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Homogenization of Maxwell's Equations in Domains with Dense Perfectly Conducting Grids

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Abstract. We consider Maxwell's equations in domains that are complements to connected, grid-like sets formed by intersecting thin wires. We impose the boundary conditions that correspond to perfectly conducting wires, and study the asymptotic behavior of solutions as grids are becoming thinner and denser. We derive a homogenized system of equations describing the leading term of the asymptotics. Assuming that a Korn-type inequality holds, we validate the homogenization procedure.

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Introduction

Grid structures like metal wire gratings with cells of various form are widely used as elements of radio devices, in particular, antennas, radio relays [7], various devices designed for physical experiments [12], etc. They provide shielding, polarizing, retarding (accelerating), and other types of control of electromagnetic fields. In order to study electrodynamic properties of a grid structure, it is necessary to study a boundary value problem for Maxwell's equations in a complex domain that is a complement to the grid, with certain boundary conditions on the grid surface. Often, a grid can be thought of (with good accuracy) as perfectly conducting, so that the corresponding boundary conditions must reflect the fact that the tangential component of the electric field vanishes on the grid surface.

If a grid is dense, it is practically impossible to solve the problem exactly, because of very complicated behavior of the electromagnetic field near the grid. On the other hand, an exact knowledge of the field near

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the grid is often needless, since we are usually interested, in applications, in certain integral characteristics such as eigenfrequencies, reflection and transmission coefficients, etc., which are determined by the "mean" field or by the behavior of the field far from the grid. Therefore, it is reasonable to assume that if a grid is sufficiently dense, then it acts similarly to some effective continuous medium (or film), so that its influence can be described, approximately, by homogenized differential equations (or homogenized boundary conditions). In order to derive these equations (boundary conditions), one has to analyze the asymptotic behavior of solutions of Maxwell's equations in domains with grids that are becoming denser. This is the main problem of homogenization theory for partial differential equations. It was formulated, in a rigorous mathematical form, by V.A.Marchenko in the middle of 1960s and since that, it has been attracting the attention of mathematicians thus stimulating the development of the homogenization theory as a whole.

The problem consists in the study of the asymptotic behavior of solutions of Maxwell's equations in a domain $\Omega \subset \mathbb{R}^3$ with perfectly conducting inclusions $F_\varepsilon \subset \Omega$ of arbitrary form; the inclusions depend of a small parameter ε such that, as $\varepsilon \to 0$, they are becoming "rarer" but filling Ω "denser". The problem in such a general formulation has appeared to be very difficult. Therefore, researchers (mathematicians as well as radio physicists) have being concentrated on two polar particular cases having prior interest for applications: (i) inclusions F_ε are assumed to have a fine-grained structure, i.e., they are unions of small disjoint components (grains); such structures are used in the synthesis of artificial dielectrics; (ii) inclusions are connected domains like grids formed by thin intersecting wires.

For the case of fine-grained periodic inclusions, a rigorous solution of the problem was obtained by V. V. Zhikov and O. A. Nazarova in [11] and [14] (see also [13]). They proved that in this case, the homogenized models are described by the standard Maxwell's equations for (non-conducting) continuous media with constant effective parameters: the dielectric permeability ε and the magnetic permittivity μ. This result was completely in accordance with the physical intuition and was not unexpected: the same results had been obtained earlier for dielectric inclusions [1] and for perfectly conducting inclusions of small concentration [4].

A different situation occurs in the case of grid-type inclusions. Despite of the fact that grid structures are of much higher interest for radio physicists then the fine-grained ones, at present there is no rigorous mathematical solution in this case. Moreover, the type of homogenized equations is unclear, even intuitively, although radio physicists and engineers are using, quite effectively, various approximate models [7].

In the paper, we propose a conditional solution of the problem. Namely, we derive homogenized equations describing the main term of the asymptotics as $\varepsilon \to 0$, under the assumption that a Korn-type inequality holds true. This inequality is to be satisfied (uniformly with respect to ε) by vector functions of a special class introduced for domains $\Omega \setminus F_\varepsilon$. The question whether there exists a grid structure $\{F_\varepsilon\}$ for which this inequality is satisfied, remains open. But, assuming that this inequality is satisfied for grids of a certain type, the derived equations are in fact the effective equations of electrodynamics of a continuous medium equivalent to the grid structure.

1. Problem Statement and Qualitative Description of Main Result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth connected boundary $\partial\Omega$ and let $\{F_\varepsilon, \varepsilon > 0\}$ be a family of closed sets in Ω depending on a small parameter $\varepsilon > 0$. We assume that the structure of sets of this family is as follows:

1. For all $\varepsilon > 0$, the sets F_ε are connected and belong to a fixed subdomain G that is compact in Ω and has a smooth boundary ∂G; the boundary ∂F_ε of F_ε is also smooth (i.e., it is a smooth manifold of dimension 2);

2. As $\varepsilon \to 0$, the intersection of F_ε and of its complement $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ with any cube $K(x,h)$ of size $h > 0$ centered at $x \in G$ (i.e. the sets $F_\varepsilon \cap K(x,h)$ and $\Omega_\varepsilon \cap K(x,h)$), are becoming non-empty and connected.

Consider in $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$ the initial boundary value problem for Maxwell's equations

$$\frac{\partial H_\varepsilon}{\partial t} + \mathrm{rot} E_\varepsilon = 0, \qquad x \in \Omega_\varepsilon, t > 0, \tag{1.1}$$

$$-\frac{\partial E_\varepsilon}{\partial t} + \mathrm{rot} H_\varepsilon = J, \qquad x \in \Omega_\varepsilon, t > 0, \tag{1.2}$$

$$n \wedge E_\varepsilon = 0, \qquad x \in \partial F_\varepsilon \cup \partial\Omega, \tag{1.3}$$

$$E_\varepsilon(x,0) = E^0(x), \qquad H_\varepsilon(x,0) = H^0(x), \qquad x \in \Omega_\varepsilon, \tag{1.4}$$

where $E_\varepsilon = E_\varepsilon(x,t)$ and $H_\varepsilon = H_\varepsilon(x,t)$ are the vectors of electric and magnetic fields, respectively, $E^0(x)$ and $H^0(x)$ are the given vectors of the initial distribution of these fields, $J = J(x,t)$ is the given current, and n is the normal vector to ∂F_ε or $\partial\Omega$ (the symbol $\wedge$ denotes the vector product). The boundary condition (1.3) means that the tangential component of the electric field vanishes on the perfectly conducting grid F_ε as well as on the external boundary $\partial\Omega$. Notice that in this problem, the boundary conditions on F_ε are essential whereas the boundary conditions on $\partial\Omega$ are irrelevant: one can assume any boundary conditions on $\partial\Omega$, including nonhomogeneous ones.

Remark 1.1. Equations (1.1) and (1.2) yield the following equations

$$\mathrm{div} E_\varepsilon = \rho, \tag{1.5}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div} J = 0, \qquad \rho(x,0) = \mathrm{div} E^0(x), \tag{1.6}$$

and, if $\mathrm{div} H^0(x) = 0$ (which is naturally assumed), then

$$\mathrm{div} H_\varepsilon = 0. \tag{1.7}$$

For the sake of simplicity, we will assume that the given $E^0(x)$, $H^0(x)$, and $J(x,t)$ are sufficiently smooth, that their supports (with respect to x) are compact in $\Omega \setminus \bar{G}$, and that $|J(x,t)| < C$ for all t.

It is known (see, e.g., [2]) that there exists a unique solution $\{E_\varepsilon(x,t), H_\varepsilon(x,t)\}$ of problem (1.1)–(1.4). The present paper aims at describing the asymptotic behavior of this solution as $\varepsilon \to 0$. Notice that assumptions 1 and 2 above (which are assumptions about the geometrical structure of sets $\{F_\varepsilon, \varepsilon > 0\}$) do not guarantee the existence of the asymptotics of $\{E_\varepsilon(x,t), H_\varepsilon(x,t)\}$. In Section 2 we will formulate additional (quantitative) conditions which do guarantee the existence of the (weak) limit $\lim\limits_{\varepsilon\to 0}\{E_\varepsilon(x,t), H_\varepsilon(x,t)\} = \{E(x,t), H(x,t)\}$. These conditions have to be verified in each particular case (for a particularly given system of sets $\{F_\varepsilon, \varepsilon > 0\}$), which would give, simultaneously, the coefficients of the homogenized equations for the limiting field $\{E(x,t), H(x,t)\}$.

In Section 2 we will also formulate the main result of the paper. But it seems reasonable to give first a qualitative description of the expected result. Namely, we give the form of the homogenized equations, leaving the precise formulation of conditions under which these equations are valid, to Section 3.

It turns out that the limiting field $\{E(x,t), H(x,t)\}$ satisfies the following system of equations in Ω:

$$\frac{\partial H}{\partial t} + \mathrm{rot} E = 0, \qquad x \in \Omega, t > 0, \tag{1.8}$$

$$-\frac{\partial E}{\partial t} + \mathrm{rot} H - \int_0^t \mathbf{L}(E(\cdot,\tau) - \mathrm{grad}\Phi(\cdot,\tau))d\tau = J \qquad x \in \Omega, t > 0, \tag{1.9}$$

$$C\frac{\partial^2\Phi}{\partial t^2} - \mathrm{div}(\mathbf{L}\mathrm{grad}\Phi) - \mathrm{div}(\mathbf{L}E) = 0, \qquad x \in G, t > 0 \tag{1.10}$$

and the following boundary and initial conditions:

$$n \wedge E = 0, \qquad x \in \partial\Omega, t > 0, \tag{1.11}$$

$$(\mathbf{L}\mathrm{grad}\Phi)_n = (\mathbf{L}E)_n, \qquad x \in \partial G, t > 0, \tag{1.12}$$

$$E(x,0) = E^0(x), \quad H(x,0) = H^0(x), \quad \Phi(x,0) = \frac{\partial\Phi}{\partial t}(x,0) = 0 \tag{1.13}$$

(the index n denotes the normal component of a vector). Here $C = C(x)$ is a positive function and $\mathbf{L} = \mathbf{L}(x)$ is a symmetric nonnegative tensor; they are given on G and are extended by zero outside G. The additional unknown function $\Phi(x,t)$ (the potential) can be uniquely determined only on $G \subset \Omega$. For definiteness, we assume that $\Phi(x,t)$ is extended by zero outside G; then the initial boundary value problem (1.8)–(1.13) has a unique solution $\{E(x,t), H(x,t), \Phi(x,t)\}$ (in an appropriate class of functions).

The system of equations (1.8)–(1.10) can be written in the following, more physical, form:

$$\frac{\partial H}{\partial t} + \mathrm{rot}\, E = 0,$$
$$-\frac{\partial E}{\partial t} + \mathrm{rot}\, H = J + J_{\mathrm{eff}},$$
$$\mathrm{div}\, E = \rho + \rho_{\mathrm{eff}},$$
$$\frac{\partial \rho_{\mathrm{eff}}}{\partial t} + \mathrm{div}\, J_{\mathrm{eff}} = 0,$$

where

$$J_{\mathrm{eff}} = \int_0^t \mathbf{L}(E - \mathrm{grad}\,\Phi)d\tau \tag{1.14}$$

and

$$\rho_{\mathrm{eff}} = C\Phi. \tag{1.15}$$

The physical meaning of this system is as follows: perfectly conducting grids F_ε behave, as $\varepsilon \to 0$, as a continuous medium, in which a current and a charge are induced, with the densities $J_{\mathrm{eff}}(x,t)$ and $\rho_{\mathrm{eff}}(x,t)$, respectively, determined by (1.14) and (1.15). Equalities (1.14) and (1.15) play the role of the constitutive equations for an effective medium. But, unlike the classical constitutive equations for electromagnetic media, these equations involve an additional scalar field, $\Phi(x,t)$ (the electric potential), which cannot be excluded from them. Such a medium can be called *inductive-capacitive*.

Remark 1.2. The homogenized equations (1.8)–(1.10) describing an effective continuous medium are relevant in the case of a "bulk" distribution of grids F_ε in $G \subset \Omega$ (see assumption 1). A "surface" distribution of grids is also of great interest; here the sets F_ε are concentrating, as $\varepsilon \to 0$, in an arbitrarily small neighborhood of some surface $\Gamma \subset \Omega$ [7]. In this case, grids behave like a continuous film, and the limiting field can be described by using homogenized boundary conditions on Γ. We will not consider this case in the present paper.

2. Local Quantitative Characteristics of Grids. Main Result

Let us introduce needed quantitative characteristics of sets F_ε: scalar quantities $C(x,h,\varepsilon)$ characterizing the capability of the grid F_ε to concentrate on it the electric charge, and tensors $\mathbf{L}(x,h,\varepsilon)$ characterizing the capability of the grid to keep the magnetic field coupled with it. These quantities must characterize the structure of F_ε is some small neighborhood of every $x \in G$. We take these neighborhoods in the form of cubes $K(x,h)$ of size $h > 0$ centered at $x \in \bar{G}$ and oriented along the coordinate axes (a particular orientation is irrelevant but it has to be the same for all x and all h). The length of edges has to be sufficiently small but it must be much larger than the characteristic scale ε (which can be, for example, the size of the grid cells): $0 < \varepsilon \ll h \ll 1$. Therefore, these characteristics are called *mesoscopic*.

Denote by $H_0(x,h,\varepsilon)$ the class of functions in the Sobolev space $W_2^1(K(x,h))$ equal to zero on $F_\varepsilon \cap K(x,h)$. Set

$$C(x,h,\varepsilon) = \inf_{v_\varepsilon \in H_0(x,h,\varepsilon)} \int\limits_{K(x,h)} \left\{ |\nabla v_\varepsilon|^2 + h^{-2-\gamma} |v_\varepsilon - 1|^2 \right\} dx, \tag{2.1}$$

where $\gamma > 0$ is a penalty parameter. In what follows, we will choose it in the interval $0 < \gamma < 2$.

Obviously, if $F_{\varepsilon_1} \cap K(x,h) \subseteq F_{\varepsilon_2} \cap K(x,h)$, then $C(x,h,\varepsilon_1) \leq C(x,h,\varepsilon_2)$; therefore, $C(x,h,\varepsilon)$ characterizes the massiveness (capacity) of the set $F_\varepsilon \cap K(x,h)$. The quantities $C(x,h,\varepsilon)$ depend on the parameter γ but one can show that, as $\varepsilon \to 0$ and $h \to 0$, this dependence is vanishing, so that

$$C(x,h,\varepsilon) \sim \mathrm{Cap}(F_\varepsilon \cap K(x,h)), \qquad \varepsilon \ll h \to 0,$$

where $\mathrm{Cap}(F)$ denotes Newton's capacity of F [8]. Therefore, $C(x,h,\varepsilon)$ is equivalent to Newton's capacity of the sets $F_\varepsilon \cap K(x,h)$ and thus characterizes the capability of F_ε to concentrate a charge on it, in some neighborhood of x.

Denote by $R_0(x,h,\varepsilon)$ the closure, with respect to the norm

$$\|v_\varepsilon\|^2 = \int_{K(x,h)} \{|\mathrm{rot}\, v_\varepsilon|^2 + |v_\varepsilon|^2\} dx, \tag{2.2}$$

of the set of 3-component vector functions in $(W_2^1(K(x,h)))^3$ equal to zero on $F_\varepsilon \cap K(x,h)$. It is known [3] that this set coincides with the set of vector functions, with finite norm (2.2), which equal 0 on $F_\varepsilon \cap K(x,h)$ and have vanishing tangential components on $\partial F_\varepsilon \cap K(x,h)$.

Set

$$L(x,h,\varepsilon,l) = \inf_{v_\varepsilon \in R_0(x,h,\varepsilon)} \int_{K(x,h)} \{|\mathrm{rot} v_\varepsilon|^2 + h^{-2-\gamma}|v_\varepsilon - l|^2\} dx \tag{2.3}$$

for all $l = \{l_1, l_2, l_3\} \in \mathbb{R}^3$ and some $0 < \gamma < 2$. It is easy to show (see Section 4) that $L(x,h,\varepsilon,l)$ is a homogeneous quadratic function with respect to l, which can be expressed in the form

$$L(x,h,\varepsilon,l) = \sum_{i,j=1}^{3} \mathbf{L}_{ij}(x,h,\varepsilon) l_i l_j, \tag{2.4}$$

where the system of numbers $\{\mathbf{L}_{ij}(x,h,\varepsilon)\}_{i,j=1}^3$ constitutes a symmetric non-negative tensor in $\mathbb{R}^3$ of the second rang. By analogy with the electrotechnical terminology, this tensor can be called *tensor of back induction* of the grid $F_\varepsilon \cap K(x,h)$.

We assume that, for all $x \in \bar{G}$, the following limits exist:

(c$_1$) $\quad \lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \frac{C(x,h,\varepsilon)}{h^3} = \lim_{h\to 0} \underline{\lim_{\varepsilon\to 0}} \frac{C(x,h,\varepsilon)}{h^3} = C(x),$

(c$_2$) $\quad \lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \frac{\mathbf{L}_{ij}(x,h,\varepsilon)}{h^3} = \lim_{h\to 0} \underline{\lim_{\varepsilon\to 0}} \frac{\mathbf{L}_{ij}(x,h,\varepsilon)}{h^3} = \mathbf{L}_{ij}(x),$

where $C(x)$ and $\mathbf{L}_{ij}(x)$ $(i,j = 1,2,3)$ are functions bounded in $\bar{G}$ and continuous in G such that $C(x) > 0$ and $\{\mathbf{L}_{ij}(x)\}$ is a nonnegative tensor in G.

Remark 2.1. It suffices that the limits in (c$_1$) and (c$_2$) be finite for some value of the penalty parameter γ in the interval $0 < \gamma < 2$. Then one can show that the limits are finite as well for all $\gamma > 0$ and that the limiting functions $C(x)$ and $\mathbf{L}_{ij}(x)$ $(i,j = 1,2,3)$ are independent of γ.

Now we make one more assumption about the structure of sets F_ε. Consider the Hilbert space of real-valued vector functions $W_0(\Omega_\varepsilon) = (\overset{\circ}{W}{}_2^1(\Omega_\varepsilon))^3$ defined in $\Omega_\varepsilon = \Omega \setminus F_\varepsilon$, with components $u_k(x)$ $(k = 1, 2, 3)$ in the Sobolev space $\overset{\circ}{W}{}_2^1(\Omega_\varepsilon)$ which are equal to zero on the boundary $\partial\Omega_\varepsilon = \partial F_\varepsilon \cup \partial\Omega$. Define an inner product $(\cdot,\cdot)_\varepsilon$ (and, consequently, a norm $\|\cdot\|_\varepsilon = (\cdot,\cdot)_\varepsilon^{1/2}$) in $W_0(\Omega_\varepsilon)$ by

$$(u, v)_\varepsilon = \int_{\Omega_\varepsilon} \left\{ \sum_{i,k=1}^{3} \frac{\partial u_k}{\partial x_i} \frac{\partial v_k}{\partial x_i} + \lambda^2 \sum_{k=1}^{3} u_k v_k \right\} dx,$$

where $\lambda^2 > 0$. Let $G_0(\Omega_\varepsilon)$ be a subspace in $W_0(\Omega)$ consisting of gradients of functions in the Sobolev space $W_2^2(\Omega_\varepsilon)$ which are constants on ∂F_ε and $\partial\Omega$ and the normal derivatives of which vanish on ∂F_ε and $\partial\Omega$. Without loss of generality, we can choose these functions to be equal to zero on ∂F_ε, i.e.,

$$\begin{aligned} G_0(\Omega_\varepsilon) \quad = \quad & \Big\{ u_\varepsilon(x) = \operatorname{grad} \varphi_\varepsilon(x), \quad \varphi_\varepsilon(x) \in W_2^2(\Omega_\varepsilon); \\ & \varphi_\varepsilon(x) = \frac{\partial \varphi_\varepsilon}{\partial n}(x) = 0, \quad x \in \partial F_\varepsilon; \\ & \varphi_\varepsilon(x) = \text{const}, \ \frac{\partial \varphi_\varepsilon}{\partial n}(x) = 0, \quad x \in \Omega_\varepsilon \Big\}. \end{aligned}$$

Denote by $W_1(\Omega_\varepsilon)$ the orthogonal complement to $G_0(\Omega_\varepsilon)$ in $W_0(\Omega_\varepsilon)$: $W_1(\Omega_\varepsilon) = W_0(\Omega_\varepsilon) \ominus G_0(\Omega_\varepsilon)$. We will assume that the following condition holds true:

(c_3) for all $u_\varepsilon \in W_1(\Omega_\varepsilon)$,

$$\|u_\varepsilon\|_\varepsilon^2 < C \int_{\Omega_\varepsilon} |\operatorname{rot} u_\varepsilon|^2 dx,$$

where the constant C is independent of ε.

Now we are at a position to formulate the main result.

Theorem 2.1. *Assume that conditions* (c_1)–(c_3) *are satisfied. Then solutions* $\{E_\varepsilon(x,t), H_\varepsilon(x,t)\}$ *of problem* (1.1)–(1.4)*, being extended by zero, with respect to* x*, on* F_ε*, converge, as* $\varepsilon \to 0$*, weakly in* $(L_2(\Omega \times [0,T]))^3 \times (L_2(\Omega \times [0,T]))^3$ *(for all* T*) to a solution* $\{E(x,t), H(x,t)\}$ *of problem* (1.8)–(1.13)*.*

The proof of Theorem 2.1 will be given in Sects. 3–5, the scheme of the proof being as follows. By using the Laplace transform, we reduce problem (1.1)–(1.4) to two stationary boundary value problems in

$\Omega_\varepsilon = \Omega \backslash F_\varepsilon$, which in turn are reduced, for $\lambda > 0$, to associated variational problems. The main part of the proof consists in the study of the asymptotic behavior of these variational problems and in the derivation of the homogenized equations (for $\lambda > 0$). Then we study analytical properties of solutions of the original and the homogenized problems. Finally, the application of the inverse Laplace transform completes the proof.

3. The Stationary Problem

Let us apply the Laplace transform, with respect to t, to problem (1.1)–(1.4). Introduce the vector functions

$$E_\varepsilon(x,\lambda) = \int_0^\infty E_\varepsilon(x,t)e^{-\lambda t}dt, \qquad H_\varepsilon(x,\lambda) = \int_0^\infty H_\varepsilon(x,t)e^{-\lambda t}dt, \quad (3.1)$$

which depend on $x \in \Omega_\varepsilon$ and a complex variable λ with $\mathrm{Re}\lambda > 0$. For the sake of simplicity, we keep the original notations: $E_\varepsilon = E_\varepsilon(x,\lambda)$ and $H_\varepsilon = H_\varepsilon(x,\lambda)$. Then we arrive at the following stationary boundary value problem in Ω_ε:

$$\mathrm{rot}\, E_\varepsilon + \lambda H_\varepsilon = H^0, \qquad x \in \Omega_\varepsilon, \quad (3.2)$$

$$\mathrm{rot}\, H_\varepsilon - \lambda E_\varepsilon = J - E^0, \qquad x \in \Omega_\varepsilon, \quad (3.3)$$

$$n \wedge E_\varepsilon = 0, \qquad x \in \partial\Omega_\varepsilon. \quad (3.4)$$

Therefore, $E_\varepsilon(x,\lambda)$ has to solve the following boundary value problem:

$$\mathrm{rot}\,\mathrm{rot}\, E_\varepsilon + \lambda^2 E_\varepsilon = J^0, \qquad x \in \Omega_\varepsilon, \quad (3.5)$$

$$n \wedge E_\varepsilon = 0, \qquad x \in \partial\Omega_\varepsilon, \quad (3.6)$$

where

$$J^0(x,\lambda) = -\lambda\big(J(x,\lambda) - E^0(x)\big) + \mathrm{rot}\, H^0(x). \quad (3.7)$$

In turn, $H_\varepsilon(x,\lambda)$ is determined by $E_\varepsilon(x,\lambda)$:

$$H_\varepsilon(x,\lambda) = -\frac{1}{\lambda}(\mathrm{rot}\, E_\varepsilon(x,\lambda) - H^0(x)). \quad (3.8)$$

For all λ with $\mathrm{Re}\lambda > 0$, problem (3.5)–(3.6) has a unique solution $E_\varepsilon(x,\lambda)$.

Our primary goal is to study the asymptotic behavior of this solution as $\varepsilon \to 0$. Consider first the case of real λ ($\lambda > 0$). Denote by $W_0(\Omega_\varepsilon, \mathrm{rot})$

the closure of the set of vector functions in $(\overset{\circ}{W}{}_2^1(\Omega_\varepsilon))^3$ with respect to the norm

$$\|v_\varepsilon\|^2 = \int\limits_{\Omega_\varepsilon} \{|\mathrm{rot}\, v_\varepsilon|^2 + \lambda^2 |v_\varepsilon|^2\} dx. \tag{3.9}$$

Here and below, $|u_\varepsilon|$ denotes the Euclidean norm of u_ε in $\mathbb{R}^3$.

It is known [8] that the tangential components of vector functions with finite norms (3.9) have traces on $\partial\Omega_\varepsilon$ as elements of the space $W_2^{-1/2}(\partial\Omega_\varepsilon)$. Therefore, $W_0(\Omega, \mathrm{rot})$ consists of vector functions $v_\varepsilon(x)$ with finite norm (3.9), the tangential component of which equals zero on $\partial\Omega_\varepsilon$, i.e., $n \wedge v_\varepsilon = 0$.

Let us define in $W_0(\Omega_\varepsilon, \mathrm{rot})$ the functional

$$\Phi_\varepsilon[v_\varepsilon] = \int\limits_{\Omega_\varepsilon} \{|\mathrm{rot}\, v_\varepsilon|^2 + \lambda^2 |v_\varepsilon|^2 + (J_0, v_\varepsilon)\} dx, \tag{3.10}$$

where $\lambda > 0$ and $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^3$. Consider the minimization problem for this functional in the class $W_0(\Omega_\varepsilon, \mathrm{rot})$. In a standard way (see, e.g., [10]), one can show that there exists a unique vector function $v_\varepsilon = E_\varepsilon(x, \lambda) \in W_0(\Omega_\varepsilon, \mathrm{rot})$ such that

$$\Phi[E_\varepsilon] = \min_{v_\varepsilon \in W_0(\Omega_\varepsilon, \mathrm{rot})} \Phi_\varepsilon[v_\varepsilon] \tag{3.11}$$

and that this function solves (in the sense of distributions) the boundary value problem (3.5)–(3.6) for $\lambda > 0$. Conversely, the solution of (3.5)–(3.6) minimizes Φ_ε in the variational problem (3.11).

We want to obtain for $E_\varepsilon(x, \lambda)$ an appropriate representation that will be convenient for the study of its asymptotic behavior as $\varepsilon \to 0$. Introduce the Hilbert space $W(\Omega_\varepsilon)$ as follows: it consists of vector functions v_ε defined in Ω_ε such that the integral

$$\int\limits_{\Omega_\varepsilon} \{|\mathrm{rot}\, v_\varepsilon|^2 + (\mathrm{div}\, v_\varepsilon)^2 + \lambda |v_\varepsilon|^2\} dx$$

is finite, the tangential component vanishes on the boundary of Ω_ε:

$$n \wedge v_\varepsilon = 0 \qquad \text{for} \quad x \in \partial\Omega_\varepsilon,$$

and the flux through each connected component of the boundary equals zero:

$$\int\limits_{\partial F_\varepsilon} (v_\varepsilon)_n ds = \int\limits_{\partial\Omega} (v_\varepsilon)_n ds = 0.$$

Here ds denotes the area of the surface element of ∂F_ε or $\partial\Omega$.

We define an inner product in $W(\Omega_\varepsilon)$ by

$$(u_\varepsilon, v_\varepsilon)_\varepsilon = \int_{\Omega_\varepsilon} \{(\operatorname{rot} u_\varepsilon, \operatorname{rot} v_\varepsilon) + \operatorname{div} u_\varepsilon \operatorname{div} v_\varepsilon + \lambda^2 (u_\varepsilon, v_\varepsilon)\} dx. \tag{3.12}$$

Introduce in $W(\Omega_\varepsilon)$ the following subspaces:

- $W_0(\Omega_\varepsilon)$ is the subspace of vector functions equal to zero on ∂F_ε and $\partial\Omega$;
- $G(\Omega_\varepsilon)$ is the subspace of vector functions such that they are gradients of functions $\varphi_\varepsilon(x) \in W_2^2(\Omega_\varepsilon)$ equal to constants on ∂F_ε and $\partial\Omega$, and that the fluxes through each connected component of $\partial\Omega_\varepsilon$ vanish:
$$\int_{\partial F_\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial n} ds = \int_{\partial \Omega} \frac{\partial \varphi_\varepsilon}{\partial n} ds = 0.$$

Set
$$G_0(\Omega_\varepsilon) = W_0(\Omega_\varepsilon) \cap G(\Omega_\varepsilon)$$
and introduce the subspaces
$$W_1(\Omega_\varepsilon) = W_0(\Omega_\varepsilon) \cap G_0^\perp(\Omega_\varepsilon)$$
and
$$G_1(\Omega_\varepsilon) = G(\Omega_\varepsilon) \cap G_0^\perp(\Omega_\varepsilon),$$
where $G_0^\perp(\Omega_\varepsilon)$ is the orthogonal complement to $G_0(\Omega_\varepsilon)$ in $W(\Omega_\varepsilon)$ with respect to the inner product (3.12). It is easily seen that the subspaces $W_0(\Omega_\varepsilon)$, $G_0(\Omega_\varepsilon)$, and $W_1(\Omega_\varepsilon)$ in $W(\Omega_\varepsilon)$ are the same spaces as those introduced above, in the formulation of condition (c_3) (that is why we use for them the same notations).

Lemma 3.1. *Every $v_\varepsilon \in W(\Omega_\varepsilon)$ can be represented in the form*

$$v_\varepsilon(x) = h_\varepsilon(x) + g_\varepsilon(x) + p_\varepsilon(x), \tag{3.13}$$

where $h_\varepsilon(x) \in W_1(\Omega_\varepsilon)$, $g_\varepsilon(x) \in G_1(\Omega_\varepsilon)$, and $p_\varepsilon(x) \in G_0(\Omega_\varepsilon)$.

Proof. First, let us show that the representation (3.13) is unique. Indeed, if it is not true, then there must exist vectors $h'_\varepsilon \in W_1(\Omega_\varepsilon)$, $g'_\varepsilon \in G_1(\Omega_\varepsilon)$, and $p'_\varepsilon \in G_0(\Omega_\varepsilon)$ such that

$$h'_\varepsilon + g'_\varepsilon + p'_\varepsilon = 0, \tag{3.14}$$

where not all the terms in this equation equal zero. Since $W_1(\Omega_\varepsilon)$ and $G_1(\Omega_\varepsilon)$ are orthogonal to $G_0(\Omega_\varepsilon)$, it follows that $p'_\varepsilon = 0$ and thus $h'_\varepsilon + g'_\varepsilon = 0$. The latter equality yields $h'_\varepsilon = -g'_\varepsilon \in W_1(\Omega_\varepsilon) \cap G_1(\Omega_\varepsilon) \subset W_0(\Omega_\varepsilon) \cap G(\Omega_\varepsilon) = G_0(\Omega_\varepsilon)$ and thus $h'_\varepsilon = 0$ and $g'_\varepsilon = 0$. Therefore, all the vectors in (3.14) equal 0, and we have arrived at a contradiction. Therefore, the representation (3.13) is unique.

In order to prove (3.13), it suffices to show that $[W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon), G_0(\Omega_\varepsilon)] = W(\Omega_\varepsilon)$ (here the square brackets denote a linear span of the corresponding subspaces). From the definitions of the subspaces $W_1(\Omega_\varepsilon)$ and $G_1(\Omega_\varepsilon)$ it follows that

$$[W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon), G_0(\Omega_\varepsilon)] = [W_0(\Omega_\varepsilon), G(\Omega_\varepsilon)]$$

and thus

$$[W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon), G_0(\Omega_\varepsilon)]^\perp = [W_0(\Omega_\varepsilon), G(\Omega_\varepsilon)]^\perp. \tag{3.15}$$

By the duality principle,

$$[W_0(\Omega_\varepsilon), G(\Omega_\varepsilon)]^\perp = W_0^\perp(\Omega_\varepsilon) \cap G^\perp(\Omega_\varepsilon). \tag{3.16}$$

Now, by the definitions of $W(\Omega_\varepsilon)$ and $W_0(\Omega_\varepsilon) \subset W(\Omega_\varepsilon)$, we conclude that $W_0^\perp(\Omega_\varepsilon)$ consists of vector functions $v_\varepsilon(x)$ that satisfy in Ω_ε the differential equation

$$\operatorname{rot}\operatorname{rot} v_\varepsilon - \operatorname{grad}\operatorname{div} v_\varepsilon + \lambda^2 v_\varepsilon = 0, \qquad x \in \Omega_\varepsilon, \tag{3.17}$$

have on $\partial\Omega_\varepsilon$ a vanishing tangential component

$$n \wedge v_\varepsilon = 0, \qquad x \in \partial F_\varepsilon \cup \Omega_\varepsilon, \tag{3.18}$$

and have vanishing fluxes through each connected component of the boundary

$$\int_{\partial F_\varepsilon} (v_\varepsilon)_n ds = \int_{\partial\Omega} (v_\varepsilon)_n ds = 0. \tag{3.19}$$

In a similar way, from the definition of $G(\Omega_\varepsilon)$ it follows that $G^\perp(\Omega_\varepsilon)$ consists of vector functions $v_\varepsilon(x)$ satisfying in Ω_ε the differential equation

$$\Delta \operatorname{div} v_\varepsilon - \lambda^2 \operatorname{div} v_\varepsilon = 0$$

and the boundary condition

$$\operatorname{div} v_\varepsilon = \text{const} \tag{3.20}$$

on each connected component of $\partial\Omega_\varepsilon$.

Let $v_\varepsilon \in W_0^\perp(\Omega_\varepsilon) \cap G^\perp(\Omega_\varepsilon)$. Multiply (3.17) by $v_\varepsilon(x)$ and integrate by parts; then, using (3.18) – (3.19), we obtain

$$\int\limits_\Omega \{|\mathrm{rot}\, v_\varepsilon|^2 + (\mathrm{div}\, v_\varepsilon)^2 + \lambda^2|v_\varepsilon|^2\}dx = 0.$$

Therefore, $v_\varepsilon(x) \equiv 0$ ($x \in \Omega_\varepsilon$). Finally, by (3.16) and (3.15) we have

$$[W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon), G_0(\Omega_\varepsilon)]^\perp = 0$$

and thus

$$[W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon), G_0(\Omega_\varepsilon)] = W(\Omega_\varepsilon).$$

Lemma 3.1 is proved. □

Notice that the subspaces $W_1(\Omega_\varepsilon)$ and $G_1(\Omega_\varepsilon)$ are orthogonal to the subspace $G_0(\Omega_\varepsilon)$ (with respect to the inner product (3.12)) but do not orthogonal to one another. However, if condition (c_3) is satisfied, then the mutual slope of these subspaces is bounded form below uniformly with respect to ε. More precisely, the following lemma holds true.

Lemma 3.2. *Let condition* (c_3) *be satisfied. Then, for all* $v_\varepsilon \in W_1(\Omega_\varepsilon)$ *and all* $u_\varepsilon \in G_\varepsilon(\Omega_\varepsilon)$*, the following inequality holds true:*

$$|(u_\varepsilon, v_\varepsilon)_\varepsilon| \le \alpha \|u_\varepsilon\|_\varepsilon \|v_\varepsilon\|_\varepsilon,$$

where the constant α *is independent of* ε *and* $\alpha < 1$.

Proof. Since $u_\varepsilon \in G_1(\Omega_\varepsilon)$ and thus $\mathrm{rot}\, u_\varepsilon = 0$, by the Schwartz inequality we obtain

$$\begin{aligned} |(u_\varepsilon, v_\varepsilon)_\varepsilon| &= \left| \int\limits_{\Omega_\varepsilon} \{\mathrm{div}\, u_\varepsilon, \mathrm{div}\, v_\varepsilon + \lambda^2(u_\varepsilon, v_\varepsilon)\}dx \right| \\ &\le \left\{ \int\limits_\Omega \left[(\mathrm{div}\, u_\varepsilon)^2 + \lambda^2|u_\varepsilon|^2\right] dx \right\}^{1/2} \left\{ \int\limits_\Omega \left[(\mathrm{div}\, v_\varepsilon)^2 + \lambda^2|v_\varepsilon|^2\right] dx \right\}^{1/2} \\ &= \|u_\varepsilon\|_\varepsilon \left\{ \int\limits_\Omega \left[(\mathrm{div}\, v_\varepsilon)^2 + \lambda^2|v_\varepsilon|^2\right] dx \right\}^{1/2}. \end{aligned} \tag{3.21}$$

By using condition (c_3) for $v_\varepsilon \in W_1(\Omega_\varepsilon)$, we can write

$$\int\limits_{\Omega_\varepsilon} \left\{|\mathrm{rot}\, v_\varepsilon|^2 + (\mathrm{div}\, v_\varepsilon)^2 + \lambda^2|v_\varepsilon|^2\right\} dx = \|v_\varepsilon\|_\varepsilon^2 \le C \int\limits_{\Omega_\varepsilon} |\mathrm{rot}\, v_\varepsilon|^2 dx,$$

where C is independent of ε and $C > 1$. This implies that

$$\int\limits_{\Omega_\varepsilon} \left[(\operatorname{div} v_\varepsilon)^2 + \lambda^2 |v_\varepsilon|^2\right] dx \leq \frac{C-1}{C} \|v_\varepsilon\|_\varepsilon^2. \tag{3.22}$$

Now (3.21) and (3.22) yield the sought inequality, with the constant $\alpha = \left(\frac{C-1}{C}\right)^{1/2} < 1$ independent of ε. Lemma 3.2 is proved. □

Lemma 3.2 allows us to make the assertion of Lemma 3.1 stronger. Namely, the following lemma holds true.

Lemma 3.3. *Every $v_\varepsilon(x) \in W(\Omega_\varepsilon)$ can be uniquely represented in the form*

$$V_\varepsilon(x) = h_\varepsilon(x) + g_\varepsilon(x) + p_\varepsilon(x),$$

where $p_\varepsilon = P_{0\varepsilon} v_\varepsilon \in G_0(\Omega_\varepsilon)$, $h_\varepsilon \in W_1(\Omega_\varepsilon)$, and $g_\varepsilon \in G_1(\Omega_\varepsilon)$, with

$$\|p_\varepsilon\|_\varepsilon \leq \|v_\varepsilon\|, \qquad \|h_\varepsilon\|_\varepsilon \leq C\|v_\varepsilon\|_\varepsilon, \qquad \|g_\varepsilon\|_\varepsilon \leq C\|v_\varepsilon\|_\varepsilon. \tag{3.23}$$

Here $P_{0\varepsilon}$ is the operator of the orthogonal projection on $G_0(\Omega_\varepsilon) \subset W(\Omega_\varepsilon)$ and the constant C is independent of ε.

Proof. By the definition of the subspaces $G_0(\Omega_\varepsilon)$, $W_1(\Omega_\varepsilon)$, and $G_1(\Omega_\varepsilon)$ in $W(\Omega_\varepsilon)$ and Lemma 3.1, we can write the following orthogonal decomposition:

$$W(\Omega_\varepsilon) = [W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon)] \oplus G_0(\Omega_\varepsilon).$$

Therefore, in (3.13) we have $p_\varepsilon = P_{0\varepsilon} v_\varepsilon$ and $h_\varepsilon + g_\varepsilon \in [W_1(\Omega_\varepsilon), G_1(\Omega_\varepsilon)]$, with

$$\|p_\varepsilon\|_\varepsilon \leq \|v_\varepsilon\|_\varepsilon, \qquad \|h_\varepsilon + g_\varepsilon\|_\varepsilon \leq \|v_\varepsilon\|_\varepsilon. \tag{3.24}$$

Hence, we have established the first inequality in (3.23). In order to prove the other two, we use the second inequality in (3.24) and Lemma 3.2; in this way we obtain

$$\begin{aligned}
\|v_\varepsilon\|_\varepsilon^2 \geq \|h_\varepsilon + g_\varepsilon\|_\varepsilon^2 &= \|h_\varepsilon\|_\varepsilon^2 + \|g_\varepsilon\|_\varepsilon^2 + 2(h_\varepsilon, g_\varepsilon)_\varepsilon \\
&\geq \|h_\varepsilon\|_\varepsilon^2 + \|g_\varepsilon\|_\varepsilon^2 - 2|(h_\varepsilon, g_\varepsilon)_\varepsilon| \geq \|h_\varepsilon\|_\varepsilon^2 + \|g_\varepsilon\|_\varepsilon^2 - 2\alpha\|h_\varepsilon\|_\varepsilon \|g_\varepsilon\|_\varepsilon \\
&= (\|h_\varepsilon\|_\varepsilon - \|g_\varepsilon\|_\varepsilon)^2 + 2(\alpha - 1)\|h_\varepsilon\|_\varepsilon \|g_\varepsilon\|_\varepsilon.
\end{aligned}$$

This implies the inequalities

$$2\|h_\varepsilon\|_\varepsilon \|g_\varepsilon\|_\varepsilon \leq \frac{1}{(1-\alpha)} \|v_\varepsilon\|_\varepsilon^2$$

and

$$\|h_\varepsilon\|_\varepsilon^2 + \|g_\varepsilon\|_\varepsilon^2 \leq \|v_\varepsilon\|_\varepsilon^2 + 2\|h_\varepsilon\|_\varepsilon \|g_\varepsilon\|_\varepsilon^2$$

which in turn yield the remaining two inequalities in (3.23), with the constant $C = \left(\frac{2-\alpha}{1-\alpha}\right)^{1/2}$ independent of ε. Lemma 3.3 is proved. □

Now return to the solution $E_\varepsilon(x,\lambda)$ of the variational problem (3.11) or, equivalently, the boundary value problem (3.5)–(3.6) for $\lambda > 0$. Since (by (3.11)) $\Phi_\varepsilon[E_\varepsilon] \le \Phi_\varepsilon[0] = 0$, using (3.10) and (3.7) gives

$$\int\limits_{\Omega_\varepsilon} \{|\mathrm{rot}\, E_\varepsilon|^2 + \lambda^2 |E_\varepsilon|^2\} dx < C_1.$$

From (3.5) and (3.7) it follows that

$$\int\limits_{\Omega_\varepsilon} (\mathrm{div}\, E_\varepsilon)^2 dx < C_2,$$

where C_1 and C_2 are independent of ε.

These two inequalities mean that $E_\varepsilon = E_\varepsilon(x,\lambda)$ belongs to the space $W(\Omega_\varepsilon)$ and that the norm of E_ε is bounded uniformly with respect to ε:

$$\|E_\varepsilon\|_\varepsilon < C.$$

By Lemma 3.3, E_ε can be uniquely represented in the form

$$E_\varepsilon(x) = u_\varepsilon(x) + \mathrm{grad}\, \varphi_\varepsilon(x), \tag{3.25}$$

where $u_\varepsilon(x) \in W_1(\Omega_\varepsilon)$ and $\varphi_\varepsilon(x)$ is a function in $W_2^2(\Omega_\varepsilon)$ equal to constants on ∂F_ε and $\partial\Omega$ and having vanishing fluxes through ∂F_ε and $\partial\Omega$ (i.e., $\mathrm{grad}\, \varphi_\varepsilon \in G_1(\Omega_\varepsilon) \oplus G_0(\Omega_\varepsilon) = G(\Omega_\varepsilon)$); moreover, the following estimates hold:

$$\int\limits_{\Omega_\varepsilon} \{|\mathrm{rot}\, u_\varepsilon|^2 + (\mathrm{div}\, u_\varepsilon)^2 + \lambda^2 u_\varepsilon^2\} dx < C, \tag{3.26}$$

$$\int\limits_{\Omega_\varepsilon} \{(\Delta\varphi_\varepsilon)^2 + \lambda^2 |\mathrm{grad}\, \varphi_\varepsilon|^2\} dx < C, \tag{3.27}$$

with C independent of ε. Without loss of generality, we may assume that $\varphi_\varepsilon(x)$ equals zero on $\partial\Omega$.

From (3.25) and (3.5)–(3.6) it follows that the vector function $u_\varepsilon(x)$ and the function $\varphi_\varepsilon(x)$, being considered together, solve the following boundary value problem (I)–(II) in Ω_ε:

$$\left.\begin{array}{lr} \mathrm{rot}\,\mathrm{rot}\, u_\varepsilon + \lambda^2 u_\varepsilon = J^0 - \lambda^2 \mathrm{grad}\, \varphi_\varepsilon, & x \in \Omega_\varepsilon \\ n \wedge u_\varepsilon = 0, & x \in \partial\Omega_\varepsilon \end{array}\right\} \tag{I}$$

$$\left.\begin{array}{ll}\Delta\varphi_\varepsilon = \frac{1}{\lambda^2}\operatorname{div} J^0 - \operatorname{div} u_\varepsilon, & x \in \Omega_\varepsilon \\[2ex] \varphi_\varepsilon = \text{const}(= A_\varepsilon), & x \in \partial F_\varepsilon \\[2ex] \varphi_\varepsilon = 0, & x \in \partial\Omega \\[2ex] \int\limits_{\partial F_\varepsilon} \frac{\partial \varphi_\varepsilon}{\partial n}\, ds = 0. & \end{array}\right\} \qquad (II)$$

Due to the calibration $u_\varepsilon \to u_\varepsilon + \nabla\varphi'_\varepsilon$, $\varphi_\varepsilon \to \varphi_\varepsilon - \varphi'_\varepsilon$, with $\varphi'_\varepsilon \in G_0(\Omega_\varepsilon)$, the boundary value problem (I)–(II) has many solutions, but we can single out the unique solution $\{u_\varepsilon, \varphi_\varepsilon\}$ specified by the representation (3.25). Our aim is to study the asymptotic behavior of this solution as $\varepsilon \to 0$.

Let us extend $u_\varepsilon(x) \in W_1(\Omega_\varepsilon) = (\overset{\circ}{W}{}_2^1(\Omega_\varepsilon))^3$ on F_ε by zero and let us extend $\varphi_\varepsilon(x)$ on F_ε by the constants A_ε. Then the inequality (3.26) implies that the set of extended vector functions $\{u_\varepsilon(x) \in (\overset{\circ}{W}{}_2^1(\Omega))^3$, $\varepsilon > 0\}$ is weakly compact in $(\overset{\circ}{W}{}_2^1(\Omega))^3$. Since $\varphi_\varepsilon(x) = 0$ on $\partial\Omega$, from (3.27) it follows that the set of extended functions $\{\varphi_\varepsilon(x) \in \overset{\circ}{W}{}_2^1(\Omega),\ \varepsilon > 0\}$ is weakly compact in $\overset{\circ}{W}{}_2^1(\Omega)$. Extract a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$ such that the corresponding subsequences of vector functions $\{u_{\varepsilon_k}(x) \in (\overset{\circ}{W}{}_2^1(\Omega))^3\}$ and of functions $\{\varphi_{\varepsilon_k}(x) \in \overset{\circ}{W}{}_2^1(\Omega)\}$ converge weakly to $u(x) \in (\overset{\circ}{W}{}_2^1(\Omega))^3$ and $\varphi(x) \in \overset{\circ}{W}{}_2^1(\Omega)$, respectively.

In order to describe the limiting pair $\{u(x), \varphi(x)\}$, we proceed as follows. Consider the boundary value problem (I) with respect to $u_\varepsilon(x)$, assuming that $\varphi_\varepsilon(x)$ is known. Then the following theorem holds.

Theorem 3.1. *Assume that conditions (c_1) and (c_2) are satisfied. Then solutions $u_\varepsilon(x)$ of problem (I) (extended by zero on F_ε) converge weakly in $(\overset{\circ}{W}{}_2^1(\Omega))^3$, on a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$, to a vector function $u(x)$, which solves the following boundary value problem:*

$$\left.\begin{array}{lr}\operatorname{rot}\operatorname{rot} u + \lambda^2 u + \mathbf{L}(x)u = J^0 - \lambda^2 \operatorname{grad}\varphi, & x \in \Omega, \\[2ex] n \wedge u_\varepsilon = 0, & x \in \partial\Omega, \end{array}\right\} \qquad (I')$$

where $\varphi(x)$ is the weak limit of $\varphi_\varepsilon(x)$ on the same subsequence and $\mathbf{L} = \{\mathbf{L}_{ik}(x)\}_{i,k=1}^3$ is the limiting tensor in condition (c_2) extended by zero outside $G \subset \Omega$.

The proof of this theorem will be given in the next section.

Now consider the boundary value problem (II) with respect to the function $\varphi_\varepsilon(x)$, assuming that the vector function $u_\varepsilon(x)$ is known.

Theorem 3.2. *Assume that condition* (c_1) *is satisfied. Then solutions* $\varphi_\varepsilon(x)$ *of problem* (II) *(extended on* F_ε *by the constants* A_ε*) converge weakly in* $\overset{\circ}{W}{}_2^1(\Omega)$*, on a subsequence* $\{\varepsilon = \varepsilon_k \to 0\}$*, to a function* $\varphi(x)$*, which solves the boundary value problem*

$$\left.\begin{aligned} \Delta\varphi - C(x)\left[\varphi(x) - \left(\int_\Omega C(x)dx\right)^{-1}\int_\Omega C(x)\varphi(x)dx\right]& \\ = \frac{1}{\lambda^2}\operatorname{div} J^0 - \operatorname{div} u, \quad x \in \Omega,& \\ \varphi(x) = 0, \quad x \in \partial\Omega,& \end{aligned}\right\} \qquad (II')$$

where $u(x)$ *is the weak limit of* $u_\varepsilon(x)$ *on the same subsequence and* $C(x)$ *is the limiting function in condition* (c_1) *extended by zero outside* $G \subset \Omega$*.*

The proof of this theorem is, in fact, given in [5]; only minor modifications are required, so we will not dwell on this.

By using Theorems 3.1 and 3.2, it is easy now to obtain the main result concerning solutions $E_\varepsilon(x,\lambda)$ of problem (3.5)–(3.6). These solutions are assumed to be extended, with respect to $x \in \Omega$, by zero on $F_\varepsilon \subset \Omega$; we will keep the same notation E_ε for these extensions.

Theorem 3.3. *Assume that conditions* (c_1)–(c_3) *are satisfied. Then* $E_\varepsilon(x,\lambda)$ *converge weakly in* $(L_2(\Omega))^3$ *to a vector function* $E(x,\lambda)$*, which, being considered together with* $\Phi(x,\lambda)$*, solves the following boundary value problem:*

$$\operatorname{rot}\operatorname{rot} E + \lambda^2(E - \operatorname{grad}\Phi) = J^0, \qquad x \in \Omega, \tag{3.28}$$

$$n \wedge E = 0, \qquad x \in \partial\Omega, \tag{3.29}$$

$$\operatorname{div}(\mathbf{L}\operatorname{grad}\Phi) - \lambda^2 C\Phi = \operatorname{div}(\mathbf{L}E), \qquad x \in G, \tag{3.30}$$

$$(\mathbf{L}\operatorname{grad}\Phi)_n = (\mathbf{L}E)_n, \qquad x \in \partial G, \tag{3.31}$$

Proof. From Theorems 3.1 and 3.2 and the representation (3.25) it follows that, for $\lambda > 0$, $E_\varepsilon(x,\lambda)$ converge weakly in $(L_2(\Omega))^3$, by a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$, to a vector function

$$E(x,\lambda) = u(x,\lambda) + \operatorname{grad}\varphi(x,\lambda) \in (W_2^1(\Omega))^3.$$

Introduce in the subdomain $G \subset \Omega$ (outside of which $C(x) \equiv 0$ and $\mathbf{L}(x) \equiv 0$) the function

$$\Phi(x, \lambda) = \varphi(x, \lambda) - \left(\int_\Omega C(x)\, dx \right)^{-1} \int_\Omega C(x) \varphi(x)\, dx \in W_2^1(G).$$

Then, by (I') and (II'), we see that $E(x, \lambda)$ and $\Phi(x, \lambda)$ have to satisfy the differential equations (3.28) and (3.30) and the boundary conditions (3.29) and (3.31).

Let us show that $E_\varepsilon(x, \lambda)$ converge weakly in $(L_2(\Omega))^3$ to $E(x, \lambda)$ as $\varepsilon \to 0$ (and not only on a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$). Notice, first of all, that the set $\{E_\varepsilon(x, \lambda), \varepsilon > 0\}$ is bounded and, therefore, is weakly compact in $(L_2(\Omega))^3$. Hence, it suffices to show that the boundary value problem (3.22)–(3.25), describing the limits on subsequences, has a unique solution.

Consider the corresponding homogeneous system of equations (i.e., the system (3.28)–(3.31) with $J^0 = 0$). Multiply the homogeneous analogue of (3.28) by E (in the sense of the inner product in $\mathbb{R}^3$) and integrate the resulting equation over Ω. Then, integration by parts gives (taking into account (3.29) and the fact that the tensor $\mathbf{L}$ is symmetric)

$$\int_\Omega \{(\operatorname{rot} E)^2 + \lambda^2 (E, E) + (\mathbf{L}E, E) - (\mathbf{L}E, \operatorname{grad} \Phi)\} dx = 0. \tag{3.32}$$

Now multiply (3.28) by Φ and integrate the result over G. Then, integrating by parts and using the boundary condition (3.31), we obtain

$$\int_G \{(\mathbf{L} \operatorname{grad} \Phi, \operatorname{grad} \Phi) + \lambda^2 C \Phi^2 - (\mathbf{L}E, \operatorname{grad} \Phi)\} dx = 0. \tag{3.33}$$

Since $\mathbf{L}$ is symmetric and nonnegative, it follow that

$$2|(\mathbf{L}E, \operatorname{grad} \Phi)| \leq (\mathbf{L}E, E) + (\mathbf{L} \operatorname{grad} \Phi, \operatorname{grad} \Phi).$$

Using this inequality and the fact that $\mathbf{L}(x) = 0$ and $C(x) = 0$ outside G, by (3.32) and (3.33) we conclude that

$$\int_\Omega \{|\operatorname{rot} E|^2 + \lambda^2 |E|^2 + \lambda^2 C \Phi^2\} dx \leq 0.$$

This implies that $E(x) = 0$ for $x \in \Omega$ and, since $C(x) > 0$ for $x \in G$, $\Phi(x) = 0$ for $x \in G$. We have shown that the homogeneous problem has only trivial solution; hence, the solution of problem (3.28)–(3.31) is unique. This completes the proof of Theorem 3.3 for $\lambda > 0$. $\square$

Now consider problem (3.5)–(3.6) for complex λ. Using standard technique of the perturbation theory for the operator equations (see, e.g., [6]), it is easy to show that if $\arg\lambda^2 \neq \pi$, then this problem has the unique solution $E_\varepsilon(x,\lambda)$, which is an analytic function of λ in the half-plane $\mathrm{Re}\lambda > 0$.

Multiply (3.5) by $\overline{E_\varepsilon(x,\lambda)}$ and integrate over Ω_ε. Then, integrating by parts, using (3.6), and separating the real and imaginary parts, we obtain two equations:

$$(\mu^2-\nu^2)\int\limits_{\Omega_\varepsilon}|E_\varepsilon|^2dx+\int\limits_{\Omega_\varepsilon}|\mathrm{rot}\,E_\varepsilon|^2dx$$
$$=\mu\mathrm{Re}\int\limits_{\Omega_\varepsilon}(g_1,\overline{E_\varepsilon})\,dx-\nu\mathrm{Im}\int\limits_{\Omega_\varepsilon}(g_1,\overline{E_\varepsilon})\,dx+\mathrm{Re}\int\limits_{\Omega_\varepsilon}(g_2,\overline{E_\varepsilon})\,dx,$$

$$2\mu\nu\int\limits_{\Omega_\varepsilon}|E_\varepsilon|^2dx=\mu\mathrm{Im}\int\limits_{\Omega_\varepsilon}(g_1,\overline{E_\varepsilon})\,dx+\nu\mathrm{Re}\int\limits_{\Omega_\varepsilon}(g_1,\overline{E_\varepsilon})\,dx+\mathrm{Im}\int\limits_{\Omega_\varepsilon}(g_2,\overline{E_\varepsilon})\,dx,$$

where the following notations have been introduced: $\lambda=\mu+i\nu$, $g_1 = J(x,\lambda)-E^0(x)$, and $g_2=\mathrm{rot}\,H^0(x)$.

These equalities yield the following estimates for $E_\varepsilon(x.\lambda)$ (which are uniform with respect to ε):

$$\int\limits_{\Omega_\varepsilon}|E_\varepsilon(x,\lambda)|^2dx<\frac{C_1}{\mathrm{Re}\lambda} \tag{3.34}$$

and

$$\int\limits_{\Omega_\varepsilon}|\mathrm{rot}\,E_\varepsilon|^2dx<C_2, \tag{3.35}$$

where $\mathrm{Re}\,\lambda\geq\mu_0>0$. In a similar way one can show that if $\arg\lambda^2\neq\pi$, then problem (3.28)–(3.31) has the unique solution $(E(x,\lambda),\Phi(x,\lambda))$, which is analytic with respect to λ in the half-plane $\mathrm{Re}\,\lambda>0$, and that the following inequality holds true:

$$\int\limits_{\Omega}|E(x,\lambda)|^2dx<\frac{C}{\mathrm{Re}\lambda}. \tag{3.36}$$

Now, since the convergence of $E_\varepsilon(x,\lambda)$ to $E(x,\lambda)$ for $\lambda>0$ has already been proved, by the estimate (3.34) and the Vitalli theorem [9] we conclude that $E_\varepsilon(x,\lambda)$ converge to $E(x,\lambda)$ as $\varepsilon\to 0$ in the whole right half-plane λ ($\mathrm{Re}\,\lambda>0$) uniformly in every its compact subdomain. Theorem 3.3 is proved.

4. Proof of Theorem 3.1

First, we give the brief scheme of the proof based on the variational principle. Namely, for all $\lambda > 0$ and the given $J^0(x, \lambda) \in (L_2(\Omega_\varepsilon))^3$ and $\varphi_\varepsilon(x, \lambda) \in W_2^1(\Omega)$, the solution $u_\varepsilon(x, \lambda)$ of problem (I) minimizes the functional

$$J_\varepsilon[w_\varepsilon] = \int_\Omega \{\mathrm{rot}\, w_\varepsilon|^2 + \lambda^2|w_\varepsilon|^2 + 2(J^0 - \lambda^2 \mathrm{grad}\, \varphi_\varepsilon, w_\varepsilon)\}\, dx \tag{4.1}$$

in the class $R_0(\Omega, F_\varepsilon) = \{w_\varepsilon(x) : \Omega \to \mathbb{R}^3;\ w_\varepsilon(x) \equiv 0,\ x \in F_\varepsilon;\ n \wedge w_\varepsilon = 0,\ x \in \partial\Omega_\varepsilon;\ w_\varepsilon(x) \in (L_2(\Omega))^3;\ \mathrm{rot}\, w_\varepsilon \in (L_2(\Omega))^3\}$. This means that for all $w_\varepsilon(x) \in R_0(\Omega, F_\varepsilon)$, the following inequality holds true:

$$J_\varepsilon[u_\varepsilon] \le J_\varepsilon[w_\varepsilon]. \tag{4.2}$$

In Section 4.2 we will construct special test vector functions $w_{\varepsilon h}(x)$ (which depend on $w(x) \in (C_0^2(\Omega))^3$, the problem parameter $\varepsilon > 0$, and an auxiliary parameter $h > 0$) and show that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon=\varepsilon_k\to 0}} J_\varepsilon[w_{\varepsilon h}] \le J[w], \tag{4.3}$$

where the functional $J[w]$ is defined by

$$J[w] = \int_\Omega \{|\mathrm{rot}\, w|^2 + \lambda^2|w|^2 + (\mathbf{L}w, w) + 2(J^0 + \lambda\, \mathrm{grad}\, \varphi, w)\}\, dx. \tag{4.4}$$

Here $\varphi(x)$ is the weak limit in $W_2^1(\Omega)$ of the functions $\varphi_\varepsilon(x)$ on a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$, and the tensor $\mathbf{L} = \mathbf{L}(x)$ is defined in condition (c_2).

From (4.2) and (4.3) it follows that

$$\overline{\lim_{\varepsilon=\varepsilon_k\to 0}} J_\varepsilon[u_\varepsilon] \le J[w], \tag{4.5}$$

for all $w(x) \in (C_0^2(\Omega))^3$. Since $(C_0^2(\Omega))^3$ is dense in $(\overset{\circ}{W}{}_2^1(\Omega))^3$, this inequality holds true for all $w \in (\overset{\circ}{W}{}_2^1(\Omega))^3$.

In Section 4.3, by using the fact that $u_\varepsilon(x, \lambda)$ converge to $u(x, \lambda)$ weakly in $(\overset{\circ}{W}{}_2^1(\Omega))^3$ on a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$, we will establish the "converse" inequality

$$\underline{\lim}_{\varepsilon=\varepsilon_k\to 0} J_\varepsilon[u_\varepsilon] \ge J[u]. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$J[u] \leq J[w] \qquad \text{for all } w \in (\overset{\circ}{W}{}_2^1(\Omega))^3;$$

hence, $u \in (\overset{\circ}{W}{}_2^1(\Omega))^3$ minimizes the functional (4.4) in the class $(\overset{\circ}{W}{}_2^1(\Omega))^3$. This, by standard arguments, implies that $u(x,\lambda)$ is a (generalized) solution of problem (I').

To implement this scheme, we will need special ("coordinate") vector functions $v^i_{\varepsilon h}(x)$ $(i = 1,2,3)$ satisfying certain estimates.

4.1. Coordinate Vector Functions $v^{ix}_{\varepsilon h}(\xi)$.

Let $v^i_{\varepsilon h} \equiv v^i_{\varepsilon h}(\xi)$, $i = 1,2,3$, be vector functions giving the infimum in (2.3) when l coincides with the unit vector e^i of the coordinate axis x_i. It is easily seen that for an arbitrary $l = \{l_1, l_2, l_3\} \in \mathbb{R}^3$, the minimizer $v^{l(x)}_{\varepsilon h}(\xi)$ of (2.3) is a (generalized) solution of the following boundary value problem in $\Omega_\varepsilon \cap K(x,h)$:

$$\operatorname{rot}\operatorname{rot} v^{l(x)}_{\varepsilon h}(\xi) + h^{-2-\gamma} v^{l(x)}_{\varepsilon h}(\xi) = h^{-2-\gamma} l, \qquad \xi \in \Omega_\varepsilon \cap K(x,h),$$

$$n \wedge v^{l(x)}_{\varepsilon h}(\xi) = 0, \qquad \xi \in \partial F_\varepsilon \cap K(x,h) \cup \partial K(x,h),$$

$$v^{l(x)}_{\varepsilon h}(\xi) = 0, \qquad x \in F_\varepsilon \cap K(x,h).$$

Since this problem is linear, it follows that

$$v^{l(x)}_{\varepsilon h}(\xi) = \sum_{i=1}^{3} l_i v^{ix}_{\varepsilon h}(\xi),$$

from which, by (2.3), we obtain the representation (2.4) with $\mathbf{L}_{ij}(x,h,\varepsilon)$ defined by

$$\mathbf{L}_{ij}(x,h,\varepsilon) = \int\limits_{K(x,h)} \{(\operatorname{rot} v^{ix}_{\varepsilon h}, \operatorname{rot} v^{jx}_{\varepsilon h}) + h^{-2-\gamma}(v^{ix}_{\varepsilon h} - e^i, v^{jx}_{\varepsilon h} - e^j)\}\, dx, \quad (4.7)$$

$i,j = 1,2,3$.

Let us derive some estimates for $v^{ix}_{\varepsilon h}(\xi)$, which will be used in what follows.

Lemma 4.1. *Let $K^x_{h_1} = K(x,h_1)$ be a cube of size $h_1 = h - 2r$ $(r = o(h))$ centered at $x \in \Omega$ and oriented as the cube $K(x,h) = K^x_h$.*

For sufficiently small $\varepsilon < \hat{\varepsilon}(h,x)$, the following estimates hold true as $h \to 0$:

$$\int_{K_h^x} |\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 d\xi = O(h^3), \tag{4.8}$$

$$\int_{K_h^x} |v_{\varepsilon h}^{ix} - e^i|^2 d\xi = O(h^{5+\gamma}), \tag{4.9}$$

$$\int_{(K_h^x \setminus K_{h_1}^x)} |\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 d\xi = o(h^3), \tag{4.10}$$

$$\int_{(K_h^x \setminus K_{h_1}^x)} |v_{\varepsilon h}^{ix} - e^i|^2 d\xi = o(h^{5+\gamma}). \tag{4.11}$$

Moreover, if condition (c_2) *is satisfied uniformly with respect to $x \in G$, then these estimates are also satisfied uniformly with respect to x (i.e., for $\varepsilon < \hat{\varepsilon}(h)$ for all x).*

Proof. The estimates (4.8) and (4.9) follow immediately from condition (c_2). By (4.9), we have

$$\begin{aligned}
&\int_{(K_h^x \setminus K_{h_1}^x)} \{|\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 + h^{-2-\gamma}|v_{\varepsilon h}^{ix} - e^i|^2\}\, d\xi \\
&\qquad = \int_{K_h^x} \{|\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 + h^{-2-\gamma}|v_{\varepsilon h}^{ix} - e^i|^2\}\, d\xi \\
&\qquad - \int_{K_{h_1}^x} \{|\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 + h_1^{-2-\gamma}|v_{\varepsilon h}^{ix} - e^i|^2\}\, d\xi + O(rh^2).
\end{aligned} \tag{4.12}$$

According to (2.3), the first term in the right-hand side of (4.12) equals $L(x,h,\varepsilon,e^i)$, and the second term is not less than $L(x,h_1,\varepsilon,e^i)$. Hence,

$$\begin{aligned}
&\int_{(K_h^x \setminus K_{h_1}^x)} \{|\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 + h^{-2-\gamma}|v_{\varepsilon h}^{ix} - e^i|^2\}\, d\xi \\
&\qquad\qquad \le L(x,h,\varepsilon,e^i) - L(x,h_1,\varepsilon,e^i) + O(rh^2),
\end{aligned}$$

from which, taking into account condition (c_2) and the fact that $h_1 = h - 2r$ with $r = o(h)$, we obtain the estimate

$$\int_{(K_h^x \setminus K_{h_1}^x)} \{|\operatorname{rot} v_{\varepsilon h}^{ix}(\xi)|^2 + h^{-2-\gamma}|v_{\varepsilon h}^{ix} - e^i|^2\}\, d\xi = o(h^3), \quad h \to 0,\ \varepsilon < \hat{\varepsilon}(h,x).$$

Finally, this estimate yields the required estimates (4.10) and (4.11), which completes the proof of the lemma. □

4.2. Construction of Test Vector Functions $w_{\varepsilon h}$ and Proof of Inequality (4.3).

Consider the covering of Ω by cubes $K_h^\alpha = K(x^\alpha, h)$ of size $h>0$ centered at points x^α forming a cubic lattice of period $h-r$ with $r = h^{1+\gamma/2} = o(h)$ and oriented along the coordinate axes. Construct a partition of unity associated with this covering, i.e., a system of functions $\{\varphi_\alpha(x)\}$ satisfying the following conditions: $\varphi_\alpha(x) \in C_0^2(\mathbb{R}^3)$; $0 \le \varphi_\alpha(x) \le 1$ and $\sum_\alpha \varphi_\alpha = 1$; $\varphi_\alpha(x) = 1$ for $x \in K_h^\alpha \setminus \bigcup_{\beta\neq\alpha} K_h^\beta$; $\varphi_\alpha(x) = 0$ for $x \notin K_h^\alpha$; $|\nabla\varphi_\alpha(x)| \le Cr^{-1}$.

Let $w(x)$ be an arbitrary vector function with components $w_i(x) \in C_0^2(\Omega)$ $(i = 1, 2, 3)$. Set

$$w_{\varepsilon h}(x) = \sum_\alpha \sum_i w_i(x) v_{\varepsilon h}^{i\alpha}(x)\varphi_\alpha(x), \tag{4.13}$$

where $v_{\varepsilon h}^{i\alpha}(x)$ denote "coordinate" vector functions of cubes K_h^α introduced in Section 4.1.

Taking into account specific properties of the vector functions $v_{\varepsilon h}^{i\alpha}(x)$ and the functions $\{\varphi_\alpha(x)\}$, it is easy to see that $w_{\varepsilon h}(x)$ belongs to $R_0(\Omega, F_\varepsilon)$ and thus can be taken as a test vector function for the estimation of the functional (4.1).

Let us estimate $J_\varepsilon(w_{\varepsilon h})$. First, we notice that, by the properties of the partition of unity, formula (4.13) can be transformed into the form

$$w_{\varepsilon h}(x) = w(x) + \sum_\alpha \sum_{i=1}^3 w_i(x)[v_{\varepsilon h}^{i\alpha}(x) - e^i]\varphi_\alpha(x), \tag{4.14}$$

which yields

$$\begin{aligned}\operatorname{rot} w_{\varepsilon h}(x) = \operatorname{rot} w(x) &+ \sum_\alpha \sum_{i=1}^3 w_i(x) \operatorname{rot} v_{\varepsilon h}^{i\alpha}(x)\varphi_\alpha(x)\\ &+ \sum_\alpha \sum_{i=1}^3 \operatorname{grad} w_i(x) \wedge [v_{\varepsilon h}^{i\alpha}(x) - e^i]\varphi_\alpha(x)\\ &+ \sum_\alpha \sum_{i=1}^3 w_i(x)[v_{\varepsilon h}^{i\alpha}(x) - e^i] \wedge \operatorname{grad}\varphi_\alpha(x). \end{aligned}\tag{4.15}$$

The second term in the right-hand side of (4.14) (the sum with respect to α), as well as the third and the fourth terms in the right-hand side of (4.15) give a vanishing (as $\varepsilon \to 0$ and $h \to 0$) contribution to the functional (4.1). Indeed, setting in (4.1) $w_\varepsilon = w_{\varepsilon h}(x)$, by (4.14), (4.15), and the estimates (4.9) and (4.11) we obtain

$$J_\varepsilon[w_{\varepsilon h}] = \int_\Omega \{|\mathrm{rot}\, w|^2 + \lambda^2|w|^2 + 2(J^0 + \lambda^2 \mathrm{grad}\, \varphi, w)\} dx$$
$$+ I_{\varepsilon h}^{(1)} + I_{\varepsilon h}^{(2)} + O(h^{2+\gamma}) + o(r^{-2}h^{2+\gamma}) \quad (4.16)$$

for $\varepsilon = \varepsilon_k \leq \hat{\varepsilon}(h)$ and $h \to 0$, where

$$I_{\varepsilon h}^{(1)} = \sum_{\alpha,\beta}\sum_{i,j} \int_{K_h^\alpha \cap K_h^\beta} w_i w_j \varphi_\alpha \varphi_\beta \,\mathrm{rot}\, v_{\varepsilon h}^{i\alpha}\, \mathrm{rot}\, v_{\varepsilon h}^{j\beta}\, dx,$$

$$I_{\varepsilon h}^{(2)} = \int_\Omega \left(\mathrm{rot}\, w, \sum_\alpha \sum_{i=1}^3 w_i \varphi_\alpha \,\mathrm{rot}\, v_{\varepsilon h}^{i\alpha}\right) dx.$$

We have taken into account also that $w \in (C_0^2(\Omega))^3$ and that $\mathrm{grad}\, \varphi_\varepsilon$ converge weakly in $(W_2^1(\Omega))^3$, on a subsequence $\varepsilon = \varepsilon_k \to 0$, to $\mathrm{grad}\, \varphi$.

The term $I_{\varepsilon h}^{(2)}$ can be transformed (by integration by parts) to the form

$$I_{\varepsilon h}^{(2)} = \int_\Omega \left(\mathrm{rot}\, w, \sum_\alpha \sum_{i=1}^3 w_i \varphi_\alpha \,\mathrm{rot}\, v_{\varepsilon h}^{i\alpha}\right) dx$$
$$= \sum_\alpha \sum_{i=1}^3 \int_{K_h^\alpha} (\mathrm{rot}\,\mathrm{rot}\, w + \mathrm{rot}\, w \wedge \mathrm{grad}\,(w_i\varphi_\alpha), [v_{\varepsilon h}^{i\alpha} - e^i])\, dx.$$

This, by the estimates (4.9) and (4.11), gives

$$I_{\varepsilon h}^{(2)} = O(h^{2+\gamma}) + o(r^{-2}h^{2+\gamma}). \quad (4.17)$$

Now consider the term $I_{\varepsilon h}^{(1)}$. As above, one can transform this term (using the construction of the partition of unity, the smoothness of $w(x)$, and the estimates (4.8) and (4.10)) to the form

$$I_{\varepsilon h}^{(1)} = \sum_\alpha \sum_{i=1}^3 w_i(x^\alpha) w_j(x^\alpha) \int_{K_h^\alpha} (\mathrm{rot}\, v_{\varepsilon h}^{i\alpha}, \mathrm{rot}\, v_{\varepsilon h}^{j\alpha})\, dx + o(1), \qquad h \to 0,$$

which, by the definition of the tensor $\mathbf{L} = \{\mathbf{L}_{ij}(x,h,\varepsilon)\}_{i,j=1}^3$ (see (4.7)), gives

$$I_{\varepsilon h}^{(1)} \leq \sum_\alpha \sum_{i=1}^3 w_i(x^\alpha) w_j(x^\alpha) \mathbf{L}_{ij}(x^\alpha, h, \varepsilon) + o(1), \qquad h \to 0.$$

In view of the smoothness of $w(x)$ and condition (c_2), it follows that

$$\lim_{h\to 0}\varlimsup_{\varepsilon\to 0} I^{(1)}_{\varepsilon h} \le \int_\Omega (\mathbf{L}w, w)\, dx. \tag{4.18}$$

Now, combining (4.16), (4.17), and (4.18) and taking into account that $r = h^{1+\gamma/2}$, we obtain the sought inequality (4.3).

4.3. Proof of Inequality (4.6).

First, we prove an auxiliary statement. Denote by $R(\Omega)$ the Hilbert space of vector functions $w(x) : \Omega \to \mathbb{R}^3$ which is the closure of the set of vector functions in $(C_0^2(\Omega))^3$ with respect to the norm $\|w\|_R = (w, w)_R^{1/2}$ generated by the inner product

$$(w, v)_R = \int_\Omega \{(\operatorname{rot} w, \operatorname{rot} v) + (w, v)\}\, dx.$$

It is known [3] that the tangential components of vectors of $R(\Omega)$ vanish on the boundary of Ω.

It turns out that the class $R_0(\Omega, F_\varepsilon)$ introduced in Section 3, is weakly compact in $R(\Omega)$ and strongly dense in $R(\Omega)$ with respect to the norm of $L_2(\Omega)$. More precisely, the following lemma holds true:

Lemma 4.2. *Assume that condition (c_2) is satisfied. Then, for any $w(x) \in R(\Omega)$ one can construct vector functions $w_\varepsilon(x) \in R_0(\Omega, F_\varepsilon)$, depending of the parameter ε ($0 < \varepsilon < \varepsilon_0(w)$), which converge, as $\varepsilon \to 0$, to $w(x)$ weakly in $R(\Omega)$ and strongly in $L_2(\Omega)$ and such that*

$$\varlimsup_{\varepsilon\to 0} \|w_\varepsilon\|_R^2 \le C\|w\|_R^2,$$

where the constant C is independent of w.

Proof. Since the class $(C_0^2(\Omega))^3$ is dense in $R(\Omega)$, it suffices to prove the assertion of the lemma for $w(x) \in (C_0^2(\Omega))^3$.

Given $w(x) \in (C_0^2(\Omega))^3$, we construct $w_{\varepsilon h}(x)$ by (4.13). By the same arguments as in Section 4.2 (by using (4.14), (4.15), Lemma 4.1, the properties of functions constituting the partition of unity $\{\varphi_\alpha(x)\}$, and condition (c_2)) we conclude that

$$\|w_{\varepsilon h}\|_R^2 \le \|w\|_R^2 + \int_\Omega (\mathbf{L}w, w)dx + o(1) \tag{4.19}$$

for sufficiently small h ($h \le \hat{h}(w)$), $r = h^{1+\gamma/2}$ and sufficiently small ε ($\varepsilon < \hat{\varepsilon}(h)$). In this way, we define a monotone decreasing, as $h \to 0$, function $\hat{\varepsilon}(h)$: $(0, \hat{h}(w)) \to R_+$. Introduce a step-like monotone function $h(\varepsilon)$ by setting $h(\varepsilon) = \frac{1}{k}$ when ε is in the interval $\hat{\varepsilon}(\frac{1}{k+1}) \le \varepsilon \le \hat{\varepsilon}(\frac{1}{k})$, $k \in \mathbb{N}$, and define the sought vector function $w_\varepsilon(x)$ by $w_\varepsilon(x) = w_{\varepsilon h}(x)|_{h=h(\varepsilon)}$ ($0 < \varepsilon < \hat{\varepsilon}(\hat{h}(w))$). Then from (4.19) it follows that

$$\overline{\lim_{\varepsilon \to 0}} \|w_\varepsilon\|_R^2 \le C\|w\|_R^2,$$

with $C = 1 + \max_{x \in \Omega} \|\mathbf{L}(x)\|$. □

It remains to show that $w_\varepsilon(x)$ converge, as $\varepsilon \to 0$, to $w(x)$ weakly in $R(\Omega)$ and strongly in $L_2(\Omega)$. But this can be achieved by using the same arguments as in Section 4.2, with the help of formulas (4.14), (4.15), lemma 4.1, and taking into account the properties of the partition of unity $\{\varphi_\alpha(x)\}$. Lemma 4.2 is proved.

Now we are at a position to prove the inequality (4.6). Denote by $u(x) \in \overset{\circ}{W}{}_2^1(\Omega) \subset R(\Omega)$ the weak limit in $\overset{\circ}{W}{}_2^1(\Omega)$, on a subsequence $\{\varepsilon = \varepsilon_k \to 0\}$, of solutions $u_\varepsilon(x) \in \overset{\circ}{W}{}_2^1(\Omega_\varepsilon) \subset R_0(\Omega, F_\varepsilon)$ of problem (I). Let $u^\delta(x)$ be a vector function in $(C_0^2(\Omega))^3$ such that

$$\|u^\delta - u\|_R < \delta, \qquad \delta > 0. \tag{4.20}$$

Since $u^\delta - u \in R(\Omega$ and $u_\varepsilon \in R_0(\Omega, F_\varepsilon)$, we can construct, by using Lemma 4.2, a sequence of vector functions $\{u_\varepsilon^\delta \in R_0(\Omega, F_\varepsilon), \varepsilon = \varepsilon_k \to 0\}$ such that it converges to $u^\delta(x)$ weakly in $R(\Omega)$ and strongly in $L_2(\Omega)$ and that the following inequality holds true:

$$\overline{\lim_{\varepsilon = \varepsilon_k \to 0}} \|u_\varepsilon^\delta\|_R^2 \le C\|u^\delta - u\|_R^2, \tag{4.21}$$

where C is independent of δ.

Split Ω into disjoint (i.e., having no common interior points) cubes $K_h^\alpha = K(x^\alpha, h)$ of sufficiently small size h ($0 < h \ll \delta$). Let $\varphi_{\varepsilon h}^\alpha(x)$ be the function minimizing the functional (2.1) in K_h^α. In what follows, we will use some properties of this function, namely: $\varphi_{\varepsilon h}^\alpha(x) = 0$ for $x \in F_\varepsilon \cap K_h^\alpha$ and

$$\int_{K_h^\alpha} |\nabla \varphi_{\varepsilon h}^\alpha(x)|^2 dx \le Ch^3, \qquad \int_{K_h^\alpha} |\nabla \varphi_{\varepsilon h}^\alpha - 1|^2 dx \le Ch^{5+\gamma}$$

for sufficiently small ε ($\varepsilon < \hat{\varepsilon}(h)$).

The last two inequality follow immediately from condition (c_1). In each cube K_h^α, consider the vector function

$$v_{\varepsilon h}^\delta = u_\varepsilon^\delta(x) - [u^\delta(x) - u^\delta(x^\alpha)]\varphi_{\varepsilon h}^\alpha(x). \tag{4.22}$$

By the properties of $u_\varepsilon^\delta(x) \in R_0(\Omega, F_\varepsilon)$ and the function $\varphi_{\varepsilon h}^\alpha(x)$ ($\varphi_{\varepsilon h}^\alpha(x) = 0$ for $x \in F_\varepsilon \cap K_h^\alpha$), $v_{\varepsilon h}^\delta(x)$ belongs to the class $R_0(x^\alpha, h, \varepsilon)$, in which the infimum in (2.3) is sought. Therefore, by (2.3), we have the inequality

$$\int_{K_h^\alpha} \{|\operatorname{rot} v_{\varepsilon h}^\delta|^2 + h^{-2-\gamma}|v_{\varepsilon h}^\delta - l|^2\}\,dx \geq \sum_{i,j=1}^3 \mathbf{L}_{ij}(x^\alpha, h, \varepsilon)l_i l_j,$$

where $l \in \mathbb{R}^3$. Setting $l = u^\delta(x^\alpha)$, we obtain

$$\int_{K_h^\alpha} \{|\operatorname{rot} v_{\varepsilon h}^\delta|^2 + h^{-2-\gamma}|v_{\varepsilon h}^\delta - u^\delta(x^\alpha)|^2\,dx$$
$$\geq \sum_{i,j=1}^3 \mathbf{L}_{ij}(x^\alpha, h, \varepsilon)u_i^\delta(x^\alpha)u_j^\delta(x^\alpha). \tag{4.23}$$

By (4.22), we can write

$$\operatorname{rot} v_{\varepsilon h}^\delta = \operatorname{rot} u_\varepsilon^\delta - \operatorname{rot} u^\delta + \operatorname{rot} u^\delta(\varphi_{\varepsilon h}^\alpha - 1) + (u^\delta - u^\delta(x^\alpha)) \wedge \operatorname{grad} \varphi_{\varepsilon h}^\alpha.$$

Substituting this expression into (4.23), we obtain the inequality

$$\int_{K_h^\alpha} |\operatorname{rot} u_\varepsilon^\delta|^2\,dx \geq \int_{K_h^\alpha} \{2(\operatorname{rot} u_\varepsilon^\delta, \operatorname{rot} u^\delta) - |\operatorname{rot} u^\delta|^2\}\,dx$$
$$+ \sum_{i,j=1}^3 \mathbf{L}_{ij}(x^\alpha, h, \varepsilon)u_i^\delta(x^\alpha)u_j^\delta(x^\alpha) - I_1^\alpha(\varepsilon, h, \delta) + h^{-2-\gamma}I_2^\alpha(\varepsilon, h, \delta), \tag{4.24}$$

where

$$I_1^\alpha(\varepsilon, h, \delta) = 2\int_{K_h^\alpha} (\operatorname{rot}[u_\varepsilon^\delta - u^\delta], \operatorname{rot} u^\delta)(\varphi_{\varepsilon h}^\alpha - 1)\,dx$$
$$+ 2\int_{K_h^\alpha} (\operatorname{rot}[u_\varepsilon^\delta - u^\delta], [u^\delta - u^\delta(x^\alpha)] \wedge \operatorname{grad} \varphi_{\varepsilon h}^\alpha)\,dx$$
$$+ 2\int_{K_h^\alpha} (\operatorname{rot} u^\delta, [u^\delta - u^\delta(x^\alpha)] \wedge \operatorname{grad} \varphi_{\varepsilon h}^\alpha)(\varphi_{\varepsilon h}^\alpha - 1)dx, \tag{4.25}$$

$$I_2^\alpha(\varepsilon,h,\delta) = 2\int\limits_{K_h^\alpha} (u_\varepsilon^\delta - u^\delta, u^\delta - u^\delta(x^\alpha))(\varphi_{\varepsilon h}^\alpha - 1)\,dx \\ - \int\limits_{K_h^\alpha} |u_\varepsilon^\delta - u^\delta|^2\,dx - \int\limits_{K_h^\alpha} |u^\delta - u^\delta(x^\alpha)|^2(\varphi_{\varepsilon h}^\alpha - 1)^2\,dx. \quad (4.26)$$

Let us estimate from below the value of the functional J_ε, defined by (4.1), on the vector function $u_\varepsilon^\delta(x)$. Representing this functional in the form

$$J_\varepsilon[u_\varepsilon^\delta] = \sum_\alpha \int\limits_{K_h^\alpha} |\mathrm{rot}\, u_\varepsilon^\delta|^2 dx + \lambda^2 \int\limits_\Omega |u_\varepsilon^\delta|^2 dx \\ + \int\limits_\Omega (J^0 + \lambda^2 \mathrm{grad}\,\varphi_\varepsilon, u_\varepsilon^\delta)\,dx,$$

by (4.24) we obtain

$$J_\varepsilon[u_\varepsilon^\delta] \geq \int\limits_\Omega \{2(\mathrm{rot}\, u_\varepsilon^\delta, \mathrm{rot}\, u^\delta) - |\mathrm{rot}\, u_\varepsilon^\delta|^2\}\,dx \\ + \lambda^2 \int\limits_\Omega |u_\varepsilon^\delta|^2 dx + \int\limits_\Omega (J^0 - \lambda^2 \mathrm{grad}\,\varphi_\varepsilon, u_\varepsilon^\delta)\,dx \\ + \sum_\alpha \sum_{i,j=1}^3 \mathbf{L}_{ij}(x^\alpha,h,\varepsilon) u_i^\delta(x^\alpha) u_j^\delta(x^\alpha) \\ - \sum_\alpha |I_1^\alpha(\varepsilon,h,\delta)| - h^{-2-\gamma} \sum_\alpha |I_2^\alpha(\varepsilon,h,\delta)|. \quad (4.27)$$

Since the norms of u_ε^δ and u^δ in $R(\Omega)$ are bounded uniformly with respect to ε and δ, and $|u^\delta(x) - u^\delta(x^\alpha)| \leq C(\delta)$ for $x \in K_h^\alpha$, it follows from (4.25), (4.26), and the properties of $\varphi_{\varepsilon h}^\alpha(x)$ that

$$\overline{\lim_{\varepsilon\to 0}} \sum_\alpha |I_1^\alpha(\varepsilon,h,\delta)| \leq C_1(\delta) h \quad (4.28)$$

and

$$\lim_{\varepsilon\to 0} \sum_\alpha |I_2^\alpha(\varepsilon,h,\delta)| \\ \leq C_2(\delta) h^{2+\gamma/2} \|u_\varepsilon^\delta - u^\delta\|_{L_2(\Omega)} + \|u_\varepsilon^\delta - u^\delta\|_{L_2(\Omega)}^2 + C_3(\delta) h^{4+\gamma}, \quad (4.29)$$

where the constants $C_i(\delta)$ $(i = 1,2,3)$ depend only on $u(x)$ and δ.

Now, taking into account that, as $\varepsilon = \varepsilon_k \to 0$, $u_\varepsilon^\delta(x)$ converge to $u^\delta(x)$ weakly in $R(\Omega)$ and strongly in $L_2(\Omega)$, and $\operatorname{grad}\varphi_\varepsilon(x)$ converge to $\operatorname{grad}\varphi(x)$ weakly in $L_2(\Omega)$, by (4.27), (4.28), (4.29), and condition (c_2) we conclude that

$$\lim_{h\to 0} \varliminf_{\varepsilon=\varepsilon_k\to 0} J_\varepsilon[u_\varepsilon^\delta] \geq J[u^\delta]$$

for any fixed $\delta > 0$, where the functional $J[w]$ is defined by (4.4). Passing in this inequality to the limit as $\delta \to 0$ and taking into account (4.20) and (4.21), we obtain the sought inequality (4.6), which completes the proof of Theorem 3.1.

5. Completion of Proof of Theorem 2.1

First of all, we notice that, in view of the estimates (3.34) and (3.35) and Theorem 3.3, solutions $E_\varepsilon(x, \lambda)$ of problem (3.5)–(3.6) (the electric field in problem (3.2)–(3.4)) converge, for $\operatorname{Re}\lambda > 0$, to $E(x, \lambda)$ (the component of the solution (E, Φ) of problem (3.28)–(3.31)) weakly in $R(\Omega)$. This, in view of (3.8), implies that the magnetic fields $H_\varepsilon(x, \lambda)$ converge weakly in $L_2(\Omega)$ to the vector function

$$H(x, \lambda) = -\frac{1}{\lambda}(\operatorname{rot} E(x, \lambda) - H^0(x)).$$

By this, problem (3.28)–(3.31) can be rewritten in the form

$$\left.\begin{array}{lr} \operatorname{rot} H - \lambda E - \frac{1}{\lambda}\mathbf{L}(E - \operatorname{grad}\Phi) = J - E^0, & x \in \Omega, \\ \operatorname{rot} E + \lambda H = H^0, & x \in \Omega, \\ \operatorname{div}(\mathbf{L}\operatorname{grad}\Phi) - \lambda^2 C\Phi = \operatorname{div}(\mathbf{L}E), & x \in G, \\ n \wedge E = 0, & x \in \partial\Omega, \\ (\mathbf{L}\operatorname{grad}\Phi)_n = (\mathbf{L}E)_n, & x \in \partial G. \end{array}\right\} \tag{5.1}$$

Therefore, solutions $(E_\varepsilon, H_\varepsilon)$ of problem (3.2)–(3.4) with $\operatorname{Re}\lambda > 0$ converge weakly in $(L_2(\Omega))^3 \times (L_2(\Omega))^3$ to a pair of functions (E, H) such that the triple (E, H, Φ) solves problem (5.1). Moreover, the convergence is uniform in every compact part of the half-plane $\operatorname{Re}\lambda > 0$.

Now consider the solution $(E_\varepsilon(x, t), H_\varepsilon(x, t))$ of the original (nonstationary) problem (1.1)–(1.4). By standard arguments, one can easily obtain the estimate

$$\int_0^T \int_\Omega \{|E_\varepsilon(x, t)|^2 + |H_\varepsilon(x, t)|^2\} dx dt < C, \tag{5.2}$$

where the constant C is independent of ε (it depends on J, E^0, H^0, and T). This estimate implies that the set of solutions $\{E_\varepsilon, H_\varepsilon, \varepsilon > 0\}$ of problem (1.1)–(1.4) is weakly compact in $(L_2(\Omega \times [0,T]))^3 \times (L_2(\Omega \times [0,T]))^3$ for all $T > 0$. By (3.1), these solutions can be expressed in terms of $\{E_\varepsilon(x,\lambda), H_\varepsilon(x,\lambda)\}$ by using the inverse Laplace transform

$$E_\varepsilon(x,t) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} E_\varepsilon(x,\lambda)e^{\lambda t}d\lambda, \tag{5.3}$$

$$H_\varepsilon(x,t) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} H_\varepsilon(x,\lambda)e^{\lambda t}d\lambda, \quad \sigma > 0.$$

The integrals here converge in the sense of distributions $D'((L_2(\Omega))^3;(0,T))$ for all T.

Let $\varphi(x) \in (L_2(\Omega))^3$ and $\psi \in C_0^\infty(0,T)$. Using (5.3) and changing the order of integration, we can write

$$\int\limits_0^T \int\limits_\Omega E_\varepsilon(x,t)\varphi(x)\,dx\psi(t)\,dt$$
$$= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\left(\int\limits_\Omega E_\varepsilon(x,\lambda)\varphi(x)\,dx\int\limits_0^T e^{\lambda t}\psi(t)\,dt\right)d\lambda. \tag{5.4}$$

Here the change of order of integration is allowed, since, by (3.34),

$$\left|\int\limits_\Omega E_\varepsilon(x,\lambda)\varphi(x)\,dx\right| < C, \qquad \operatorname{Re}\lambda = \sigma$$

and

$$\int\limits_0^T e^{\lambda t}\psi(t)dt = O\left(\frac{1}{|\lambda|^2}\right).$$

Using these estimates and taking into account that $\{E_\varepsilon(x,\lambda), H_\varepsilon(x,\lambda)\}$ converge weakly to $\{E(x,\lambda, H(x,\lambda)\}$, from (5.4) we obtain

$$\lim_{\varepsilon\to 0}\int\limits_\Omega E_\varepsilon(x,t)\varphi(x)\psi(t)\,dx\,dt$$
$$= \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\left(\int\limits_\Omega E(x,\lambda)\varphi(x)\,dx\int\limits_0^T e^{\lambda t}\psi(t)\,dt\right)d\lambda$$

$$= \int_0^T \int_\Omega E(x,t)\varphi(x)\psi(t)\,dx\,dt.$$

where

$$E(x,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} E(x,\lambda) e^{\lambda t} d\lambda. \tag{5.5}$$

Similarly,

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega H_\varepsilon(x,t)\varphi(x)\psi(t)\,dx\,dt = \int_0^T \int_\Omega H(x,t)\varphi(x)\psi(t)\,dt,$$

where

$$H(x,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} H(x,\lambda) e^{\lambda t} d\lambda. \tag{5.6}$$

Here the integrals (5.5) and (5.6) converge in the sense of distributions in $D'(L_2(\Omega))^3; [0,T])$.

Since the set of vector functions of the form $\sum_k \varphi_k(x)\psi_k(t)$, with $\varphi_k(x) \in (L_2(\Omega))^3$ and $\psi_k(t) \in C_0^\infty(0,T)$, is dense in $(L_2(\Omega) \times (0,T))^3$, and the estimate (5.2) holds uniformly with respect to ε, it follows that $\{E_\varepsilon(x,t), H_\varepsilon(x,t)\}$ converge, as $\varepsilon \to 0$, weakly in $(L_2(\Omega) \times [0,T])^3 \times (L_2(\Omega) \times [0,T])^3$ to $\{E(x,t), H(x,t)\}$, where $E(x,t)$ and $H(x,t)$ are defined by (5.5) and (5.6), respectively. Finally, introducing the function

$$\Phi(x,t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Phi(x,\lambda) e^{\lambda t} d\lambda \tag{5.7}$$

and taking into account that $(E(x,\lambda), H(x,\lambda), \Phi(x,\lambda))$ solves the boundary value problem (5.1), by (5.5)–(5.7) we conclude that $\{E(x,t), H(x,t), \Phi(x,t)\}$ solves the initial boundary value problem (1.8)–(1.13), which completes the proof of Theorem 2.1.

Remark 5.1. Theorems 2.1 and 3.1 (the main results of the paper) are conditional, since we assume that conditions (c_1)–(c_3) are satisfied; moreover, at present we know no grid structure $\{F_\varepsilon, \varepsilon > 0\}$ for which condition (c_3) is established to be true. As for conditions (c_1) and (c_2), one can verify whether they are true or not, for every particular case. For instance, let sets F_ε are formed by thin wires of diameter $d_\varepsilon \ll \varepsilon$, the axes of which form a periodic lattice in $\mathbb{R}^3$, with periods $h_i\varepsilon$ ($0 < h_i \leq 1$, $i = 1, 2, 3$), and let the following limit exists:

$$d = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2 \ln \frac{\varepsilon}{d_\varepsilon}}.$$

Then conditions (c_1) and (c_2) are satisfied; moreover, the limiting density of capacity $C(x)$ and the limiting tensor of back induction $\mathbf{L}(x)$ are constants:

$$C(x) \equiv C = \frac{2\pi d(h_1 + h_2 + h_3)}{h_1 h_2 h_3},$$

$$\mathbf{L}(x) \equiv \mathbf{L} = \frac{2\pi d}{h_1 h_2 h_3} \operatorname{diag}\{h_i, i = 1, 2, 3\}.$$

The proof of this fact as well as the discussion of condition (c_3) will be presented elsewhere.

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