{im} F^{\cdot} = \{f \in \mathfrak{F} : \lim_{t \to 0+} F^t f = f\},$$

and let us set

$$\mathfrak{U}^0 = \ker R^L_{(\mu,p)}(M), \quad \mathfrak{F}^0 = \ker L^L_{(\mu,p)}(M),$$
$$\mathfrak{U}^1 = \overline{\mathrm{im} R^L_{(\mu,p)}(M)}, \quad \mathfrak{F}^1 = \overline{\mathrm{im} L^L_{(\mu,p)}(M)}.$$

Theorem 2.3. *Suppose that an operator M is (L, p)-sectorial. Then*

$$\mathfrak{U}^0 = \ker U^{\cdot}, \quad \mathfrak{F}^0 = \ker F^{\cdot}, \quad \mathfrak{U}^1 = \mathrm{im} U^{\cdot}, \quad \mathfrak{F}^1 = \mathrm{im} F^{\cdot}.$$

Now we are introducing the operators

$$L_k = L\Big|_{\mathfrak{U}^k}, \quad M_k = M\Big|_{\mathrm{dom}M\cap\mathfrak{U}^k}, \quad k=0,1.$$

Theorem 2.4. *Assume that an operator M is (L,p)-sectorial. Then*
(i) $L_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $k=0,1$;
(ii) $M_k : \mathrm{dom}M\cap\mathfrak{U}^k \to \mathfrak{F}^k$, $k=0,1$;
(iii) there exists the operator $M_0^{-1}\in\mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$;
(iv) the operator $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ is nilpotent with degree of nilpotency not greater than p.

Later on we are interesting in the cases

$$\mathfrak{U} = \mathfrak{U}^0\oplus\mathfrak{U}^1 \quad \text{and} \quad \mathfrak{F} = \mathfrak{F}^0\oplus\mathfrak{F}^1. \tag{2.6}$$

Theorem 2.5. *(Yagi–Fedorov* [9], [10]*). Let an operator M be (L,p)-sectorial, and let Banach space $\mathfrak{U}$ ($\mathfrak{F}$) be reflexive. Then $\mathfrak{U}=\mathfrak{U}^0\oplus\mathfrak{U}^1$ ($\mathfrak{F}=\mathfrak{F}^0\oplus\mathfrak{F}^1$).*

Further we need the condition:

$$\textit{there exists the operator } L_1^{-1}\in\mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1). \tag{2.7}$$

Definition 2.5. *Operator M is called strongly (L,p)-sectorial if it is (L,p)-sectorial and*
(i) there exists a dense in $\mathfrak{F}$ subspace $\overset{\circ}{\mathfrak{F}}$ such that for all $f\in\overset{\circ}{\mathfrak{F}}$

$$\left\|M(\lambda L-M)^{-1}\prod_{q=0}^{p}L(\mu_qL-M)^{-1}f\right\|_{\mathfrak{F}} \lim_{\epsilon\to 0}\frac{\mathrm{const}(f)}{|\lambda-a|\prod\limits_{q=0}^{p}|\mu_q-a|};$$

$$\text{(ii)} \quad \left\|(\lambda L-M)^{-1}\prod_{q=0}^{p}L(\mu_qL-M)^{-1}\right\|_{\mathcal{L}(\mathfrak{F};\mathfrak{U})} \lim_{\epsilon\to 0}\frac{\mathrm{const}}{|\lambda-a|\prod\limits_{q=0}^{p}|\mu_q-a|}$$

for all $\lambda,\mu_0,\mu_1,\dots,\mu_p\in S^L_{a,\theta}(M)$.

Theorem 2.6. *Let an operator M be strongly (L,p)-sectorial. Then the conditions* (2.6), (2.7) *are fulfilled.*

Now we can construct the operator $S=L_1^{-1}M_1$.

Theorem 2.7. *Let an operator M be (L,p)-sectorial, and the conditions* (2.6), (2.7) *are fulfilled. Then*
(i) the operator $S\in\mathcal{C}l(\mathfrak{U}^1)$ is sectorial;
(ii) $M_k\in\mathcal{C}l(\mathfrak{U}^k;\mathfrak{F}^k)$.

3. Quasistationary semitrajectories

Suppose that an operator M is (L,p)-sectorial. Then by the Theorem 2.2 let us put $\mathfrak{U}_0^k = \mathfrak{U}^k$, and $\mathfrak{U}_1^k = \mathrm{dom} M \cap \mathfrak{U}^k$, $k = 0,1$. The spaces $\mathfrak{U}_1^k$, $k = 0,1$, equipped by the "graphic norm" is Banach spaces. If an operator M is (L,p)-sectorial and the condition (2.6) is fulfilled, then the embeddings $\mathfrak{U}_1^k \hookrightarrow \mathfrak{U}_0^k$, $k = 0,1$, are dense and continuous. Denote by $\mathfrak{U}_\alpha^1 = [\mathfrak{U}_0^1, \mathfrak{U}_1^1]_\alpha$, $\alpha \in [0,1)$, an interpolation space [6], and by $\mathfrak{U}_\alpha$ the direct sum $\mathfrak{U}_1^0 \oplus \mathfrak{U}_\alpha^1$. A vector-function $u : (0,T) \to \mathfrak{U}_\alpha$ is called a *solution of the Eq.* (1.2) if it satisfies to this Eq. (1.2). A solution $u = u(t)$ of the Eq. (1.2) is called *solution of the problem* (1.1), (1.2), if $\lim\limits_{t\to 0+} u(t) = u_0$ in $\mathfrak{U}_\alpha$.

Return to the Eq. (1.2). Under assumptions of the Theorem 2.2 and the conditions (2.6), (2.7) we can reduce this equation to the equivalent system

$$H\dot{u}^0 = u^0 + M_0^{-1}(I - Q)N(u), \tag{3.1}$$

$$\dot{u}^1 = Su^1 + L_1^{-1}QN(u), \tag{3.2}$$

where $u^1 = Pu$, $u^0 = u - u^1$, the operator P (Q) is the projection onto $\mathfrak{U}^1$ $(\mathfrak{F}^1)$ along $\mathfrak{U}^0$ $(\mathfrak{F}^0)$, the operator $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ is nilpotent, and the operator $S = L_1^{-1}M_1 \in \mathcal{C}l(\mathfrak{U}^1)$ is sectorial.

Definition 3.1. *A solution $u = u(t)$ of the problem* (1.1), (1.2) *is called a quasistationary semitrajectory of the Eq.* (1.2) *passing through the point u_0 if $H\dot{u}^0(t) = 0$ for every $t \in (0,T)$.*

Recall that a stationary solution of the Eq. (1.2) is a quasistationary semitrajectory, but the converse is false. In the above mentioned example (1.7) the quasistationary semitrajectories coincide with the stationary one, i. e. with the point $(0,0)$. In general $(0,0)$ is not the unique solution of the problem $\dot{\eta} = \xi$, $0 = \eta - \xi^2$, $\xi(0) = 0$, $\eta(0) = 0$; there is one more: $(t/2, t^2/4)$; however as a quasistationary semitrajectory the point $(0,0)$ is a unique solution of this problem. Remark also that the example (1.8) has not quasistationary semitrajectory passing through the point $(0,0)$.

To find qusistationary semitrajectories of the Eq. (1.2) we introduce in consideration a set

$$\mathfrak{M} = \{u \in \mathfrak{U}_\alpha : (I - Q)(Mu + N(u)) = 0\}.$$

It is obvious (see (3.1)) that if $u = u(t)$ is a quasistationary semitrajectory then it lies in $\mathfrak{M}$ (i. e. $u(t) \in \mathfrak{M}$ for every $t \in [0,T)$). Let a point $u_0 \in \mathfrak{M}$. Set $u_0^1 = Pu_0$ and by $O_0^1 \subset \mathfrak{U}_\alpha^1$ define a neighborhood of the point $u_0^1 \in \mathfrak{U}_\alpha^1$. If there exists a C^∞-diffeomorphism $\delta : O_0^1 \to \mathfrak{M}$ such that $\delta^{-1} = P$, then we shall call the set $\mathfrak{M}$ a *Banach C^∞-manifold at the point u_0*. If the set $\mathfrak{M}$ is a Banach C^∞-manifold at every point $u_0 \in \mathfrak{M}$, then we shall call the set $\mathfrak{M}$ a *Banach C^∞-manifold modeling by the subspace $\mathfrak{U}_\alpha^1$*. Connected Banach C^∞-manifold is called a simple Banach C^∞-manifold if every its atlas is equivalent to the atlas containing only a map.

Theorem 3.1. *Let an operator M be (L,p)-sectorial, and an operator $N \in C^\infty(\mathfrak{U}_\alpha^1;\mathfrak{F})$. Let the conditions* (1.3), (1.4) *be fulfilled, and the set $\mathfrak{M}$ be a Banach C^∞-manifold at the point u_0. Then for any $T \in \mathbb{R}_+$ there exists a unique quasistationary semitrajectory of the Eq.* (1.2) *passing through the point u_0.*

Proof. In the neighborhood O_0^1 of the point u_0^1 the Eq. (3.2) may be written in the form
$$\dot{u}^1 = Su^1 + F(u^1), \tag{3.3}$$
where the operator $F = L_1^{-1}QN\delta \in C^\infty(O_0^1;\mathfrak{U}^1)$, and the operator $S \in \mathcal{C}l(\mathfrak{U}^1)$ is sectorial. The existence of a unique solution of the Cauchy problem $u^1(0) = u_0^1$ for the Eq. (3.3) for some $T \in \mathbb{R}_+$ is the classical result [6]. Required qusistationary semitrajectory $u = u(t)$ has the form $u(t) = \delta(u^1(t)) + u^1(t)$. □

In conclusion let us consider the problem (1.1), (1.2) where an operator M is (L,σ)-bounded. A vector-function $u \in C^\infty((-T,T);\mathfrak{U})$ is called a *solution of the equation* if it satisfies to this equation for any $T \in \mathbb{R}_+$. A solution of the Eq. (1.2) is called a *solution of the problem* (1.1), (1.2) if it satisfies to (1.2). By analogy with above mentioned set $\mathfrak{M}$ we introduce in consideration a set
$$\mathfrak{M}' = \{u \in \mathfrak{U} : (I-Q)(Mu + N(u)) = 0\}.$$

Definition 3.2. *A solution $u = u(t)$ of the problem* (1.1), (1.2) *is called a quasistationary trajectory of the Eq.* (1.2) *passing through the point u_0 if $H\dot{u}^0(t) = 0$ for every $t \in (-T,T)$.*

Remark that in the case of (L,σ)-boudedness of an operator M quasistationary semitrajectory may be continued "back" by the time.

Theorem 3.2. *Let an operator M be (L,σ)-bounded, moreover, ∞ be a removable singular point or a pole of the order $p \in \mathbb{N}$. Let an operator $N \in C^\infty(\mathfrak{U};\mathfrak{F})$, and the set $\mathfrak{M}'$ be a Banach C^∞-manifold at the point u_0. Then for any $T \in \mathbb{R}_+$ there exists a unique quasistationary trajectory $u = u(t)$, $t \in (-T,T)$, of the Eq.* (1.2) *passing through the point u_0.*

Proof. In the neighborhood $O_0^1 \subset \mathfrak{U}^1$ of the point $u_0^1 = Pu_0$ the Eq. (1.2) may be written in the form (3.3). Multiplying (3.3) by δ'_{u^1} on the left we can take the equation

$$\dot{u} = G(u), \tag{3.4}$$

where the operator $G = \delta'_{u^1}(S + L_1^{-1}QN\delta) : u \to T_u\mathfrak{M}$, the operator δ'_{u^1} is the Frechet derivative of C^∞-diffeomorphism δ at the point $u^1 = Pu$, $T_u\mathfrak{M}$ is the tangent space. The existence of a unique solution of the Cauchy problem (1.1) for the Eq. (3.4) for any $T \in \mathbb{R}_+$ is the classical Cauchy theorem [11]. $\square$

Remark 3.1. *If ∞ is essential singular point then the Theorem* 3.2 *is false even for the linear Eq.* (1.3) [7].

4. Phase spaces

Let us return to the Eq. (1.2).

Definition 4.1. *A set $\mathfrak{P} \subset \mathfrak{U}$ is called a phase space of the Eq.* (1.2), *if*

(i) every solution $u = u(t)$ of the Eq. (1.2) *lies in $\mathfrak{P}$, i. e. $v(t) \in \mathfrak{P}$ $\forall t \in \mathbb{R}_+$;*

(ii) for every $u_0 \in \mathfrak{P}$ there exists a unique solution of the problem (1.1), (1.2).

In this section we shall consider such examples, in which the phase space is simple Banach C^∞-manifold and coincides with the set $\mathfrak{M}$.

Example 4.1 The hybrid of Oskolkov system and heat equation in the Oberbeck–Boussinesque approximation

$$\begin{aligned} &(1-\kappa\nabla^2)v_t = \nu\nabla^2 v - (v\cdot\nabla)v - \nabla p + g\gamma S,\\ &\nabla\cdot v = 0,\\ &S_t = \delta\nabla^2 S - v\cdot\nabla S + \gamma v \end{aligned} \tag{4.1}$$

describes the thermal convection of viscoelastic incompressible fluid (see [12] for more details). If one of horizontal components of the velocity is equal to zero, then the system (4.1) is reduced to a system

$$\begin{aligned} &(1-\kappa\Delta)\Delta\frac{\partial\psi}{\partial t} = \nu\Delta^2\psi - \frac{\partial(\psi,\Delta\psi)}{\partial(x,y)} + g\gamma\frac{\partial\theta}{\partial x},\\ &\frac{\partial\theta}{\partial t} = \delta\Delta\theta - \frac{\partial(\psi,\theta)}{\partial(x,y)} + \beta\frac{\partial\psi}{\partial x}, \end{aligned} \tag{4.2}$$

which models plane-parallel thermal convection.

Set $\Omega = (0,l) \times (0,h)$, where $l, h \in \mathbb{R}_+$. At the domain $\Omega \times \mathbb{R}_+$ we shall consider the Cauchy–Bernard problem for the system (4.2)

$$\psi(x,y,0) = \psi_0(x,y), \quad \theta(x,y,0) = \theta_0(x,y), \quad (x,y) \in \Omega; \tag{4.3}$$

$$\psi(x,0,t) = \Delta\psi(x,0,t) = \psi(x,h,t) = \Delta\psi(x,h,t) = 0, \tag{4.4}$$

$$\theta(x,0,t) = \theta(x,h,t) = 0, \quad x \in (0,l), \quad t \in \mathbb{R}_+; \tag{4.5}$$

$$\text{functions } \psi \text{ and } \theta \text{ are periodic with respect to } x. \tag{4.6}$$

We shall reduce the problem (4.2)–(4.6) to the problem (1.1), (1.2). To this end we set $\mathfrak{U} = \mathfrak{U}_\psi \times \mathfrak{U}_\theta$, $\mathfrak{F} = \mathfrak{F}_\psi \times \mathfrak{U}_\theta$, where $\mathfrak{U}_\psi = \{\psi \in W_2^4(\Omega) :$ ψ satisfies (4.4), (4.6)$\}$, $\mathfrak{U}_\theta = \mathfrak{F}_\psi = L^2(\Omega)$. We define the operators

$$L = \begin{pmatrix} (I-\kappa\Delta)\Delta & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix}, \kappa \in \mathbb{R}; \quad M = \begin{pmatrix} \nu\Delta^2 & \mathbb{O} \\ \mathbb{O} & \delta\Delta \end{pmatrix}, \kappa \in \mathbb{R};$$

where dom$M = \mathfrak{U}_\psi \times \{\theta \in W_1^2(\Omega) : \theta$ satisfies (4.5), (4.6)$\}$. It is clear that the operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$.

Lemma 4.1. [12]. *For every $\kappa \in \mathbb{R} \setminus \{0\}$, $\nu, \delta \in \mathbb{R}_+$ the operator M is strongly $(L,0)$-sectorial.*

Further we set

$$\mathrm{dom}N = \mathfrak{U}_\psi \times \{\theta \in W_1^2(\Omega) : \theta \text{ satisfies (4.5), (4.6)}\}$$

and by the formula

$$N(u) = \begin{pmatrix} g\gamma\dfrac{\partial\theta}{\partial x} & \dfrac{\partial(\psi,\Delta\psi)}{\partial(x,y)} \\ \dfrac{\partial\psi}{\partial x} & \dfrac{\partial(\psi,\theta)}{\partial(x,y)} \end{pmatrix}$$

we define the operator $N : \mathrm{dom}N \to \mathfrak{F}$. It is obviously that dom$N \subset [\mathfrak{U}_0, \mathfrak{U}_1]_\alpha$, $\alpha \in (0,1)$.

Lemma 4.2. [12]. *For every $g, \gamma \in \mathbb{R}$, $\alpha \in (0,1)$ the operator $N \in C^\infty(\mathfrak{U}_\alpha;\mathfrak{F})$.*

It is easy to show that all Frechet derivatives of the operator N are equal to zero, when the order of derivative is greater than two.

Denote by $\sigma(A)$ the spectrum of the homogeneous Dirichlet problem for the Laplace operator Δ in the domain Ω. Using the methods of [12] one can obtain the following result.

Theorem 4.1. *For every $\gamma, g \in \mathbb{R}$, $\delta, \nu \in \mathbb{R}_+$, $\alpha \in (0,1)$ and*

(i) $\kappa^{-1} \notin \sigma(\Delta)$ the phase space of the problem (4.2), (4.4)–(4.6) *is whole space $\mathfrak{U}_\alpha$;*

(ii) $\kappa^{-1} \in \sigma(\Delta)$ the phase space of the problem (4.2), (4.4)–(4.6) *is a simple Banach C^∞-manifold $\mathfrak{M} = \{u \in \mathfrak{U}_\alpha : \langle Mu + N(u), \varphi_l\rangle = 0, \kappa^{-1} = \lambda_l\}$ that is modelling by the subspace $\mathfrak{U}^1_\alpha = \{u \in \mathfrak{U}_\alpha : \langle u, \varphi_l\rangle = 0, \kappa^{-1} = \lambda_l\}$.*

Here $\langle\cdot,\cdot\rangle$ is inner product in $\mathfrak{F}$, $\{\lambda_k\} = \sigma(\Delta)$, $\{\varphi_k\}$ is orthonormal family of corresponding eigenfunctions.

Example 4.2 Let $\Omega \subset \mathbb{R}^n$ be bounded domain with a boundary $\partial\Omega$ of the class C^∞. In the semicylinder $\Omega \times \mathbb{R}_+$ we consider a system of the equations of the reaction-diffusion type

$$0 = \alpha_1 \Delta u_1 + f_1(u_1, u_2), \quad u_{2t} = \alpha_2 \Delta u_2 + f_2(u_1, u_2), \tag{4.7}$$

where one of the concentration (namely, $u_1 = u_1(x,t)$) "varies faster than the other" ($u_2 = u_2(x,t)$). We suppose that functions f_k, $k = 1, 2$, have the form of so-called Lefever–Prigogine model (see for example [13]) $f_1(u_1, u_2) = Au_2 - u_1u_2^2$, $f_2(u_1, u_2) = B - (A+1)u_2 + u_1u_2^2$, where A and B are any constants. Starting to the reduction of the Cauchy–Dirichlet problem

$$u_k(x, 0) = u_{k0}(x), \quad x \in \Omega, \quad k = 0, 1, \tag{4.8}$$

$$u_k(x, t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}, \quad k = 0, 1, \tag{4.9}$$

for the system (4.7) to the problem (1.1), (1.2) we recall the following result from Appendix C of [13].

Lemma 4.3. *Suppose $f \in C^\infty(\Omega; \mathbb{R}^m)$ and $l > n/2$. Then the operator $F : \oplus W^l_2 \to W^l_2$ given by $F : u \to f(u)$ is well defined and belongs to the class C^∞.*

Assuming that $l > n/2$, we set $\mathfrak{U} = \mathfrak{F} = W^l_2 \oplus W^l_2$, and we define the operators

$$L = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & I \end{pmatrix}, \quad M = \begin{pmatrix} \alpha_1\Delta & \mathbb{O} \\ \mathbb{O} & \alpha_2\Delta \end{pmatrix}, \quad N : u \to \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix},$$

$\mathrm{dom}M = (W^{l+2}_2 \cap \overset{\circ}{W}{}^1_2) \oplus (W^{l+2}_2 \cap \overset{\circ}{W}{}^1_2)$, $u = (u_1, u_2)$. It is clear that the operator M is strongly $(L, 0)$-sectorial for all $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$, and the operator $N \in C^\infty(\mathfrak{U}_\alpha; \mathfrak{F})$ for all $(A, B) \in \mathbb{R}^2$, $\mathfrak{U}_\alpha = \mathfrak{U}$. Since the projectors $P = Q = L$, we can construct the set

$$\mathfrak{M} = \{u \in \mathfrak{U} : \alpha_1\Delta u_1 + Au_2 - u_1u_2 = 0\}.$$

Lemma 4.4. *The set $\mathfrak{M}$ is a simple Banach C^∞-manifold that is modeling by the subspace $\{0\} \oplus W_2^l$ for all $\alpha_1 \in \mathbb{R}_+$, $A \in \mathbb{R}$.*

Proof. First we show that $\mathfrak{M} \neq \emptyset$. Really, for every $u_2 \in W_2^l$ there exists a unique $u_1 \in W_2^{l+2} \cap \overset{\circ}{W}{}_2^1$ such that $u = (u_1, u_2) \in \mathfrak{M}$.

Next, let us consider the operator $D : u_2 \to -A(\alpha_1 \Delta - u_2^2)^{-1}(u_2)$, where $(\alpha_1 \Delta - u_2^2)^{-1}$ is a the Green operator. The operator $D \in C^\infty(\mathfrak{U}^1; \mathfrak{U}^0)$, where $\mathfrak{U}^1 = \{0\} \oplus W_2^l$, $\mathfrak{U}^0 = W_2^l \oplus \{0\}$. Desired C^∞-diffeomorphism $\delta : \mathfrak{U}^1 \to \mathfrak{M}$ has the form

$$\delta(u_2) = \begin{pmatrix} D(u_2) \\ u_2 \end{pmatrix}.$$

□

By the lemma (4.4) and the theorem (3.1) we have

Theorem 4.2. *Suppose $(\alpha_1, \alpha_2) \in \mathbb{R}_+^2$, $(A, B) \in \mathbb{R}^2$, then for every $(u_{10}, u_{20}) \in \mathfrak{M}$, and for any $T \in \mathbb{R}_+$ there exists a unique solution $(u_1, u_2) \in C^\infty((0, T); \mathfrak{M})$ of the problem* (4.8), (4.9).

Remark 4.1. *See in* [14] *another view-point on the problem* (4.7)–(4.9).

Example 4.3 Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be bounded domain with a boundary $\partial\Omega$ of the class C^∞. In the semicylinder $\Omega \times \mathbb{R}_+$ we consider the Cauchy–Dirichlet problem

$$\vec{v}(x, 0) = \vec{v}_0(x), \quad x \in \Omega, \tag{4.10}$$

$$\vec{v}(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+ \tag{4.11}$$

for the Navier–Stokes equations

$$\vec{v}_t = \nu \nabla^2 \vec{v} - (\vec{v} \cdot \nabla)\vec{v} - \nabla p + \vec{f}, \quad 0 = \nabla \cdot \vec{v}. \tag{4.12}$$

Before the beginning of the reduction of the problem (4.10)–(4.12) to the problem (1.1), (1.2) we replace the equation $0 = \nabla \cdot \vec{v}$ to the system of equations $0 = \nabla(\nabla \cdot \vec{v})$. By the Gauss theorem we obtain the system which is equivalent to (4.12). Besides that we set $\vec{p} = \nabla p$, and later on we consider the problem (4.10), (4.11) for the system

$$\vec{v}_t = \nu \nabla^2 \vec{v} - (\vec{v} \cdot \nabla)\vec{v} - \vec{p} + \vec{f}, \quad 0 = \nabla(\nabla \cdot \vec{v}). \tag{4.13}$$

Denote by $\mathbb{H}^2 = (\overset{\circ}{W}{}_2^2)^n$, $\mathbb{H}^1 = (\overset{\circ}{W}{}_2^1)^n$, $\mathbb{L}^2 = (L^2)^n$ and consider lineal $\mathfrak{G} = \{\vec{v} \in (C^\infty)^n : \nabla \cdot v = 0\}$. The closure of $\mathfrak{G}$ by the norm of $\mathbb{L}^2$ we denote by $\mathbb{H}_\sigma$. The space $\mathbb{H}_\sigma$ is Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ from $\mathbb{L}^2$, besides that, there exists a decomposition $\mathbb{L}^2 = \mathbb{H}_\sigma \oplus \mathbb{H}_\pi$, where $\mathbb{H}_\pi$ is the orthogonal complement to $\mathbb{H}_\sigma$. Denote by $\Pi : \mathbb{L}^2 \to \mathbb{H}_\pi$ the corresponding orthoprojector. The restriction of the projector Π on $\mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1$ is a linear bounded operator $\Pi :\to \mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1 \to \mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1$. Therefore we can represent $\mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1$ as the direct sum $\mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1 = \mathbb{H}^2_\sigma \oplus \mathbb{H}^2_\Pi$, where $\mathbb{H}^2_\sigma = \ker \Pi$, $\mathbb{H}^2_\sigma = \mathrm{im}\Pi$. There exist dense and continuous imbeddings $\mathbb{H}^2_\sigma \subset \mathbb{H}_\sigma$, and $\mathbb{H}^2_\pi \subset \mathbb{H}_\pi$. The space $\mathbb{H}^2_\pi$ contains such vector functions, that $u(x) = 0$, $x \in \partial\Omega$, and $u = \nabla\varphi$, $\varphi \in W^3_2$.

Let us set $\mathfrak{U} = \mathfrak{F} = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}_p$, $\mathbb{H}_p = \mathbb{H}_\pi$. Every vector $u \in \mathfrak{U}$ has the form $u = (u_\sigma, u_\pi, u_p)$, and every vector $f \in \mathfrak{F}$ has the form $f = (f_\sigma, f_\pi, f_p)$. It is easy to show that the operator, given by the formula $L = \mathrm{diag}\{\Sigma, \Pi, \mathbb{O}\}$ is linear and bounded with $\ker L = \{0\} \times \{0\} \times \mathbb{H}_p$, $\mathrm{im}L = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \{0\}$; and the operator, given by the formula

$$M = \begin{pmatrix} -\nu\Sigma A & -\nu\Sigma A & \mathbb{O} \\ -\nu\Pi A & -\nu\Pi A & -\Pi \\ \mathbb{O} & B & \mathbb{O} \end{pmatrix},$$

is linear, closed and dense defined operator $M : \mathrm{dom}M \to \mathfrak{F}$, $\mathrm{dom}M = \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \times \mathbb{H}_p$. Here $A = \mathrm{diag}\{-\nabla^2, \ldots, -\nabla^2\} : \mathbb{H}^2_\sigma \times \mathbb{H}^2_\pi \to \mathbb{L}^2$, $B : \vec{v} \to \nabla(\nabla \cdot \vec{v})$, $B \in \mathcal{L}(\mathbb{H}^2_\sigma \oplus \mathbb{H}^2_\pi; \mathbb{H}_\pi)$, $\ker B = \mathbb{H}^2_\sigma$.

Lemma 4.5. *Suppose $\nu \in \mathbb{R}_+$, then the operator M is strongly $(L,1)$-sectorial.*

The projectors P and Q have the forms

$$P = \begin{pmatrix} \Sigma & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \Sigma & \mathbb{O} & -\nu\Sigma A B_\pi^{-1} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix},$$

where B_π is the restriction of the operator B on $\mathbb{H}^2_\pi$.

It is easy to show, that the operator, given by the formula

$$N : u \to \begin{pmatrix} -\Sigma(v \cdot \nabla)v \\ -\Pi(v \cdot \nabla)v \\ \mathbb{O} \end{pmatrix}, \quad v = u_\sigma + u_\pi,$$

lies in $C^\infty(\overset{\circ}{\mathbb{H}}{}^1; \mathbb{L}^2)$, if $n = 2, 3$. Using the projectors P and Q we construct the set

$$\mathfrak{M} = \{u \in \mathfrak{U}_\alpha : u_\pi = 0,\ u_p = \Pi(f - (u_\sigma \cdot \nabla)u_\sigma)\},$$

where $f = (f_\sigma, f_\pi, 0) \in \mathfrak{F}$, $\mathfrak{U}_\alpha = \mathbb{H}^1_\sigma \times \overset{\circ}{\mathbb{H}}{}^1_\pi \times \mathbb{H}_p$, $\overset{\circ}{\mathbb{H}}{}^1{}_{\sigma(\pi)} = \overset{\circ}{\mathbb{H}}{}^1 \cap \mathbb{H}_{\sigma(\pi)}$.

Theorem 4.3. *Suppose $\nu \in \mathbb{R}_+$, $n = 2, 3$, $f \in \mathfrak{F}$, $f = (f_\sigma, f_\pi, 0)$, then for every $u_0 \in \mathfrak{M}$ there exists a unique solution $u \in C^\infty((0,T);\mathfrak{M})$ for any $T \in \mathbb{R}_+$ of the problem* (4.10), (4.11), (4.13). *The set $\mathfrak{M}$ is a Banach C^∞-manifold modeled by the subspace $\overset{\circ}{\mathbb{H}}{}^1_\sigma \times \{0\} \times \{0\}$.*

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