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Asymptotic Solution to a Mixed Boundary-Value Problem in a Thick Multi-Structure of Type 3:2:2

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(*Presented by E. Ya. Khruslov*)

Abstract. The leading terms of the asymptotic expansion for the solution to a mixed boundary value problem for the Poisson equation in a thick multi-structure, which is the union of some domain and a large number N of ε-periodically situated thin annular disks with variable thickness of order $\varepsilon = \mathcal{O}(N^{-1})$, are constructed and the corresponding estimates in the Sobolev space H^1 are proved as $\varepsilon \to 0$.

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1. Introduction and Statement of the Problem

It is an interesting problem to study the behaviour of solutions to boundary-value problems when the domain is perturbed. There are many kinds of the domain perturbations and we need different asymptotic methods to study boundary-value problems in perturbed domains. Numerous monographs and papers (see, e.g., [3, 5, 9, 10, 12, 16, 17, 19, 31–33] and references there) are devoted to asymptotic methods for the investigation of boundary-value problems in domains with complex dependence on a parameter of perturbation (perforated domains, partly perforated domains, lattice frames, junctions of domains with different limit dimensions, etc.).

Boundary-value problems in thick multi-structures (components of such junctions infinitely increases as the perturbation parameter $\varepsilon \to 0$) have own specific difficulties and such problems deserve special attention. As it was shown in E. Sanchez-Palencia's papers [13, 32], such problems lose the coercitivity as $\varepsilon \to 0$ and this creates special difficulties in the

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asymptotic investigation. In [21]– [28], a classification of such thick multi-structures was given and basic results were obtained for boundary-value problems in thick junctions of different types. It was shown that qualitative properties of solutions essentially depend on the junction type and on the conditions given on the boundaries of the attached thin domains. There and in [11] a survey of results obtained in this direction is presented. Here we mention only the pioneer papers [15, 34], where the asymptotic behaviour of Green's function of the Neumann problem for the Helmholtz equation in an unbounded junction body of type 3:2:1 was studied, and the papers [6,7], where the homogenization of nonlinear problems in thick junctions of types 3:2:1, 2:2:1 was made.

Such thick junctions are prototypes of widely used engineering constructions, industrial installations, spaceship grids as well as many other physical and biological systems with very different characteristic scales.

Despite the enormous growth in computational power, it is often impossible to represent a complete system at the finest scale for which the various constitutive elements may suitably be represented. Increase in the size of computational domains for thick multi-structures naturally leads to longer computing times and makes it very difficult to maintain an acceptable level of accuracy.

Thus, asymptotic analysis of boundary value problems in such domains is an important task for applied mathematics. The aim of the analysis is to develop rigorous asymptotic methods for boundary value problems in thick junctions of different types as $\varepsilon \to 0$, i.e., when the number of attached thin domains increases and their thickness decreases.

In this paper we consider a model thick multi-structure (thick junction) Ω_ε of type 3:2:2. It consists of a junction's body

$$\Omega_0 = \Big\{ x \in \mathbb{R}^3 : \ 0 < x_2 < l, \quad r := \sqrt{x_1^2 + x_3^2} < a_0 \Big\}$$

and a large number N of thin annular disks $G(\varepsilon) = \bigcup_{j=0}^{N-1} G_j(\varepsilon)$,

$$G_j(\varepsilon) = \{ x \in \mathbb{R}^3 : \ -\varepsilon\, h_-(r) < x_2 - \varepsilon(j+1/2) < \varepsilon\, h_+(r), \quad a_0 \le r < a_1 \},$$

i.e.,

$$\Omega_\varepsilon = \Omega_0 \bigcup G(\varepsilon).$$

Here h_- and h_+ are piecewise smooth functions on $[a_0, a_1]$, $0 < h_\pm(r) < \frac{1}{2}$ for $r \in [a_0, a_1]$, and the functions h_- and h_+ are locally constant and equal at an enough small neighborhood of the point a_0; the number of the thin disks is equal to a large even integer N, therefore, $\varepsilon = l/N$ is a small parameter, which characterizes the distance between the neighboring thin disks and their thickness. The type 3:2:2 of the junction refers to the limit dimensions of the junction's body, the joint zone (the lateral surface of Ω_0), and each attached thin disk.

In Ω_ε we consider the mixed boundary value problem

$$\begin{array}{rcll} -\Delta_x u_\varepsilon(x) &=& f_\varepsilon(x), & x \in \Omega_\varepsilon, \\ u_\varepsilon(x) &=& 0, & x \in S^{(0)} \cup S^{(l)}, \\ \partial_\nu u_\varepsilon(x) &=& 0, & x \in \partial\Omega_\varepsilon \setminus \big(S^{(0)} \cup S^{(l)}\big), \end{array} \tag{1.1}$$

where $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; $S^{(0)} = \{x \in \partial\Omega_0 : x_2 = 0\}$, $S^{(l)} = \{x \in \partial\Omega_0 : x_2 = l\}$ and the right-hand side $f_\varepsilon \in L^2(\Omega_\varepsilon)$.

The aim of the paper is to construct the leading terms of the asymptotic expansion for the solution to problem (1.1) and to prove the corresponding asymptotic estimates as $\varepsilon \to 0$, i.e., when the number of attached thin disks increases and their thickness decreases.

There are two ways of investigation of boundary-value problems in perturbed domains. The first one lies in the proofs of convergence theorems with the help of special extension operators. The second one consists in the construction the leading terms of the asymptotic expansions for the solutions and proving the corresponding asymptotic estimates. For these two ways we need different assumptions for the right-hand sides and the geometrical structure of the perturbed domains.

To construct the leading terms of asymptotics of solutions to boundary-value problems in thick junctions, the method of matched asymptotic expansions and asymptotic methods for thin domains were used in [21]– [28]. It is turned out that the corresponding limit problem is derived from limit problems for each domain forming the thick junction with the help of the solutions to junction-layer problems around the joint zone. However, the junction-layer solutions behave as powers (or logarithm) at infinity and do not decrease exponentially. Therefore, they influence directly the leading terms of the asymptotics.

Regarding the approximations of the solutions it should be mentioned that in [1, 2] the correctors for the solutions to the Laplace equation in a plane thick junction of type 2:1:1 with the Dirichlet condition were constructed outside a layer of width 2ε in a neighborhood of the joint zone. But for applied problems, it is very important to construct the asymptotic expansion for the solution and to prove the asymptotic estimates in all thick junction, especially in a neighborhood of the joint zone, since the solution has singularities in the joint zone.

It should be emphasized new moments in the investigation of problem (1.1). The first one is type 3:2:2; only spectral problems were considered in such multi-structures (see [4, 27]). Secondly, only boundary value problems in thick junctions with attached thin domains whose thickness are unvarying or problems in domains with rapidly oscillating smooth boundaries (see [8]) were considered till now. Our thick junction Ω_ε has the Lipschitz boundary and the thickness of each thin rings is equal to the value $\varepsilon h_0(r),\ r \in [a_0, a_1]$. Because of this, the coefficients of the corresponding limit problem depend on the function $h_0 = h_- + h_+$. Finally, we require for the right-hand side f_ε of problem (1.1) only the minimal condition, namely, $f_\varepsilon \in L^2(\Omega_\varepsilon)$. For comparison, in [8] the authors assumed that the right-hand side f does not depend on ε and $f \in H^1(\Omega_\varepsilon)$ to prove only the convergence theorem for the solution of the Neumann problem in a plane thick junction of type 2:1:1; in [2] to construct the leading terms of asymptotics for the solution to the Dirichlet problem in a plane thick junction of type 2:1:1, the right-hand side f must vanish on the thin rods and $f \in H^4$ in the junction's body, in [23] f_ε has special form like in Corollary 2.

The convergence theorem for the solution to the Neumann problem for the Ukawa equation in Ω_ε, when $h_- = h_+$, was proved by authors in [11].

2. Formal Asymptotic Representation for the Solution

In this section, we construct the leading terms of outer expansions both in the junction's body and in each thin disks as well as the leading terms of an inner expansion in neighborhood of the joint zone. Then using the method of matched asymptotic expansions, the corresponding limit problem is derived. Also we assume in this section that the right-hand side in (1.1) is independent of ε, i.e., $f_\varepsilon = f_0$ and f_0 is smooth in $\overline{\Omega}_1$, where Ω_1 is the interior of $\overline{\Omega}_0 \cup \overline{D}$. The domain $D = \{x : 0 < x_2 < l,\ a_0 < r < a_1\}$ is filled up by the thin disks in the limit passage.

2.1. Outer Expansions

We seek the leading terms of the asymptotics for the solution u_ε, restricted to Ω_0, in the form

$$u_\varepsilon(x) \approx v_0^+(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^+(x, \varepsilon), \tag{2.1}$$

and, restricted to $G_j(\varepsilon)$, in the form

$$u_\varepsilon(x) \approx v_0^-(x) + \sum_{k=1}^{\infty} \varepsilon^k v_k^-(x, \xi_2 - j), \qquad \xi_2 = \varepsilon^{-1} x_2\,. \tag{2.2}$$

The expansions (2.1) and (2.2) are usually called *outer expansions.*

Substituting the series (2.1) in the equation of problem (1.1) and in the boundary conditions on the bases $S^{(0)}, S^{(l)}$ of the cylinder Ω_0 and collecting the coefficients of the same powers of ε, we get the following relations for the function v_0^+ :

$$-\Delta_x v_0^+(x) = f_0(x), \quad x \in \Omega_0, \qquad v_0^+(x) = 0, \quad x \in S^{(0)} \cap S^{(l)}. \quad (2.3)$$

Now we find limit relations in the domain D. Assuming for the moment that the functions v_k^- in (2.2) are smooth, we write their Taylor series with respect to the x_2 at the point $x_2 = \varepsilon(j + \frac{1}{2})$ and pass to the "fast" variable $\xi_2 = \varepsilon^{-1} x_2$. Then (2.2) takes the form

$$u_\varepsilon(x) = v_0^-\big(x_1, \varepsilon(j+1/2), x_3\big) + \sum_{k=1}^{+\infty} \varepsilon^k V_k^j(x_1, \xi_2, x_3), \quad x \in G_j(\varepsilon), \quad (2.4)$$

where

$$V_k^j = v_k^-\big(x_1, \varepsilon(j+1/2), x_3, \xi_2\big) + \sum_{m=1}^{k} \frac{(\xi_2 - j - \frac{1}{2})^m}{m!} \, \frac{\partial^m v_{k-m}^-}{\partial x_2^m}\big(x_1, \varepsilon(j+1/2), x_3, \xi_2\big). \quad (2.5)$$

Let us substitute (2.4) into (1.1) instead of u_ε. Since the Laplace operator takes the form $\Delta_x = \varepsilon^{-2} \frac{\partial^2}{\partial \xi_2^2} + \Delta_{\widetilde{x}}$, where $\widetilde{x} = (x_1, x_3)$, and the outward normal to the lateral surface of the thin disk $G_j(\varepsilon)$ (beside some set of zero measure) has the form

$$\nu_\pm(\widetilde{x}, \varepsilon) = \frac{1}{\sqrt{1 + \varepsilon^2 |h'_\pm(r)|^2}} \Big(-\varepsilon h'_\pm(r) \frac{x_1}{r}, \ \pm 1, \ -\varepsilon h'_\pm(r) \frac{x_3}{r}\Big),$$

the collection of coefficients of the same power of ε gives us one dimensional Neumann problems with respect to ξ_2. Write the first two ones:

$$\partial^2_{\xi_2\xi_2} V_1^j(\widetilde{x}, \xi_2) = 0, \quad \xi_2 \in I_j(r), \qquad \partial_{\xi_2} V_1^j\big(\widetilde{x}, j{+}2^{-1}{\pm}h_\pm(r)\big) = 0; \quad (2.6)$$

$$-\partial^2_{\xi_2\xi_2} V_2^j(\widetilde{x}, \xi_2) = \Delta_{\widetilde{x}}\, v_0^-(\widetilde{x}, \varepsilon(j+1/2)) + f_0(\widetilde{x}, \varepsilon(j+1/2)), \quad \xi_2 \in I_j(r), \quad (2.7)$$

$$\partial_{\xi_2} V_2^j\big(\widetilde{x}, j + 2^{-1} \pm h_\pm(r)\big) = \pm \nabla_{\widetilde{x}}\, h_\pm(r) \cdot \nabla_{\widetilde{x}}\, v_0^-(\widetilde{x}, \varepsilon(j+1/2)), \quad (2.8)$$

where $I_j(r) := \big(j + \frac{1}{2} - h_-(r)\,,\, j + \frac{1}{2} + h_+(r)\big)$, $\partial_{\xi_2} = \partial/\partial\xi_2$, $\partial^2_{\xi_2\xi_2} = \partial^2/\partial\xi_2^2$.

From (2.6), it follows that the function V_1^j does not depend on ξ_2. We restrict ourselves to the leading term of the asymptotics, and thus set $V_1^j = 0$. Then, by virtue of (2.5), we have

$$v_1^-\big(x_1, \varepsilon(j{+}1/2), x_3, \xi_2\big) = -\partial_{x_2} v_0^-\big(x_1, \varepsilon(j{+}1/2), x_3\big)\big(\xi_2{-}j{-}1/2\big), \quad (2.9)$$

where $\partial_{x_2} = \partial/\partial x_2$.

The solvability condition for the problem (2.7)–(2.8) is given by the differential equation

$$-\mathrm{div}_{\widetilde{x}}\,\big(h_0(r)\nabla_{\widetilde{x}}\, v_0^-(\widetilde{x}, \varepsilon(j+1/2))\big) \;=\; f_0(\widetilde{x}, \varepsilon(j+1/2)), \quad r \in (a_0, a_1), \qquad (2.10)$$

where $h_0(r) = h_-(r) + h_+(r)$, $r \in [a_0, a_1]$. Since these planes $x_2 = \varepsilon(j+1/2)$, $j = 0, 1, \dots, N-1$, make up the ε-net in D, we can spread this equation in all domain D. Due to the Neumann conditions for the solution to problem (1.1) we must require from v_0^- to satisfy the condition

$$\partial_r v_0^-(x) = 0, \qquad r = a_1, \quad x_2 \in (0, l), \qquad (2.11)$$

where $\partial_r = \partial/\partial r$ is the derivative with respect to polar radius r which coincides with the outward normal derivative in this case.

So, it remains to provide the continuity of the asymptotic approximation and their gradients in the joint zone $\Gamma_0 := \partial\Omega_0 \setminus \big(S^{(0)} \cup S^{(l)}\big)$. It is doubtless the condition

$$v_0^+(x) = v_0^-(x), \qquad x \in \Gamma_0. \qquad (2.12)$$

To get the second transmission condition we should use the method of matched asymptotic expansions for the outer expansions (2.1), (2.2) and an inner expansion which we will construct in the following section.

2.2. Inner Expansion

In a neighborhood of Γ_0 we consider the Laplace operator in the cylindrical coordinates r, φ, x_2, where $r = \sqrt{x_1^2 + x_3^2}$ and $\tan(\varphi) = x_3/x_1$, and then pass to the "rapid" coordinates $\xi = (\xi_1, \xi_2)$, where $\xi_1 = -\varepsilon^{-1}(r - a_0)$ and $\xi_2 = \varepsilon^{-1} x_2$. Then Laplace's operator in the coordinates $\xi = (\xi_1, \xi_2), \varphi$ has the following form

$$\varepsilon^{-2}\Big(\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2}\Big) - \frac{\varepsilon^{-1}}{a_0 - \varepsilon\xi_1}\frac{\partial}{\partial\xi_1} + \frac{1}{(a_0 - \varepsilon\xi_1)^2}\frac{\partial^2}{\partial\varphi^2}. \qquad (2.13)$$

We seek the leading terms of the inner expansion in a neighborhood of Γ_0 in the form

$$u_\varepsilon(x) \approx v_0^+(x)|_{r=a_0} + \varepsilon\big(Z_1(\xi)(\partial_{x_2} v_0^+(x))|_{r=a_0} + Z_2(\xi)(\partial_r v_0^+(x))|_{r=a_0}\big) + \dots . \quad (2.14)$$

Substituting (2.14) in (2.13) and in the Neumann condition, collecting the coefficients of the same power of ε, we arrive at junction-layer problems for the functions Z_1 and Z_2 :

$$\begin{array}{rcll} -\Delta_{\xi_1\xi_2}\, Z_i(\xi) &=& 0, & \xi\in\Pi,\\ \partial_{\xi_1} Z_i(0,\xi_2) &=& 0, & (0,\xi_2)\in\partial\Pi^+\setminus I_h,\\ \partial_{\xi_2} Z_i(\xi) &=& -\delta_{1i}, & \xi\in\partial\Pi^-\setminus I_h,\\ \partial^k_{\xi_2} Z_i(\xi_1,0) &=& \partial^k_{\xi_2} Z_i(\xi_1,1), & \xi_1>0,\quad k=0,1. \end{array} \tag{2.15}$$

Here Π is the union of semi-infinite strips $\Pi^+=(0,+\infty)\times(0,1)$ and $\Pi^-=(-\infty,0]\times I_h$, where $I_h=\big((1-h)/2\,,\,(1+h)/2\big)$, the constant h is equal to $h_0(a_0)$. The last periodic condition in (2.15) due to the periodicity of the thin disks $\{G_j(\varepsilon):\, j=0,\ldots,N-1\}$.

The same junction-layer problems were investigated in [22]. The main asymptotic relations for the functions $\{Z_i\}$ can be obtained from general results about the asymptotic behaviour of solutions to elliptic problems in domains with different exits to infinity [14, 29]. However, using the symmetry of the domain Π, we can define more exactly the asymptotic relations and detect other properties of the junction-layer solutions $Z_1,\ Z_2$ similarly as in the papers [22, 23].

Statement 2.1 ([22]). *There exist solutions* $Z_i\in H^1_{loc,\eta_2}(\Pi),\ i=1,2,$ *of problems* (2.15), *which have the following differentiable asymptotics*

$$Z_1(\xi)=\begin{cases}\mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1\to+\infty,\\ -\xi_2+\frac12+\mathcal{O}(\exp(\pi h^{-1}\xi_1)), & \xi_1\to-\infty;\end{cases} \tag{2.16}$$

$$Z_2(\xi)=\begin{cases}-\xi_1+c_h+\mathcal{O}(\exp(-2\pi\xi_1)), & \xi_1\to+\infty,\\ -h^{-1}\xi_1+\mathcal{O}(\exp(\pi h^{-1}\xi_1)), & \xi_1\to-\infty.\end{cases} \tag{2.17}$$

In addition, the function Z_1 is odd in ξ_2 and Z_2 is even in ξ_2 with respect to $1/2$.

Now we verify the matching conditions for the outer expansions (2.1), (2.2) and the inner expansion (2.14), namely, the leading terms of the asymptotics of the outer expansions as $r\to a_0\pm0$ must coincide with the leading terms of the inner expansion as $\xi_1\to\mp\infty$ respectively. Near the point $x\in\Gamma_0$ the function v_0^+ has the following asymptotics

$$v_0^+(x)=v_0^+(x)|_{r=a_0}-\varepsilon\;\xi_1\,(\partial_r v_0^+(x))|_{r=a_0}+\mathcal{O}(\varepsilon^2\xi_1^2),\quad r\to a_0-0.$$

Taking into account the asymptotics of Z_1 and Z_2 as $\xi_1\to+\infty$, we see that the matching conditions are satisfied for the expansion (2.1) and

(2.14). The asymptotics of (2.2) as $r \to a_0 + 0$ and the asymptotics of (2.14) as $\xi_1 \to -\infty$ are the following

$$
\begin{aligned}
&v_0^-(x)|_{r=a_0} + \varepsilon\big(Y(\xi_2)(\partial_{x_2} v_0^-(x))|_{r=a_0} - \xi_1\,(\partial_r v_0^-(x))|_{r=a_0}\big) + \dots,\\
&\hspace{22em} r \to a_0 + 0;\\
&v_0^+(x)|_{r=a_0} + \varepsilon\big(Y(\xi_2)(\partial_{x_2} v_0^+(x))|_{r=a_0} - h^{-1}\xi_1\,(\partial_r v_0^+(x))|_{r=a_0}\big) + \dots,\\
&\hspace{22em} \xi_1 \to -\infty.
\end{aligned}
$$

Comparing the main terms of these asymptotics, we get the first transmission condition (2.12) and the second one

$$
\partial_r v_0^+(x) = h\,\partial_r v_0^-(x), \qquad x \in \Gamma_0. \tag{2.18}
$$

So, the function

$$
v_0(x) = \begin{cases} v_0^+(x), & x \in \Omega_0,\\ v_0^-(x), & x \in D, \end{cases} \tag{2.19}
$$

must satisfy the relations (2.3), (2.10)–(2.12), (2.18), which form the limit problem

$$
\begin{aligned}
-\Delta_x v_0^+(x) &= f_0(x), && x \in \Omega_0,\\
-\mathrm{div}_{\widetilde{x}}\big(h_0(r)\nabla_{\widetilde{x}}\, v_0^-(x)\big) &= h_0(r)\, f_0(x), && x \in D,\\
\partial_r v_0^-(x) &= 0, && r = a_1,\ x_2 \in (0,l),\\
v_0^+(x) &= 0, && x \in S^{(0)} \cup S^{(l)},\\
v_0^+(x) &= v_0^-(x), && x \in \Gamma_0,\\
\partial_r v_0^+(x) &= h_0(a_0)\,\partial_r v_0^-(x), && x \in \Gamma_0,
\end{aligned} \tag{2.20}
$$

for problem (1.1).

Let us show that there exist a unique weak solution $v \in \mathcal{H}_0$ to problem (2.20) if $f_0 \in L^2(\Omega_1)$. Here

$$
\begin{aligned}
\mathcal{H}_0 := \big\{\varphi \in L^2(\Omega_1) :\ &\varphi \in H^1(\Omega_0), \quad \varphi = 0 \ \text{ on } \ S^{(0)} \cup S^{(l)},\\
&\partial_{x_1}\varphi \in L^2(D), \quad \partial_{x_3}\varphi \in L^2(D), \quad \varphi|_{r=a_0-0} = \varphi|_{r=a_0+0} \ \text{ on } \ \Gamma_0\big\}
\end{aligned}
$$

is an anisotropic Sobolev space with the scalar product

$$
(\varphi,\psi)_{\mathcal{H}_0} = \int_{\Omega_0} \nabla_x\varphi \cdot \nabla_x\psi\, dx + \int_D h_0(r)\,\nabla_{\widetilde{x}}\,\varphi\ \cdot\ \nabla_{\widetilde{x}}\,\psi\, dx.
$$

A function $v \in \mathcal{H}_0$ is called a weak solution to problem (2.20) if it satisfies the following integral identity

$$\big(v, \psi\big)_{\mathcal{H}_0} = \int\limits_{\Omega_0} f_0(x)\psi(x)\,dx + \int\limits_{D} h_0(r)\, f_0(x)\psi(x)\,dx, \quad \forall\, \psi \in \mathcal{H}_0. \quad (2.21)$$

Next using standard Hilbert space methods and the Lax–Milgram lemma, it is easy to prove the existence and uniqueness of the weak solution to problem (2.20).

3. Corrector and Asymptotic Estimates

Let f_0 be a function in $H^3(\Omega_1)$. Assume that f_0 and $\partial_{x_2} f_0$ vanish on $S^{(0)} \cup S^{(l)}$. Let $v_0 \in \mathcal{H}_0$ be the unique weak solution to problem (2.20) with the right-hand side f_0. With the help of v_0 and the junction-layer solutions Z_1, Z_2 (see Statement 2.1), we define the leading terms in (2.1), (2.2) and (2.14). Then matching these expansions, we construct an asymptotic approximation R_ε belonging to the Hilbert space

$$\mathcal{H}_\varepsilon := \{u \in H^1(\Omega_\varepsilon) : \ u = 0 \text{ on } S^{(0)} \cup S^{(l)}\}.$$

It is equal to

$$R^+_\varepsilon(x) := v^+_0(x) + \varepsilon\chi_0(r)\,\mathcal{N}^+\Big(-\frac{r-a_0}{\varepsilon}, \frac{x_2}{\varepsilon}, \varphi, x_2\Big), \qquad x \in \Omega_0; \quad (3.1)$$

and to

$$R^-_\varepsilon(x) := v^-_0(x) + \varepsilon\Big(Y\Big(\frac{x_2}{\varepsilon}\Big)\partial_{x_2} v^-_0(x) \\ + \chi_0(r)\,\mathcal{N}^-\Big(-\frac{r-a_0}{\varepsilon}, \frac{x_2}{\varepsilon}, \varphi, x_2\Big)\Big), \quad x \in D, \quad (3.2)$$

where $Y(\xi_2) = -\xi_2 + \frac{1}{2} + [\xi_2]$; $\chi_0 \in C^\infty_0(\mathbb{R})$ is a cut-off function such that

$$\chi_0(r) = \begin{cases} 1, & |r-a_0| \le \sigma/2, \\ 0, & |r-a_0| \ge \sigma, \end{cases}$$

where σ is an enough small fixed positive number such that the functions $h_\pm$ are constant and equal on the segment $[a_0, a_0+\sigma]$;

$$\mathcal{N}^+(\xi, \varphi, x_2) = Z_1(\xi)\,(\partial_{x_2} v^+_0(x))|_{r=a_0} + \big(Z_2(\xi) + \xi_1\big)\,(\partial_r v^+_0(x))|_{r=a_0},$$

$$\mathcal{N}^-(\xi, \varphi, x_2) = \big(Z_1(\xi) - Y(\xi_2)\big)\,(\partial_{x_2} v^+_0(x))|_{r=a_0} \\ + \big(Z_2(\xi) + h^{-1}\xi_1\big)\,(\partial_r v^+_0(x))|_{r=a_0},$$

$$\xi_1 = -\varepsilon^{-1}(r-a_0), \quad \xi_2 = \varepsilon^{-1}x_2, \quad r = \big(x_1^2 + x_3^2\big)^{1/2}, \quad \tan(\varphi) = x_3/x_1.$$

3.1. Discrepancies in the Domain Ω_0

Taking into account the properties of the functions Z_1 and Z_2, we conclude that

$$R^+_\varepsilon(x) = 0, \quad x \in S^{(0)} \cup S^{(l)},$$

$$\partial_r R^+_\varepsilon(x) = -\partial_{\xi_1} Z_1(0, x_2/\varepsilon)\,\big(\partial_{x_2} v^+_0(x)\big)|_{r=a_0} - \partial_{\xi_1} Z_2(0, x_2/\varepsilon)\big(\partial_r v^+_0(x)\big)|_{r=a_0} = 0$$

for any $x \in \partial\Omega_\varepsilon \cap \{r = a_0\}$.

Observing that

$$[\Delta_{\widetilde{x}}\,,\, \chi_0(r)]Y(x) = \nabla_{\widetilde{x}} \cdot \big(Y(x)\nabla_{\widetilde{x}}\,\chi_0(r)\big) + \nabla_{\widetilde{x}} Y(x) \cdot \nabla_{\widetilde{x}}\,\chi_0(r),$$

where $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ is the commutator of two operators $\mathbf{A}$ and $\mathbf{B}$, and taking into account form (2.13) of Laplace's operator, we get

$$\begin{aligned} &- \Delta_x R^+_\varepsilon(x) - f_\varepsilon(x) = f_0(x) - f_\varepsilon(x) \\ &\qquad - \chi_0(r)\big(\partial^2_{x_2\xi_2}\mathcal{N}^+(\xi,\varphi,x_2) - r^{-1}\partial_{\xi_1}\mathcal{N}^+(\xi,\varphi,x_2)\big) \\ &\qquad - \nabla_\xi \mathcal{N}^+(\xi,\varphi,x_2) \cdot \nabla_{\widetilde{x}}\,\chi_0(r) \\ &\qquad - \varepsilon\big(\nabla_{\widetilde{x}} \cdot \big(\mathcal{N}^+ \nabla_{\widetilde{x}}\,\chi_0(r)\big) + \chi_0(r)\partial^2_{x_2x_2}\mathcal{N}^+(\xi,\varphi,x_2) \\ &\qquad + r^{-2}\chi_0(r)\partial^2_{\varphi\varphi}\mathcal{N}^+(\xi,\varphi,x_2)\big), \end{aligned}$$

$$\xi_1 = -\frac{r - a_0}{\varepsilon}, \quad \xi_2 = \frac{x_2}{\varepsilon}, \quad r = \sqrt{x_1^2 + x_3^2}, \quad \tan(\varphi) = \frac{x_3}{x_1}, \quad x \in \Omega_0. \tag{3.3}$$

Further, the arguments of functions involved in calculations are indicated only if their absence may cause confusion. We multiply identity (3.3) by a test function $\psi \in \mathcal{H}_\varepsilon$ and integrate by parts in Ω_0 :

$$- \int\limits_{Q_\varepsilon} \partial_r R^+_\varepsilon\, \psi\, dS_x + \int\limits_{\Omega_0} \nabla_x R^+_\varepsilon \cdot \nabla_x \psi\, dx - \int\limits_{\Omega_0} f_\varepsilon \psi\, dx = I^+_1(\varepsilon,\psi) + \ldots + I^+_4(\varepsilon,\psi), \tag{3.4}$$

where $Q_\varepsilon = \Omega_\varepsilon \cap \{r = a_0\}$,

$$I^+_1(\varepsilon,\psi) = \int\limits_{\Omega_0} \big(f_0(x) - f_\varepsilon(x)\big)\psi\, dx,$$

$$I^+_2(\varepsilon,\psi) = \int\limits_{\Omega_0} \chi_0(r)\,\big(r^{-1}\partial_{\xi_1}\mathcal{N}^+ - \partial^2_{x_2\xi_2}\mathcal{N}^+\big)\psi\, dx,$$

$$I_3^+(\varepsilon,\psi)=\varepsilon\int\limits_{\Omega_0}\mathcal{N}^+\,\nabla_{\widetilde{x}}\,\chi_0(r)\cdot\nabla_{\widetilde{x}}\psi\,dx-\int_{\Omega_0}\psi\,\nabla_\xi\mathcal{N}^+\cdot\nabla_{\widetilde{x}}\,\chi_0(r)\,dx,$$

$$I_4^+(\varepsilon,\psi)=\varepsilon\int\limits_{\Omega_0}\chi_0(r)\,(\partial_{x_2}\psi\,\partial_{x_2}\mathcal{N}^+\;+\;r^{-2}\partial_{x_2}\psi\,\partial_\varphi\mathcal{N}^+)\,dx.$$

3.2. Discrepancies in the Thin Disks

Denote by $S_j^+(\varepsilon)$ and $S_j^-(\varepsilon)$ the right and left lateral surfaces of the thin disk $G_j(\varepsilon)$ respectively;

$$S^+(\varepsilon):=\bigcup_{j=0}^{N-1}S_j^+(\varepsilon),\quad S^-(\varepsilon):=\bigcup_{j=0}^{N-1}S_j^-(\varepsilon).$$

It is easy to calculate that $\big(\partial_r R_\varepsilon^-\big)|_{r=a_1}=0$,

$$\partial_r R_\varepsilon^-(x)=\varepsilon Y(x_2/\varepsilon)\,\partial_r(\partial_{x_2}v_0^-(x))\;+\;\partial_r R_\varepsilon^+(x),\quad x\in Q_\varepsilon,\tag{3.5}$$

and

$$\partial_\nu R_\varepsilon^-(x)=\frac{1}{\sqrt{1+\varepsilon^2|h_\pm'(r)|^2}}\big(-\varepsilon\nabla_{\widetilde{x}}(h_\pm)\cdot\nabla_{\widetilde{x}}\big(v_0^-+\varepsilon Y(\xi_2)\partial_{x_2}v_0^-\big)$$
$$\pm\,\varepsilon\big(Y(\xi_2)\partial^2_{x_2x_2}v_0^-(x)+\chi_0(r)\partial_{x_2}\mathcal{N}^-(\xi,\varphi,x_2)\big)\big),\quad x\in S^\pm(\varepsilon).\tag{3.6}$$

Putting R_ε^- in the differential equation of problem (1.1), we obtain

$$\begin{aligned}-\,\Delta_x R_\varepsilon^-(x)-f_\varepsilon(x)&=f_0(x)-f_\varepsilon(x)+\nabla_{\widetilde{x}}(\ln h_0)\cdot\nabla_{\widetilde{x}}v_0^-\\&+\chi_0(r)\big(r^{-1}\partial_{\xi_1}\mathcal{N}^--\partial_{\xi_2}\mathcal{N}^-\big)-\nabla_\xi\mathcal{N}^-\cdot\nabla_{\widetilde{x}}\,\chi_0(r)\\&-\varepsilon\partial_{x_2}\Big(Y\Big(\frac{x_2}{\varepsilon}\Big)\partial^2_{x_2x_2}v_0^-+\chi_0(r)\big(\partial_{x_2}\mathcal{N}^-\big)|_{\xi_2=x_2/\varepsilon}\Big)\\-\,\varepsilon\Big(Y\Big(\frac{x_2}{\varepsilon}\Big)\Delta_{\widetilde{x}}\big(\partial_{x_2}v_0^-\big)&+\nabla_{\widetilde{x}}\cdot\big(\mathcal{N}^-\nabla_{\widetilde{x}}\,\chi_0(r)\big)+r^{-2}\chi_0(r)\,\partial^2_{\varphi\varphi}\mathcal{N}^-\Big),\\&\qquad x\in G(\varepsilon).\end{aligned}\tag{3.7}$$

Next we will use the following identity

$$\int\limits_{S_\varepsilon^\pm}\frac{\varepsilon h_\pm(r)}{\sqrt{1+\varepsilon^2|h_\pm'(r)|^2}}\,\psi\,dS_x=\int\limits_{G_\varepsilon}\psi\,dx-\varepsilon\int\limits_{G_\varepsilon}Y\Big(\frac{x_2}{\varepsilon}\Big)\partial_{x_2}\psi\,dx\quad\forall\,\psi\in\mathcal{H}_\varepsilon.\tag{3.8}$$

To prove (3.8) it is enough to integrate by part the last integral.

Using (3.8) and taking into account the boundary values of $\partial_\nu R_\varepsilon^-$ (see (3.5), (3.6)), we multiply (3.7) by a test function $\psi \in \mathcal{H}_\varepsilon$ and integrate by parts in G_ε. This yields

$$\int\limits_{Q_\varepsilon} \partial_r R_\varepsilon^+ \psi \, dS_x + \int\limits_{G_\varepsilon} \nabla_x R_\varepsilon^- \cdot \nabla_x \psi \, dx - \int_{G_\varepsilon} f_\varepsilon \psi \, dx = I_1^-(\varepsilon, \psi) + \ldots + I_5^-(\varepsilon, \psi), \tag{3.9}$$

where

$$I_1^-(\varepsilon, \psi) = \int\limits_{G_\varepsilon} \big(f_0(x) - f_\varepsilon(x)\big) \psi \, dx,$$

$$I_2^-(\varepsilon, \psi) = \int\limits_{G_\varepsilon} \psi \, \chi_0(r) \big(r^{-1} \partial_{\xi_1} \mathcal{N}^- - \partial_{\xi_2} \mathcal{N}^-\big) \, dx,$$

$$I_3^-(\varepsilon, \psi) = \varepsilon \int\limits_{G_\varepsilon} \mathcal{N}^- \, \nabla_{\widetilde{x}} \, \chi_0(r) \cdot \nabla_{\widetilde{x}} \, \psi \, dx - \int\limits_{G_\varepsilon} \psi \, \nabla_\xi \mathcal{N}^- \cdot \nabla_{\widetilde{x}} \, \chi_0(r) \, dx,$$

$$I_4^-(\varepsilon, \psi) = \varepsilon \int\limits_{G_\varepsilon} \chi_0(r) \left(\partial_{x_2} \psi \, \partial_{x_2} \mathcal{N}^- + r^{-2} \partial_\varphi \psi \, \partial_\varphi \mathcal{N}^-\right) dx,$$

$$I_5^-(\varepsilon, \psi) = \varepsilon \int\limits_{G_\varepsilon} Y\Big(\frac{x_2}{\varepsilon}\Big) \big(\nabla_x \big(\partial_{x_2} v_0^-\big) \cdot \nabla_x \psi + \partial_{x_2} \big(\psi \, \nabla_{\widetilde{x}} \big(\ln h_0\big) \cdot \nabla_{\widetilde{x}} v_0^-\big)\big) \, dx.$$

3.3. Asymptotic Estimates

Summing (3.4) and (3.9), we see that the function R_ε constructed by formulas (3.1) and (3.2) satisfies the following integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla_x R_\varepsilon \cdot \nabla_x \psi \, dx - \int\limits_{\Omega_\varepsilon} f_\varepsilon \psi \, dx = F_\varepsilon(\psi), \qquad \forall \, \psi \in \mathcal{H}_\varepsilon,$$

where $F_\varepsilon(\psi) = I_1^\pm(\varepsilon, \psi) + \ldots + I_4^\pm(\varepsilon, \psi) + I_5^-(\varepsilon, \psi)$; $I_i^\pm = I_i^+ + I_i^-$, $i = 1, \ldots, 4$.

Since the weak solution $u_\varepsilon \in \mathcal{H}_\varepsilon$ to problem (1.1) satisfies the integral identity

$$\int\limits_{\Omega_\varepsilon} \nabla_x u_\varepsilon \cdot \nabla_x \psi \, dx - \int\limits_{\Omega_\varepsilon} f_\varepsilon \psi \, dx = 0, \qquad \forall \, \psi \in \mathcal{H}_\varepsilon,$$

we have

$$\int\limits_{\Omega_\varepsilon} \nabla_x \big(R_\varepsilon - u_\varepsilon\big) \cdot \nabla_x \psi \, dx = F_\varepsilon(\psi), \qquad \forall \, \psi \in \mathcal{H}_\varepsilon. \tag{3.10}$$

Now we are going to estimate the value $F_\varepsilon(\psi)$.

The sum $I_1^{\pm}(\varepsilon,\psi)$ is a linear bounded functional on $\mathcal{H}_\varepsilon$. Thus,

$$|I_1^{\pm}(\varepsilon,\psi)| = \|f_\varepsilon - f_0\|_* \, \|\psi\|_{H^1(\Omega_\varepsilon)},$$

where

$$\|f_\varepsilon - f_0\|_* = \sup_{\substack{\psi\in\mathcal{H}_\varepsilon,\\ \|\psi\|_{H^1(\Omega_\varepsilon)}=1}} |(f_\varepsilon - f_0,\psi)_{L^2(\Omega_\varepsilon)}|.$$

Obviously, $\|f_\varepsilon - f_0\|_* \le C_1\|f_\varepsilon - f_0\|_{L^2(\Omega_\varepsilon)}$. Here and in what follows, all constants in asymptotic inequalities are independent of the parameter ε.

In order to estimate the terms $I_2^+(\varepsilon,\psi)$, $I_2^-(\varepsilon,\psi)$, we will use the following lemma.

Lemma 3.1. *Let $\mathcal{N}$ be an 1-periodic in ξ_2 function belonging to the space $L^2(\Pi)$ and exponentially decreasing at infinity, i.e., there exist positive constants c, R, γ such that for any $|\xi_1| \ge R$*

$$\big|\mathcal{N}(\xi)\big| \le c\exp(-\gamma|\xi_1|).$$

Then for any $\delta > 0$ there exist positive constants c_1, ε_0 such that for all values $\varepsilon \in (0,\varepsilon_0)$ the following inequality is valid

$$\left|\int\limits_{\Omega_\varepsilon} \mathcal{N}\Big(-\frac{r-a_0}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\psi(x)\,dx\right| \le c_1\varepsilon^{1-\delta}\|\psi\|_{H^1(\Omega_\varepsilon)}, \qquad \forall\,\psi\in\mathcal{H}_\varepsilon.$$

Proof. Set $B_{\varepsilon,\delta} = \Omega_\varepsilon \cap \{x: \ |r-a_0| \le \varepsilon^{1-2\delta}\}$ for any $\delta>0$. Then

$$\left|\int\limits_{\Omega_\varepsilon} \mathcal{N}\Big(-\frac{r-a_0}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\psi\,dx\right| \le \left|\int\limits_{B_{\varepsilon,\delta}} \mathcal{N}\Big(-\frac{r-a_0}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\psi\,dx\right| + \left|\int\limits_{\Omega_\varepsilon\setminus B_{\varepsilon,\delta}} \mathcal{N}\Big(-\frac{r-a_0}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\psi\,dx\right|.$$

The properties of the function $\mathcal{N}$ lead us to the conclusion that the second summand in this inequality decreases exponentially as $\varepsilon\to 0$. Using Lemma 1.5 ([18]), we estimate the first summand:

$$\left|\int\limits_{B_{\varepsilon,\delta}} \mathcal{N}\Big(-\frac{r-a_0}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\psi\,dx\right| \le \left(\int\limits_{B_{\varepsilon,\delta}} \mathcal{N}^2\Big(-\frac{r-a_0}{\varepsilon},\frac{x_2}{\varepsilon}\Big)\,dx\right)^{1/2} \|\psi\|_{L^2(B_{\varepsilon,\delta})} \le c_2\,\varepsilon^{1/2}\|\mathcal{N}\|_{L^2(\Pi)}c_3\varepsilon^{-\delta+1/2}\|\psi\|_{H^1(\Omega_\varepsilon)}.$$

The lemma is proved. $\square$

Since the functions $\partial_{\xi_1}\mathcal{N}^{\pm}$, $\partial^2_{x_2\xi_2}\mathcal{N}^{-}$, $\partial_{\xi_2}\mathcal{N}^{+}$ exponentially decrease as $|\xi_1| \to +\infty$, we deduce from Lemma 3.1 that for any fixed $\delta > 0$

$$|I_2^+(\varepsilon,\psi) + I_2^-(\varepsilon,\psi)| \,\le\, \varepsilon^{1-\delta}\, C_2 \|\psi\|_{H^1(\Omega_\varepsilon)}. \tag{3.11}$$

Integrals in $I_3^+(\varepsilon,\psi)$, $I_3^-(\varepsilon,\psi)$ are, in fact, over

$$\operatorname{supp}\big(\nabla_{\widetilde{x}}\,\chi_0(r)\big) \,\cap\, \Omega_\varepsilon = \{x:\ \sigma/2 < |r-a_0| < \sigma\,\} \cap \Omega_\varepsilon,$$

where, by virtue of Statement 2.1, the functions $\mathcal{N}^{-}, \nabla_{\xi}\mathcal{N}^{\pm}$ are exponentially small and the function $\mathcal{N}^{+}$ is uniformly bounded with respect to ε. Thus,

$$|I_3^+(\varepsilon,\psi) + I_3^-(\varepsilon,\psi)| \,\le\, \varepsilon\, C_3 \|\psi\|_{H^1(\Omega_\varepsilon)}.$$

Integrals in $I_4^+(\varepsilon,\psi)$, $I_4^-(\varepsilon,\psi)$ are over $\{x:\ |r-a_0| < \sigma\,\} \cap \Omega_\varepsilon$ and they can be estimated with extracting, if necessary, the exponentially decreasing part in the corresponding integrand and then with the help of the Cauchy–Bunyakovsky inequality. Consider for example the integral

$$\begin{aligned}
\Big|\int\limits_{\Omega_0} \chi_0(r)\,\partial_{x_2}\psi\,\partial_{x_2}\mathcal{N}^{+}\,dx\Big| &= \Big|\int\limits_{\Omega_0} \chi_0(r)\,\partial_{x_2}\psi\big(Z_1\,(\partial^2_{x_2x_2}v_0^+(x))|_{r=a_0} \\
&\quad + \big(Z_2 - \varepsilon^{-1}(r-a_0)\big)(\partial_{x_2}\partial_r v_0^+(x))|_{r=a_0}\big)\,dx\Big| \\
&\le \int\limits_{\Omega_0} \chi_0(r)\,|\partial_{x_2}\psi|\,|Z_1|\,|\partial^2_{x_2x_2}v_0^+(x)|\big|_{r=a_0}\,dx \\
&\quad + \int\limits_{\Omega_0} \chi_0|\partial_{x_2}\psi|\big(|Z_2 - \varepsilon^{-1}(r-a_0) - c_h|\,|\partial_{x_2}\partial_r v_0^+(x)|\big|_{r=a_0} \\
&\quad + |c_h|\,|\partial_{x_2}\partial_r v_0^+(x)|\big|_{r=a_0}\big)dx \le c\|\psi\|_{H^1(\Omega_0)}\Big(\sqrt{\textstyle\int_{\Omega_0}\chi_0\,|Z_1|^2\,dx} \\
&\quad + \sqrt{\textstyle\int_{\Omega_0}\chi_0\,|Z_2 - \varepsilon^{-1}(r-a_0) - c_h|^2\,dx} + |c_h|\sqrt{|\Omega_0|}\Big) \\
&\le c\|\psi\|_{H^1(\Omega_0)}\Big(\sqrt{2\pi a_0 l\varepsilon\textstyle\int_{\Pi^+} Z_1^2(\xi)\,d\xi} \\
&\quad + \sqrt{2\pi a_0 l\varepsilon\textstyle\int_{\Pi^+} Z_2^2(\xi+\xi_1-c_h)\,d\xi} + |c_h|\sqrt{|\Omega_0|}\Big) \\
&\le c\|\psi\|_{H^1(\Omega_0)}\Big(\sqrt{\varepsilon}\,\|Z_1\|_{L^2(\Pi^+)} + \sqrt{\varepsilon}\,\|Z_2+\xi_1-c_h\|_{L^2(\Pi^+)} + |c_h|\sqrt{|\Omega_0|}\,\Big),
\end{aligned}$$

where $|\Omega_0|$ is the measure of the domain Ω_0. On the basis of (2.16) and (2.17) the value $\|Z_1\|_{L^2(\Pi^+)}$ and $\|Z_2+\xi_1-c_h\|_{L^2(\Pi^+)}$ are bounded. As a result, we have

$$|I_4^+(\varepsilon,\psi) + I_4^-(\varepsilon,\psi)| \ \le\ \varepsilon\, C_4 \|\psi\|_{H^1(\Omega_\varepsilon)}. \tag{3.12}$$

Remark 3.1. The constants C_2 and C_3 in (3.11) and (3.12) respectively depend on the following quantities

$$\sup_{x\in\Gamma_0} \left|\mathcal{D}^\alpha\big(v_0^+(x)\big)\right|, \qquad |\alpha| = \alpha_1+\alpha_2+\alpha_3 \le 2. \tag{3.13}$$

Applying the odd extension to the limit problem (2.20) with respect to the planes $x_2 = 0$, $x_2 = l$ and taking into account the conditions for the function f_0, we conclude that the function v_0^+ and its second derivatives have no singularities at the points on $\overline{S^{(0)}} \cap \overline{\Gamma_0}$ and on $\overline{S^{(l)}} \cap \overline{\Gamma_0}$. Thus, by virtue of classical results on the smoothness of solutions to boundary value problems, the quantities (3.13) are bounded.

Since $f_0 \in H^3(\Omega_1)$, the function $\partial_{x_2} v_0^- \in H^1(\Omega_1 \setminus \overline{\Omega_0})$. Therefore,

$$|I_5^-(\varepsilon,\psi)| \ \le\ \varepsilon\, C_5 \, \|\partial_{x_2} v_0^-\|_{H^1(\Omega_1\setminus\overline{\Omega_0})} \, \|\psi\|_{H^1(\Omega_1\setminus\overline{\Omega_0})}.$$

So, with regard to the inequalities obtained, we conclude that for the right-hand side in (3.10) the following inequality holds

$$|F_\varepsilon(\psi)| \le c(\delta)\, \varepsilon^{1-\delta} \|\psi\|_{H^1(\Omega_\varepsilon)}, \tag{3.14}$$

where δ is an arbitrary fixed positive number. From (3.10) and (3.14) it follows the following results.

Theorem 3.1. *Suppose $f_\varepsilon \in L^2(\Omega_\varepsilon)$, $f_0 \in H^3(\Omega_1)$ and f_0, $\partial_{x_2} f_0$ vanish on $S^{(0)} \cup S^{(l)}$.*

Then for any $\delta > 0$ there exist positive constants c_1, ε_0 such that for all values $\varepsilon \in (0,\varepsilon_0)$ the difference between the solution u_ε to problem (1.1) and the approximation function R_ε defined by (3.1) and (3.2) satisfies the following estimate

$$\|u_\varepsilon - R_\varepsilon\|_{H^1(\Omega_\varepsilon)} \ \le\ c_1 \left(\varepsilon^{1-\delta} \ +\ \|f_\varepsilon - f_0\|_*\right). \tag{3.15}$$

Corollary 3.1. *From (3.15) it follows that*

$$\|u_\varepsilon - v_0\|_{L^2(\Omega_\varepsilon)} \ \le\ c_2 \left(\varepsilon^{1-\delta} \ +\ \|f_\varepsilon - f_0\|_*\right),$$

where v_0 is the weak solution to the limit problem (2.20).

Corollary 3.2. *Assume $f_\varepsilon(x) = f_0(x) + \varepsilon f_1(x,\varepsilon)$, $x \in \Omega_\varepsilon$, where the norm $\|f_1(\cdot,\varepsilon)\|_{L^2(\Omega_\varepsilon)} = \mathcal{O}(1)$ as $\varepsilon \to 0$. Then for any $\delta > 0$ there exist positive constants c_3, ε_0 such that for all values $\varepsilon \in (0,\varepsilon_0)$*

$$\|u_\varepsilon - R_\varepsilon\|_{H^1(\Omega_\varepsilon)} \ \le\ c_3\, \varepsilon^{1-\delta}, \qquad \|u_\varepsilon - v_0\|_{L^2(\Omega_\varepsilon)} \ \le\ c_3\, \varepsilon^{1-\delta}.$$

Example. If the right-hand side f_0 of the limit problem (2.20) depends only on the variable r and x_2, then we can find the explicit solution in the domain $D = \{x : \ r \in (a_0, a_1), \quad x_2 \in (0, l)\}$ and reduce problem (2.20) to a problem in the junction's body Ω_0.

In this case the solution to the limit problem (2.20) depends only on the variable r and x_2 as well. So, we can rewrite the limit problem in the following form

$$\begin{aligned} -\partial_r\big(r\partial_r v_0^+\big) + r\partial^2_{x_2x_2} v_0^+ &= r f_0(r, x_2), && x \in \Omega_0, \\ -\partial_r\big(r h_0(r)\partial_r v_0^-\big) &= r h_0(r) f_0(r, x_2), && x \in D, \\ \partial_r v_0^-(a_1, x_2) &= 0, && x_2 \in (0, l), \\ v_0^+(r, 0) = v_0^+(r, l) &= 0, && x \in S^{(0)} \cup S^{(l)}, \\ v_0^+(a_0, x_2) &= v_0^-(a_0, x_2), && x_2 \in (0, l), \\ \partial_r v_0^+(a_0, x_2) &= h_0(a_0)\, \partial_r v_0^-(a_0, x_2), && x_2 \in (0, l), \end{aligned} \tag{3.16}$$

By solving the ordinary equation of problem (3.16) in the domain D with regard to the Neumann condition at $r = a_1$ and to the first transmission condition at $r = a_0$ in the joint zone Γ_0, we find that

$$v_0^-(r, x_2) = v_0^+(a_0, x_2) + \int_{a_0}^{r} \frac{1}{\rho\, h_0(\rho)} \int_{\rho}^{a_1} t\, h_0(t)\, f_0(t, x_2)\, dt\, d\rho.$$

Now, according to the second transmission condition in problem (3.16), we obtain the classical mixed boundary-value problem

$$\begin{aligned} -\partial_r\big(r\partial_r v_0^+\big) \ + \ r\partial^2_{x_2x_2} v_0^+ &= r f_0(r, x_2), && x \in \Omega_0, \\ v_0^+(r, 0) = v_0^+(r, l) &= 0, && x \in S^{(0)} \cup S^{(l)}, \\ \partial_r v_0^+(a_0, x_2) &= \widehat{F}_0(x_2), && x_2 \in (0, l), \end{aligned} \tag{3.17}$$

to find v_0^+. Here

$$\widehat{F}_0(x_2) = a_0^{-1} \int_{a_0}^{a_1} t\, h_0(t) f_0(t, x_2)\, dt, \qquad x_2 \in (0, l).$$

Problem (3.17) is called *resulting problem* for problem (1.1).

Conclusion

We assumed that the functions h_- and h_+ are locally constant and equal in an enough small neighborhood of the point a_0. This is a technical condition which allows to avoid additional bulky calculations. Because of this, the junction-layer solutions are odd or even with respect to $1/2$ (see Statement 2.1). As a result, the approximation function R_ε satisfies exactly some boundary conditions and we do not need additional boundary-layer asymptotics.

If the right-hand side has the following form $f_\varepsilon = \sum_{k=0}^{\infty} \varepsilon^k f_k(x)$, we can define the other terms in the asymptotic expansions (2.1), (2.2), (2.14) and construct an asymptotic approximation to any degree of accuracy.

From results proved in the present paper it follows that for applied problems in thick junctions we can use the corresponding limit problem or resulting problem in the junction's body, which are simpler, instead of the initial problem with the sufficient plausibility.

References

[1] Y. Amirat and O. Bodart, *Boundary layer corrector for the solution of Laplace equation in a domain with oscillating boundary*, Zeitschrift für Analysis und ihre Anwendungen // **20** (2001), N 4, 929–940.

[2] Y. Amirat, O. Bodart, U. De Maio, and A. Gaudiello, *Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary*, Preprint of Laboratoire de Mathematiques Appliquees Universite Blaise Pascal, Clemont-Ferrand, October, 2002 (to appear in SIAM, J. Math. Anal.)

[3] N. Bachvalov and G. Panasenko, *Homogenization: averaging processes in periodic media*, Kluwer Academic Publishes, series "Mathematics and its Applications". 1989.

[4] A. Benkaddour and J. Sanchez-Hubert, *Spectral study of a coupled compact-noncompact problem // M^2AN*, **26** (1992), N 6, 659–672.

[5] A. Bensoussan, J. Lions and G. Papanicolau, *Asymptotic analysis for periodic structure*, Amsterdam: North Holland, 1978.

[6] D. Blanchard, L. Carbone, and A. Gaudiello, *Homogenization of a monotone problem in a domain with oscillating boundary* // Mathematical Modelling and Numerical Analysis, **33** (1999), N 5, 1057–1070.

[7] D. Blanchard and A. Gaudiello, *Homogenization of highly oscillating boundaries and reduction of dimension for a monotone problem* // ESAIM, Control. Optim. Calc. Var., **9** (2003) 449–460.

[8] R. Brizzi and J. P. Chalot, *Homogenization and Neumann Boundary Value Problem* // Ric. Mat. **46** (1997), 347–387.

[9] P. G. Ciarlet, *Plates and Junctions in Elastic Multi-structures*, Masson, Paris, 1990.

[10] D. Cioranescu and J. Saint Jean Paulin, *Homogenization of reticulated structures*, Springer-Verlag, series "Applied Mathematical Sciences" **136**, 1999.

[11] U. De Maio and T. A. Mel'nyk, *Homogenization of the Neumann problem in thick multi-structures of type 3:2:2* // Mathematical Methods in the Applied Sciences, **28** (2005), 865–879.

[12] V. V. Jikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, Heidelberg, 1994.

[13] F. Fleury and E. Sanchez-Palencia, *Asymptotic and spectral properties of the acoustic vibrations of body perforated by narrow channels* // Bull. Sci. Math, **2** (1986), N 110, 149–176.

[14] V. A. Kondrat'ev and O. A. Oleinik, *Boundary-value problems for partial differential equations in non-smooth domains* // Uspekhi Mat. Nauk, **38** (1983), N 2, 3–76; English transl., Russian Math. Survays **38** (1983), N 2, 1–86.

[15] V. P. Kotliarov and E. Ya. Khruslov, *On a limit boundary condition of some Neumann problem* // Theor. Funkts. i Funkts. Anal. Prilozhen, Izd-vo Kharkov Univ., **10** (1970), 83–96 (in Russian).

[16] A. A. Kovalevsky, *Averaging of the Neumann problems for nonlinear elliptic equations in domains with accumulators* // Ukrainian Math. J. **47** (1995), N 2, 227–249.

[17] V. A. Kozlov, V. G. Maz'ya and A. B. Movchan, *Asymptotic representation of an elastic field in a multi-structure* // Asymptotic Analysis, **11** (1995) 343–415.

[18] V. A. Kozlov, V. G. Maz'ya and J. Rossmann, *Elliptic boundary value problems in domains with point singulaities*, Math. Surveys Monogr., **52**, AMS, Providence, RI, 1997.

[19] V. A. Marchenko and E. Ya. Khruslov, *Boundary-value problems in domains with fine-grained boundary*, Naukova Dumka, Kiev, 1974 (in Russian).

[20] V. G. Maz'ya and B. A. Plamenevskii, *The coefficients in the asymptotics of solutions of elliptic boundary value problem with conical points* // Math. Nachr. **76** (1977), 29–60 (Russian).

[21] T. A. Mel'nyk and S. A. Nazarov, *Asymptotic structure of the spectrum of the Neumann problem in a thin comb-like domain, C. R. Acad. Sci., Paris* // **319**, Serie 1 (1994), 1343–1348.

[22] T. A. Mel'nyk and S. A. Nazarov, *Asymptotics of the Neumann spectral problem solution in a domain of "thick comb" type* // Trudy Seminara imeni I. G. Petrovskogo, **19** (1996), 138–173 (in Russian); and English translation: Journal of Math. Sci. **85** (1997), N 6, 2326–2346.

[23] T. A. Mel'nyk, *Homogenization of the Poisson equation in a thick periodic junction* // Zeitschrift für Analysis und ihre Anwendungen, **18** (1999), N 4, 953–975.

[24] T. A. Mel'nyk, *Asymptotic analysis of a spectral problem in a periodic thick junction of type 3:2:1* // Mathematical Methods in the Applied sciences, **23** (2000), N 4, 321–346.

[25] T. A. Mel'nyk and S. A. Nazarov, *Asymptotic analysis of the Neumann problem of the junction of a body and thin heavy rods* // Algebra i Analiz., **12** (2000), N 2, 188–238; English transl.: St.Petersburg Math.J., **12** (2001), N 2, 317–351.

[26] T. A. Mel'nyk, *Homogenization of a singularly perturbed parabolic problem in a thick periodic junction of the type 3:2:1* // Ukrainskii Matem. Zhurnal, **52** (2000), N 11, 1524–1534 (in Ukrainian); English translation: Ukrainian Math. Journal, **52** (2001), N 11, 1737–1749.

[27] T. A. Mel'nyk, *Asymptotics of the eigenvalues and eigenfunctions of a boundary value problem in a thick periodic junction of type 3:2:2* // Bulletin of the University of Lviv, Series: Mathematics, 58 (2000), 153–160 (in Ukrainian).

[28] T. A. Mel'nyk, *Vibrations of a Thick Periodic Junction with Concentrated Masses* // Mathematical Models and Methods in Applied Sciences, **11** (2001), N 6, 1001–1029.

[29] S. A. Nazarov and B. A. Plamenevskii, *Elliptic problems in domains with piecewise smooth boundaries*, Walter de Gruyter, Berlin, 1994.

[30] S. A. Nazarov *The polynimial property selfadjoint elliptic boundary value problems and algebraic discribsion their atributes* // Uspechi Mat. Nauk, **54** (1999), N 5, 77–142; English transl., Russian Math. Surveys, **54** (1999), N 5.

[31] O. A. Oleinik, G. A. Yosifian, and A. S. Shamaev, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.

[32] J. Sanchez-Hubert and E. Sanchez-Palencia, *Vibration and Coupling of Continuous Systems*, Springer-Verlag, Berlin, Heidelberg, 1989.

[33] I. V. Skrypnik, *Methods for investigation on nonlinear elliptic boundary-value problems*, Nauka, Moscow, 1990 (in Russian).

[34] G. V. Suzikov and E. Ya. Khruslov, *On advancing sound waves through narrow channels in a reflecting layer* // Theor. Funkts., Funkts. Anal. Prilozhen, Izd-vo Kharkov Univ., **5** (1967), 35–49 (in Russian).

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