Український математичний вісник
Том 2 **(2005)**, № 1, 38 – 51

Bounded Components of Positive Solutions of Nonlinear Abstract Equations

Santiago Cano-Casanova, Julián López-Gómez
and Marcela Molina-Meyer

(*Presented by I. V. Skrypnik*)

Abstract. In this work a general class of nonlinear abstract equations satisfying a *generalized strong maximum principle* is considered in order to show that any bounded component of positive solutions bifurcating from the curve of trivial states $(\lambda, u) = (\lambda, 0)$ at a nonlinear eigenvalue $\lambda = \lambda_0$ must meet the curve of trivial states $(\lambda, 0)$ at another singular value $\lambda_1 \neq \lambda_0$. Since the unilateral theorems of P. H. Rabinowitz [13, Theorems 1.27 and 1.40] are not true as originally stated (c.f. the counterexample of E. N. Dancer [6]), in order to get our main result the unilateral theorem of J. López-Gómez [11, Theorem 6.4.3] is required.

2000 MSC. 34A34, 34C23, 35B32, 35B50, 35J25.

Key words and phrases. Positive Solutions, Compact solution components, Nonlinear abstract equations, Bifurcation theory.

1. Introduction

Throughout this work U stands for an ordered real Banach space whose positive cone, P, is normal and it has nonempty interior, and we consider the nonlinear abstract equation

$$\mathfrak{F}(\lambda, u) := \mathfrak{L}(\lambda)u + \mathfrak{R}(\lambda, u) = 0\,, \qquad (\lambda, u) \in X := \mathbb{R} \times U\,, \qquad (1.1)$$

where

(HL) The family $\mathfrak{K}(\lambda) := I_U - \mathfrak{L}(\lambda) \in \mathcal{L}(U)$, $\lambda \in \mathbb{R}$, is compact and real analytic, and $\mathfrak{L}(\hat{\lambda})$ is a linear topological isomorphism for some $\hat{\lambda} \in \mathbb{R}$, where I_U stands for the identity map of U and $\mathcal{L}(U)$ is the space of linear continuous endomorphisms of U.

Received 13.01.2004
The work has been partially supported by grants BFM2000-0797 and REN2003-00707 of the Spanish Ministry of Science and Technology.

(HR) $\mathfrak{R} \in \mathcal{C}(\mathbb{R} \times U; U)$ is compact on bounded sets and $\lim_{u\to 0} \frac{\mathfrak{R}(\lambda,u)}{\|u\|} = 0$ uniformly on compact intervals of $\mathbb{R}$.

(HP) The solutions of (1.1) satisfy the *strong maximum principle* in the sense that

$$(\lambda, u) \in \mathbb{R} \times (P \setminus \{0\}) \quad \text{and} \quad \mathfrak{F}(\lambda, u) = 0 \quad \text{imply} \quad u \in \operatorname{Int} P\,,$$

where $\operatorname{Int} P$ stands for the interior of the cone P.

Subsequently, given u_1, $u_2 \in U$, we write $u_1 > u_2$ if $u_1 - u_2 \in P \setminus \{0\}$, and $u_1 \gg u_2$ if $u_1 - u_2 \in \operatorname{Int} P$. Also, it will be said that (λ, u) is a *positive solution* of (1.1), if (λ, u) is a solution of (1.1) with $u > 0$. Thanks to Assumption (HP), any positive solution (λ, u) of (1.1) must be *strongly positive*, in the sense that $u \gg 0$.

Under Assumptions (HL) and (HR), $\mathfrak{F}(\lambda, 0) = 0$ for each $\lambda \in \mathbb{R}$. The main result of this paper concerns the bounded components of positive solutions of (1.1) emanating from $(\lambda, u) = (\lambda, 0)$ at a *nonlinear eigenvalue* $\lambda_0 \in \mathbb{R}$ with geometric multiplicity one. By a *component* of positive solutions of (1.1) it is meant a maximal (for the inclusion) relatively closed and connected subset of the set of positive solutions of (1.1) (in $\mathbb{R} \times \operatorname{Int} P$). A value $\sigma \in \mathbb{R}$ is said to be an *eigenvalue* of the family $\mathfrak{L}(\lambda)$ if $\dim N[\mathfrak{L}(\sigma)] \geq 1$. The set of eigenvalues of $\mathfrak{L}(\lambda)$ will be denoted by $\mathfrak{S}$. Thanks to (HL), $\mathfrak{L}(\lambda)$ is Fredholm of index zero for any $\lambda \in \mathbb{R}$ and, hence, $\mathfrak{S}$ provides us with the set of *singular values* of the family $\mathfrak{L}(\lambda)$. In other words, $\mathfrak{L}(\lambda)$ is a linear topological isomorphism if $\lambda \in \mathbb{R} \setminus \mathfrak{S}$. Moreover, it follows from [11, Theorem 4.4.4] that $\mathfrak{S}$ is discrete and that it consists of *algebraic eigenvalues* of $\mathfrak{L}$, i.e., for any $\sigma \in \mathfrak{S}$ there exist $C > 0$, $\varepsilon > 0$ and $\nu \geq 1$ such that for any $\lambda \in (\sigma - \varepsilon, \sigma + \varepsilon) \setminus \{\sigma\}$ the operator $\mathfrak{L}^{-1}(\lambda)$ is well defined and $\|\mathfrak{L}^{-1}(\lambda)\|_{\mathcal{L}(U)} \leq \frac{C}{|\lambda - \sigma|^\nu}$ if $0 < |\lambda - \sigma| < \varepsilon$. Thus, thanks to the abstract spectral theory developed in [11, Chapter 4], the algebraic multiplicity $\chi[\mathfrak{L}; \cdot] : \mathfrak{S} \to \mathbb{N}$ introduced by J. Esquinas and J. López-Gómez in [8] and [7] is well defined. Actually, $\chi[\mathfrak{L}; \lambda_0] = 1$ if $\lambda_0 \in \mathfrak{S}$ is a simple eigenvalue of $\mathfrak{L}(\lambda)$ as discussed by M. G. Crandall and P. H. Rabinowitz [4], i.e., if

$$\dim N[\mathfrak{L}_0] = 1 \quad \text{and} \quad \mathfrak{L}_1(N[\mathfrak{L}_0]) \oplus R[\mathfrak{L}_0] = U\,, \tag{1.2}$$

where $\mathfrak{L}_0 := \mathfrak{L}(\lambda_0)$, $\mathfrak{L}_1 := \frac{d\mathfrak{L}}{d\lambda}(\lambda_0) = -\frac{d\mathfrak{K}}{d\lambda}(\lambda_0)$, and, for any $T \in \mathcal{L}(U)$, $N[T]$ and $R[T]$ stand for the null space and the range of T. A value $\sigma \in \mathfrak{S}$ is said to be a *nonlinear eigenvalue* of $\mathfrak{L}(\lambda)$ if $(\sigma, 0)$ is a bifurcation point of (1.1) from $(\lambda, 0)$, $\lambda \in \mathbb{R}$, for any $\mathfrak{R}(\lambda, u)$ satisfying (HR). According to [11, Theorem 4.3.4], $\chi[\mathfrak{L}; \sigma] \in 2\mathbb{N}+1$ if σ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$, and, thanks to [11, Theorem 6.6.2], for any $\sigma \in \mathfrak{S}$ there is $\eta \in \{-1,1\}$ such that

$$\operatorname{Ind}(0, \mathfrak{K}(\lambda)) = \eta \operatorname{sign}(\lambda - \sigma)^{\chi[\mathfrak{L};\sigma]}, \quad \lambda \sim \sigma, \quad \lambda \neq \sigma,$$

where $\operatorname{Ind}(0, \mathfrak{K}(\lambda))$ is the local index of $\mathfrak{K}(\lambda)$ at zero (the topological degree of $\mathfrak{L}(\lambda)$ in any open bounded set containing zero). Thus, $\operatorname{Ind}(0, \mathfrak{K}(\lambda))$ changes as λ crosses σ if, and only if, $\chi[\mathfrak{L};\sigma] \in 2\mathbb{N}+1$. Consequently, thanks to [11, Theorem 6.2.1], $\sigma \in \mathfrak{S}$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$ if $\chi[\mathfrak{L};\sigma] \in 2\mathbb{N}+1$, and, therefore, $\sigma \in \mathfrak{S}$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$ if, and only if, $\chi[\mathfrak{L};\sigma] \in 2\mathbb{N}+1$. The main result of this paper can be stated as follows.

Theorem 1.1. Suppose $\lambda_0 \in \mathfrak{S}$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$ such that

$$N[\mathfrak{L}(\lambda_0)] = \operatorname{span}[\varphi_0], \qquad \varphi_0 \in P \setminus \{0\}, \tag{1.3}$$

and $\mathfrak{K}(\lambda_0)$ is strongly positive, i.e.,

$$\mathfrak{K}(\lambda_0)(P \setminus \{0\}) \subset \operatorname{Int} P. \tag{1.4}$$

Then, there exists a component $\mathfrak{C}^P_{\lambda_0}$ of the set of positive solutions of $\mathfrak{F}(\lambda, u) = 0$ emanating from $(\lambda, u) = (\lambda, 0)$ at $\lambda = \lambda_0$. Moreover, there exists $\lambda_1 \in \mathfrak{S} \setminus \{\lambda_0\}$ such that $(\lambda_1, 0) \in \bar{\mathfrak{C}}^P_{\lambda_0}$ if $\mathfrak{C}^P_{\lambda_0}$ is bounded in $X := \mathbb{R} \times U$.

Note that, thanks to (HL) and (1.4),

$$\mathfrak{K}(\lambda_0)\varphi_0 = \varphi_0 \gg 0. \tag{1.5}$$

Thus, the existence of $\mathfrak{C}^P_{\lambda_0}$ follows by adapting some of the unilateral results of P. H. Rabinowitz [13]; in Section 2 complete details will be provided.

The distribution of this paper is as follows. In Section 2 we show the existence of $\mathfrak{C}^P_{\lambda_0}$, in Section 3 we complete the proof of Theorem 1.1, and in Section 4 we derive from Theorem 1.1 a celebrated unilateral theorem attributable to E. N. Dancer [5]. Finally, in Section 5 we give an application of Theorem 1.1, and in Section 6 we construct an example showing the necessity of condition $\chi[\mathfrak{L};\lambda_0] \in 2\mathbb{N}+1$ for the validity of Theorem 1.1. Throughout the remaining of this paper it will be assumed that $\lambda_0 \in \mathfrak{S}$ satisfies all the requirements of Theorem 1.1.

2. Unilateral Bifurcation. The Existence of $\mathfrak{C}^P_{\lambda_0}$

The set of *non-trivial solutions* of (1.1) is defined through

$$\mathcal{S} := \mathfrak{F}^{-1}(0) \setminus [(\mathbb{R} \setminus \mathfrak{S}) \times \{0\}] \,.$$

Note that $(\lambda, u) \in \mathcal{S}$ if either $u \neq 0$, or else $u = 0$ and $\lambda \in \mathfrak{S}$. Since $\chi[\mathfrak{L}; \lambda_0] \in 2\mathbb{N}+1$, thanks to [11, Corollary 6.3.2] there exists a component of $\mathcal{S}$, subsequently denoted by $\mathfrak{C}_{\lambda_0}$, such that $(\lambda_0, 0) \in \mathfrak{C}_{\lambda_0}$. Subsequently, we suppose that φ_0 has been normalized so that

$$N[\mathfrak{L}(\lambda_0)] = \operatorname{span}[\varphi_0]\,, \qquad \|\varphi_0\| = 1\,, \qquad \varphi_0 > 0\,. \tag{2.1}$$

Thanks to (1.5), $\varphi_0 \gg 0$. Now, let Y be a closed subspace of U such that $U = N[\mathfrak{L}(\lambda_0)] \oplus Y$. Thanks to Hahn-Banach's theorem, there exists $\varphi_0^* \in U'$ such that

$$Y = \{\, u \in U \;:\; \langle \varphi_0^*, u\rangle = 0 \,\}\,, \qquad \langle \varphi_0^*, \varphi_0\rangle = 1\,,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality between U and U'. Now, for each $\eta \in (0,1)$ and sufficiently small $\varepsilon > 0$ we set

$$\mathcal{Q}_{\varepsilon,\eta} := \{\, (\lambda, u) \in X \;:\; |\lambda - \lambda_0| < \varepsilon\,,\;\; |\langle \varphi_0^*, u\rangle| > \eta\, \|u\| \,\}\,.$$

Since the mapping $u \mapsto |\langle \varphi_0^*, u\rangle| - \eta\,\|u\|$ is continuous, $\mathcal{Q}_{\varepsilon,\eta}$ is an open subset of X consisting of the two disjoint components $\mathcal{Q}^+_{\varepsilon,\eta}$ and $\mathcal{Q}^-_{\varepsilon,\eta}$ defined though

$$\begin{aligned} \mathcal{Q}^+_{\varepsilon,\eta} &:= \{\, (\lambda, u) \in X \;:\; |\lambda - \lambda_0| < \varepsilon\,,\;\; \langle \varphi_0^*, u\rangle > \eta\, \|u\| \,\}\,, \\ \mathcal{Q}^-_{\varepsilon,\eta} &:= \{\, (\lambda, u) \in X \;:\; |\lambda - \lambda_0| < \varepsilon\,,\;\; \langle \varphi_0^*, u\rangle < -\eta\, \|u\| \,\}\,. \end{aligned} \tag{2.2}$$

The following result collects the main consequences from [11, Theorem 6.2.1, Proposition 6.4.2], which are consequences from the reflection argument of P. H. Rabinowitz [13]. Subsequently, we denote by $B_R(x)$ the open ball of radius $R > 0$ centered at $x \in X$.

Theorem 2.1. For each sufficiently small $\delta > 0$,

$$\mathfrak{C}_{\lambda_0} \cap B_\delta(\lambda_0, 0) \subset Q_{\varepsilon,\eta} \cup \{(\lambda_0, 0)\}$$

and each of the sets $\mathcal{S} \setminus \left[\mathcal{Q}^-_{\varepsilon,\eta} \cap B_\delta(\lambda_0,0)\right]$ and $\mathcal{S} \setminus \left[\mathcal{Q}^+_{\varepsilon,\eta} \cap B_\delta(\lambda_0,0)\right]$ contains a component, denoted by $\mathfrak{C}^+_{\lambda_0}$ and $\mathfrak{C}^-_{\lambda_0}$, respectively, such that $(\lambda_0, 0) \in \mathfrak{C}^+_{\lambda_0} \cap \mathfrak{C}^-_{\lambda_0}$ and

$$\mathfrak{C}_{\lambda_0} \cap B_\delta(\lambda_0, 0) = \left(\mathfrak{C}^+_{\lambda_0} \cup \mathfrak{C}^-_{\lambda_0}\right) \cap B_\delta(\lambda_0, 0)\,. \tag{2.3}$$

Moreover, for each $(\lambda, u) \in (\mathfrak{C}_{\lambda_0} \setminus \{(\lambda_0, 0)\}) \cap B_\delta(\lambda_0, 0)$, there exists a unique pair $(s, y) \in \mathbb{R} \times Y$ such that $u = s\varphi_0 + y$ and $|s| > \eta\,\|u\|$. Furthermore, $\lambda = \lambda_0 + o(1)$ and $y = o(s)$ as $s \to 0$.

It should be noted that if $(\lambda, u) \in \mathfrak{C}^+_{\lambda_0} \cap B_\delta(\lambda_0, 0)$, $u \neq 0$, then $u = s\varphi_0 + y$ with $s > \eta \|u\| > 0$, and, hence, $\frac{u}{s} = \varphi_0 + \frac{y}{s}$. Thus, since $\lim_{s\to 0} \frac{y}{s} = 0$, for sufficiently small $s > 0$, $\frac{u}{s} \in \operatorname{Int} P$ and, consequently, $u \in \operatorname{Int} P$. Therefore, for any sufficiently small $\delta > 0$, we have that

$$\left[\mathfrak{C}^+_{\lambda_0} \setminus \{(\lambda_0, 0)\}\right] \cap B_\delta(\lambda_0, 0) \subset \mathbb{R} \times \operatorname{Int} P\,. \tag{2.4}$$

This shows the existence of the component $\mathfrak{C}^P_{\lambda_0}$ of $\mathbb{R} \times \operatorname{Int} P$ containing $(\lambda_0, 0)$ (cf. the statement of Theorem 1.1). Actually, $\mathfrak{C}^P_{\lambda_0}$ is the maximal sub-continuum of $\mathfrak{C}^+_{\lambda_0}$ in $\mathbb{R} \times \operatorname{Int} P$.

The following result, which is [11, Theorem 6.4.3], provides us with an updated version of the unilateral theorem of P.H. Rabinowitz [13, Theorem 1.27], which is not true as originally stated (cf. E. N. Dancer [6]).

Theorem 2.2. For each $* \in \{-, +\}$, the component $\mathfrak{C}^*_{\lambda_0}$ satisfies some of the following alternatives:

1. $\mathfrak{C}^*_{\lambda_0}$ is unbounded in X.
2. There exists $\lambda_1 \in \mathfrak{S} \setminus \{\lambda_0\}$ such that $(\lambda_1, 0) \in \mathfrak{C}^*_{\lambda_0}$.
3. $\mathfrak{C}^*_{\lambda_0}$ contains a point $(\lambda, y) \in \mathbb{R} \times (Y \setminus \{0\})$.

Thanks to (HL), (1.4) and (1.5), the theorem of M. G. Krein and M. A. Rutman [10] (cf. H. Amann [1, Theorem 3.2]) as well), shows the validity of the following result, which is needed to conclude the proof of Theorem 1.1.

Theorem 2.3. Let $\operatorname{Spr}(\mathfrak{K}(\lambda_0))$ denote the spectral radius of $\mathfrak{K}(\lambda_0)$. Then,

(*a*) $\operatorname{Spr}(\mathfrak{K}(\lambda_0)) = 1$ is an algebraically simple eigenvalue of $\mathfrak{K}(\lambda_0)$ and, hence,
$$N[\mathfrak{L}_0] = N[\mathfrak{L}_0^2] = \operatorname{span}[\varphi_0]\,. \tag{2.5}$$
Thus, 0 is an algebraically simple eigenvalue of $\mathfrak{L}_0$, i.e.,
$$U = N[\mathfrak{L}_0] \oplus R[\mathfrak{L}_0]\,. \tag{2.6}$$
Moreover, no other eigenvalue of $\mathfrak{K}(\lambda_0)$ admits a positive eigenvector.

(*b*) For every $y \in \operatorname{Int} P$, the equation $u - \mathfrak{K}(\lambda_0)u = y$ cannot admit a positive solution.

Proof. As $\operatorname{Spr}(\mathfrak{K}(\lambda_0))$ is the unique eigenvalue of $\mathfrak{K}(\lambda_0)$ associated with it there is a positive eigenvector, and 1 is an eigenvalue to the eigenfunction φ_0, we have that $\operatorname{Spr}(\mathfrak{K}(\lambda_0)) = 1$. Moreover, 1 is algebraically simple, i.e.,

$$N[I_U - \mathfrak{K}(\lambda_0)] = N[(I_U - \mathfrak{K}(\lambda_0))^2]\,,$$

and, hence, due to (HL), (2.5) holds. Now, since $\mathfrak{L}_0$ is Fredholm of index zero, to prove (2.6) it suffices to show that $\varphi_0 \notin R[\mathfrak{L}_0]$. On the contrary, suppose that $\varphi_0 \in R[\mathfrak{L}_0]$. Then, there exists $u \in U \setminus N[\mathfrak{L}_0]$ such that $\mathfrak{L}_0 u = \varphi_0$, and, hence, $\mathfrak{L}_0^2 u = \mathfrak{L}_0 \varphi_0 = 0$. Thus, $u \in N[\mathfrak{L}_0^2] \setminus N[\mathfrak{L}_0]$, which contradicts (2.5) and concludes the proof of (2.6). This completes the proof of Part (a). Part (b) is an straightforward consequence from H. Amann [1, Theorem 3.2]. □

As a consequence from (2.6) we can make the choice

$$Y = R[\mathfrak{L}_0]\,, \tag{2.7}$$

which will be maintained throughout the remaining of the proof of Theorem 1.1.

3. Completion of the Proof of Theorem 1.1

Assume $\mathfrak{C}^P_{\lambda_0}$ is bounded. Since $\mathfrak{C}^P_{\lambda_0} \subset \mathfrak{C}^+_{\lambda_0}$, some of the following alternatives occurs. Either

$$\mathfrak{C}^P_{\lambda_0} = \mathfrak{C}^+_{\lambda_0} \setminus \{(\lambda_0, 0)\}\,, \tag{3.1}$$

or else

$$\mathfrak{C}^P_{\lambda_0} \quad \text{is a proper subset of} \quad \mathfrak{C}^+_{\lambda_0} \setminus \{(\lambda_0, 0)\}\,. \tag{3.2}$$

Suppose (3.1). Then, $\bar{\mathfrak{C}}^P_{\lambda_0} = \mathfrak{C}^+_{\lambda_0}$ satisfies some of the alternatives of Theorem 2.2. Alternative 1 cannot be satisfied, since $\bar{\mathfrak{C}}^P_{\lambda_0}$ is compact. Suppose Alternative 3 occurs. Then, thanks to the choice (2.7), there exists $(\lambda, y) \in \mathbb{R} \times R[\mathfrak{L}_0]$, $y \neq 0$, such that $(\lambda, y) \in \bar{\mathfrak{C}}^P_{\lambda_0} \subset \mathbb{R} \times P$. Since $y \neq 0$, necessarily $y \in \operatorname{Int} P$, by (HP). Thus, there exists $u \in U$ such that

$$\mathfrak{L}_0 u = u - \mathfrak{K}(\lambda_0) u = y\,.$$

Since $\varphi_0 \in \operatorname{Int} P$, for each sufficiently large $\alpha > 0$ we have that $u_\alpha := u + \alpha \varphi_0 \gg 0$. Moreover,

$$u_\alpha - \mathfrak{K}(\lambda_0) u_\alpha = y\,,$$

because $\mathfrak{K}(\lambda_0)\varphi_0 = \varphi_0$, which contradicts Theorem 2.3(b). Therefore, Alternative 2 of Theorem 2.2 must be satisfied. This concludes the proof of Theorem 1.1 in case (3.1).

Now, suppose (3.2). Then, since $\mathfrak{C}^+_{\lambda_0} \cap B_\delta(\lambda_0, 0) = \big[\mathfrak{C}^P_{\lambda_0} \cap B_\delta(\lambda_0, 0)\big] \cup \{(\lambda_0, 0)\}$ for each sufficiently small $\delta > 0$, fixing one of these δ's, there exists $(\lambda_1, u) \notin B_\delta(\lambda_0, 0)$ such that

$$(\lambda_1, u) \in \mathfrak{C}^+_{\lambda_0} \cap (\mathbb{R} \times \partial P) \cap \partial \mathfrak{C}^P_{\lambda_0}\,.$$

Let $\{(\mu_n, u_n)\}_{n\geq 1}$ be any subsequence of $\mathfrak{C}^P_{\lambda_0}$ such that $\lim_{n\to\infty}(\mu_n, u_n) = (\lambda_1, u)$. Then,

$$\mathfrak{F}(\lambda_1, u) = 0 \qquad \text{and} \qquad u \in P\,.$$

If $u > 0$, then, thanks to (HP), $u \in \operatorname{Int} P$, which contradicts $u \in \partial P$. Thus, $u = 0$, and, hence, $(\lambda_1, 0) \in \bar{\mathfrak{C}}^P_{\lambda_0}$. Moreover, $\lambda_1 \neq \lambda_0$, since $(\lambda_1, u) = (\lambda_1, 0) \notin B_\delta(\lambda_0, 0)$, which concludes the proof of Theorem 1.1. Note that, thanks to [11, Lemma 6.1.2], $\lambda_1 \in \mathfrak{S}$, since these are the unique possible bifurcation values from $(\lambda, 0)$ for (1.1).

4. Improving Dancer's Unilateral Theorem

As an immediate consequence from Theorem 1.1, the following unilateral result holds.

Theorem 4.1. Suppose $\lambda_0 \in \mathfrak{S}$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$ satisfying (1.3) and (1.4), and, in addition, no other $\sigma \in \mathfrak{S} \setminus \{\lambda_0\}$ admits an eigenfunction in $P \setminus \{0\}$. Then, the component $\mathfrak{C}^P_{\lambda_0}$ is unbounded in X.

Proof. On the contrary, suppose that $\mathfrak{C}^P_{\lambda_0}$ is bounded. Then, thanks to Theorem 1.1, there exists $\lambda_1 \in \mathfrak{S} \setminus \{\lambda_0\}$ such that $(\lambda_1, 0) \in \bar{\mathfrak{C}}^P_{\lambda_0}$. Let $\{(\mu_n, u_n)\}_{n\geq 1}$ be any sequence of $\mathfrak{C}^P_{\lambda_0}$ such that $\lim_{n\to\infty}(\mu_n, u_n) = (\lambda_1, 0)$ and set $v_n := \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then,

$$v_n = \mathfrak{K}(\mu_n) v_n - \frac{\mathfrak{R}(\mu_n, u_n)}{\|u_n\|}\,, \qquad n \geq 1\,,$$

and, hence, by (HL) and (HR), there exists a subsequence of $\{v_n\}_{n\geq 1}$, again labeled by n, such that $\lim_{n\to\infty} v_n = \psi$. Necessarily $\psi > 0$. Moreover, passing to the limit as $n \to \infty$ gives $\psi = \mathfrak{K}(\lambda_1)\psi$, or, equivalently, $\mathfrak{L}(\lambda_1)\psi = 0$, which is impossible, since we are assuming that λ_0 is the unique element of $\mathfrak{S}$ to a positive eigenvector. This contradiction shows that $\mathfrak{C}^P_{\lambda_0}$ is unbounded and concludes the proof. $\square$

In the special case when $\mathfrak{K}(\lambda) = \lambda K$, $\lambda \in \mathbb{R}$, for some linear strongly positive compact operator $K \in \mathcal{L}(U)$, necessarily $\lambda_0 := \frac{1}{\operatorname{Spr} K}$ is the unique element of $\mathfrak{S}$ associated with it there is a positive eigenvector, and, therefore, thanks to Theorem 4.1, $\mathfrak{C}^P_{\lambda_0}$ must be unbounded. Consequently, Theorem 4.1 is a substantial improvement of [11, Theorem 6.5.5], which is a well known result attributable to E. N. Dancer [5].

5. An Application of Theorem 1.1

In this section we consider the semi-linear weighted boundary value problem

$$\begin{cases} Eu = \lambda W(x)u - a(x)u^r & \text{in } \ \Omega\,, \\ u = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{5.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^{2+\nu}$ for some $\nu \in (0,1)$, $r \in (1,\infty)$, $\lambda \in \mathbb{R}$ is regarded as a bifurcation parameter, E is a second order uniformly elliptic operator of the form

$$E := -\sum_{i,j=1}^{N} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \alpha_i \frac{\partial}{\partial x_i} + \alpha_0$$

with $\alpha_{ij} = \alpha_{ji} \in \mathcal{C}^\nu(\bar{\Omega})$, α_i, $\alpha_0 \in \mathcal{C}^\nu(\bar{\Omega})$, $1 \le i,j \le N$, and

(Ha) $a \in \mathcal{C}^\nu(\bar{\Omega})$ and, setting

$$a^+ := \max\{a, 0\}, \qquad a^- := a^+ - a,$$

$$\Omega^0_{a^+} := \Omega \setminus \operatorname{supp} a^+, \qquad \Omega^0_{a^-} := \Omega \setminus \operatorname{supp} a^-,$$

$\Omega^0_{a^+}$ and $\Omega^0_{a^-}$ are two proper open subsets of Ω of class $\mathcal{C}^{2+\nu}$ with a finite number of well separated components. Moreover, either $N \in \{1,2\}$, or else $N \ge 3$ and for some constant $\gamma > 0$ the following is satisfied

$$[\operatorname{dist}(\cdot, \partial\Omega^0_{a^-})]^{-\gamma} a^- \in \mathcal{C}\left(\operatorname{supp} a^-, (0,\infty)\right),$$

$$r < \max\left\{\frac{N+2}{N-2}, \frac{N+1+\gamma}{N-1}\right\}.$$

(Hw) $W \in \mathcal{C}^\nu(\bar{\Omega})$ changes of sign in $\Omega^0_{a^+}$ and

$$\max_{\lambda \in \mathbb{R}} \sigma[E - \lambda W; \Omega] > 0\,, \tag{5.2}$$

where, for any elliptic operator L in a bounded domain D, $\sigma[L;D]$ stands for the principal eigenvalue of L in D under homogeneous Dirichlet boundary conditions.

Thanks to (5.2), for each $D \in \{\Omega^0_{a+}, \Omega\}$ the weighted boundary value problem

$$\begin{cases} E\varphi = \lambda W \varphi & \text{in } D\,, \\ \varphi = 0 & \text{on } \partial D\,, \end{cases} \tag{5.3}$$

possesses two principal eigenvalues, $\lambda_1^D < \lambda_2^D$. By a principal eigenvalue of (5.3) it is meant a value of λ for which (5.3) possesses an eigenfunction $\varphi > 0$. It should be noted that, necessarily, $\sigma[E - \lambda_j^D W; D] = 0$ for each $(D, j) \in \{\Omega^0_{a+}, \Omega\} \times \{1, 2\}$. Moreover, by the monotonicity of $\sigma[\cdot; D]$ with respect to D, $\lambda_1^{\Omega^0_{a+}} < \lambda_1^\Omega < \lambda_2^\Omega < \lambda_2^{\Omega^0_{a+}}$, and, thanks to a celebrated result by P. Hess and T. Kato [9], setting $\Sigma(\lambda) := \sigma[E - \lambda W; \Omega]$, $\lambda \in \mathbb{R}$, one has that $\Sigma'(\lambda_1^\Omega) > 0$ and $\Sigma'(\lambda_2^\Omega) < 0$, where $' = \frac{d}{d\lambda}$. Set $\lambda_0 := \lambda_1^\Omega$ and denote by φ_0 the principal eigenfunction associated to $\Sigma(\lambda_0) = 0$, normalized so that $\|\varphi_0\| = 1$. Then, since $\Sigma(\lambda)$ is a simple eigenvalue, there is an analytic mapping $\lambda \mapsto \varphi(\lambda) \in \mathcal{C}_0^{2+\nu}(\bar\Omega)$ such that $\varphi(\lambda_0) = \varphi_0$ and

$$(E - \lambda W)\varphi(\lambda) = \Sigma(\lambda)\varphi(\lambda)\,.$$

Now, differentiating with respect to λ and particularizing at $\lambda = \lambda_0$ gives

$$(E - \lambda_0 W)\varphi'(\lambda_0) = W\varphi_0 + \Sigma'(\lambda_0)\varphi_0$$

and, hence, $\Sigma'(\lambda_0) = -\langle \varphi_0^*, W\varphi_0 \rangle$, where $N[E^* - \lambda_0 W] = \operatorname{span}[\varphi_0^*]$ with $\langle \varphi_0^*, \varphi_0 \rangle = 1$. Therefore, since $\Sigma'(\lambda_0) > 0$, we find that

$$W\varphi_0 \notin R[E - \lambda_0 W]\,. \tag{5.4}$$

Now, suppose u is a positive solution of (5.1). Then, $u|_{\partial\Omega^0_{a+}} > 0$ and

$$(E - \lambda W)u = -au^r = a^- u^r \geq 0 \qquad \text{in} \quad \Omega^0_{a+}\,.$$

Thus, thanks to characterization of the maximum principle of J. López-Gómez and M. Molina-Meyer [12], it is apparent that $\sigma[E - \lambda W; \Omega^0_{a+}] > 0$, and, therefore, $\lambda \in (\lambda_1^{\Omega^0_{a+}}, \lambda_2^{\Omega^0_{a+}})$, by the strict concavity of $\lambda \mapsto \Sigma(\lambda)$. Thus, by the a priori bounds found by S. Cano-Casanova [2], there exists a constant $C > 0$ such that for any positive solution (λ, u_λ) of (5.1), $\|u_\lambda\|_{\mathcal{C}^\nu(\bar\Omega)} \leq C$. Now, let $M > 0$ be sufficiently large so that $\sigma[E + M; \Omega] > 0$ and $\lambda_1^\Omega W(x) + M > 0$ for each $x \in \bar\Omega$. By elliptic regularity, the positive solutions of (5.1) are given by the zeroes in $U := \mathcal{C}_0^\nu(\bar\Omega)$ of the equation

$$\mathfrak{L}(\lambda)u + \mathfrak{R}(\lambda, u) = 0\,, \tag{5.5}$$

where, for each $(\lambda, u) \in \mathbb{R} \times U$, we have denoted

$$\mathfrak{L}(\lambda)u := u - (E + M)^{-1}[(\lambda W + M)\,u]\,,$$

and

$$\mathfrak{N}(\lambda, u) := (E + M)^{-1}(au^r)\,.$$

It should be noted that the inverse operator $(E + M)^{-1} \in \mathcal{L}(U)$ is compact, by elliptic regularity and Ascoli-Arzela's theorem, and strongly order preserving, by the strong maximum principle; the space U being ordered by the cone of point-wise non-negative functions. Thus, (5.5) fits into the abstract setting of Section 1 with

$$\mathfrak{K}(\lambda)u := (E + M)^{-1}[(\lambda W + M)\,u]\,, \qquad u \in U\,.$$

By the choice of M, $\mathfrak{K}(\lambda_0)$ is strongly positive. Moreover, setting

$$\mathfrak{L}_0 := \mathfrak{L}(\lambda_0)\,, \qquad \mathfrak{L}_1 := \frac{d\mathfrak{L}}{d\lambda}(\lambda_0) = -(E + M)^{-1}(W\cdot)\,,$$

one has that $N[\mathfrak{L}_0] = \mathrm{span}[\varphi_0]$ and $\mathfrak{L}_1\varphi_0 \notin R[\mathfrak{L}_0]$. Indeed, if

$$\mathfrak{L}_0 u = -(E + M)^{-1}(W\varphi_0)$$

for some $u \in U$, then

$$u - (E + M)^{-1}[(\lambda_0 W + M)u] = -(E + M)^{-1}(W\varphi_0)$$

and, by elliptic regularity, $u \in \mathcal{C}_0^{2+\nu}(\bar\Omega)$. Thus, $(E - \lambda_0 W)u = -W\varphi_0$, which contradicts (5.4). Hence, the transversality condition of M. G. Crandall and P. H. Rabinowitz [4] is satisfied and, consequently, $\chi[\mathfrak{L}; \lambda_0] = 1$. Therefore, since λ_1^Ω and λ_2^Ω are the unique values of λ where positive solutions of (5.1) can bifurcate from $(\lambda, 0)$, as an immediate consequence from Theorem 1.1 the following result is obtained.

Theorem 5.1. There is a bounded component of the set of positive solutions of (5.1), say $\mathfrak{C}^P$, such that $(\lambda_1^\Omega, 0)$, $(\lambda_2^\Omega, 0) \in \bar{\mathfrak{C}}^P$. Moreover, $\mathcal{P}_\lambda\mathfrak{C}^P \subset (\lambda_1^{\Omega^0_{a+}}, \lambda_2^{\Omega^0_{a+}})$, where $\mathcal{P}_\lambda$ stands for the λ-projection operator.

It should be noted that, thanks to Theorem 4.1 and Theorem 5.1, (5.1) cannot admit an abstract representation as a fixed point equation of the form (1.1) with $\mathfrak{K}(\lambda) = \lambda K$ for some fixed compact strongly positive operator K, and, consequently, even if the unilateral results of P. H. Rabinowitz [13] would be correct as originally stated, Theorem 5.1 could not be a consequence from them.

6. Three Different Types of Bounded Components

Under the assumptions of Theorem 1.1, $\mathfrak{L}(\lambda)$ must possess two different eigenvalues $\lambda \in \mathfrak{S}$, at least λ_0 and λ_1, associated with each the operator has a positive eigenfunction. This is far from being true if, instead of assuming that λ_0 is a nonlinear eigenvalue, one assumes that $\chi[\mathfrak{L};\lambda_0] \in 2\mathbb{N}$, since, in this case, the component $\mathfrak{C}^P_{\lambda_0}$ might emanate from the curve $(\lambda, 0)$ exclusively at λ_0. Therefore, the oddity of the multiplicity is crucial for the validity of Theorem 1.1. Actually, there are boundary value problems of the form (5.1) that possess bounded components exhibiting each of these behaviors. For example, consider the one-dimensional prototype model in $\Omega = (0,1)$

$$\begin{cases} -u'' + \mu u = \lambda \sin(2\pi x)u - a(x)u^2\,, \\ u(0) = u(1) = 0\,, \end{cases} \tag{6.1}$$

where

$$a(x) = \begin{cases} -0.2\sin\left(\frac{\pi}{0.2}(0.2 - x)\right) & \text{if } 0 \le x \le 0.2\,, \\ \sin\left(\frac{\pi}{0.6}(x - 0.2)\right) & \text{if } 0.2 < x \le 0.8\,, \\ -0.2\sin\left(\frac{\pi}{0.2}(x - 0.8)\right) & \text{if } 0.8 < x \le 1\,, \end{cases} \tag{6.2}$$

and $(\lambda,\mu) \in \mathbb{R}^2$ are regarded as two real parameters. Note that $a > 0$ in $(0.2, 0.8)$, $a < 0$ in $(0, 0.2) \cup (0.8, 1)$, and $a(0) = a(0.2) = a(0.8) = a(1) = 0$. For an adequate choice of the parameter μ, this problem fits into the abstract setting of Section 5 by choosing $E_\mu := -\frac{d^2}{dx^2} + \mu$, $W := \sin(2\pi\,\cdot\,)$ and $r = 2$. Indeed, since $N = 1$, $\Omega^0_{a^+} = (0, 0.2) \cup (0.8, 1)$ and

$$\max_{\lambda\in\mathbb{R}} \sigma[E_\mu - \lambda W;\Omega] = \sigma[-\frac{d^2}{dx^2} + \mu;\Omega] = \pi^2 + \mu\,,$$

because of the symmetry of the problem around 0 (cf. [3] for further details), it turns out that condition (5.2) holds as soon as $\mu > -\pi^2$. Actually, for each $\mu > -\pi^2$, there exist $\lambda_1^\Omega(\mu) < 0 < \lambda_2^\Omega(\mu) = -\lambda_1^\Omega(\mu)$ such that

$$\sigma[E_\mu - \lambda_1^\Omega(\mu)W;\Omega] = \sigma[E_\mu - \lambda_2^\Omega(\mu)W;\Omega] = 0\,.$$

Moreover, As μ decreases approaching $-\pi^2$, $\lambda_1^\Omega(\mu)$ increases, and, hence, $\lambda_2^\Omega(\mu)$ decreases, approaching 0, i.e., $\lim_{\mu\downarrow-\pi^2} \lambda_1^\Omega(\mu) = 0 = \lim_{\mu\downarrow-\pi^2} \lambda_2^\Omega(\mu)$. As a result, Theorem 5.1 applies when $\mu > -\pi^2$ while it cannot be applied if $\mu \le -\pi^2$. Actually, the mapping $\lambda \mapsto \Sigma_\mu(\lambda) := \sigma[E_\mu - \lambda W;\Omega]$ satisfies $\Sigma_{-\pi^2}(0) = 0$, $\Sigma'_{-\pi^2}(0) = 0$, and $\Sigma_{-\pi^2}(\lambda) < 0$ for each $\lambda \in \mathbb{R} \setminus \{0\}$. Therefore, $\chi[E_{-\pi^2} - \lambda W; 0] = 2$ and Theorem 5.1 cannot applied to cover this *transition* situation, though the problem still possesses a bounded component of positive solutions emanating from $(\lambda, 0)$ at $\lambda = 0$ if $\mu = -\pi^2$ (cf. the second plot of Figure 6.1). When $\mu < -\pi^2$ there are no bifurcation points to positive solutions from $(\lambda, 0)$ and, actually, if μ is sufficiently close to $-\pi^2$, then (3.1) exhibits an *isola* of positive solutions (cf. the third plot of Figure 6.1). The bifurcation diagrams of Figure 6.1 were computed by coupling a pseudo-spectral method with collocation and a path-continuation solver (cf. [3]). The left plot of Figure 6.1 shows the component for $\mu = 0$. In this case, $\lambda_1^\Omega \sim -28.0233$ and $\lambda_2^\Omega \sim 28.0233$. The central plot of Figure 6.1 show the perturbations of the positive solutions of the left plot as μ decreases from zero up to reach the value $\mu = -9.8693 > \pi^2 = -9.86960\ldots$. Now, $\lambda_1^\Omega \sim -0.13861$ and $\lambda_2^\Omega \sim 0.13861$; as these values are very close, the central plot of Figure 6.1 shows them super-imposed. As the computational model is discrete and π^2 is irrational there is no way to get the bifurcation diagram for $\mu = -\pi^2$, though it must be very similar to the central diagram. The right plot shows the isola of solutions obtained for $\mu = -40$, for which $(\lambda, 0)$ always is linearly unstable.

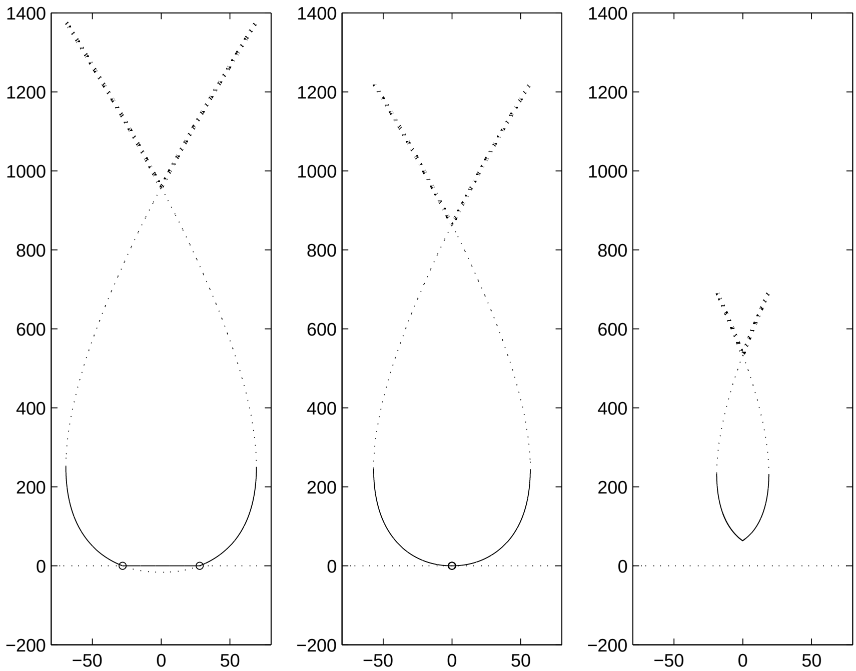

Figure 6.1 Three components of positive solutions for $\mu = 0, -9.8693, -40$, respectively.

In Figure 6.1 we are plotting the value of λ against the L_∞-norm of the corresponding positive solution. Stable solutions are indicated by solid lines, unstable by dotted lines. As there are some ranges of values of λ where the model possesses at least two solutions with very similar L_∞-norms, the plot did not allow us distinguishing them, but rather plotted twice these pieces. This is why the bifurcation diagrams exhibit a darker arc of curve.

It should be clear that in case $\mu = -\pi^2$ the component of positive solutions of (3.1) bifurcating from $(\lambda, 0)$ at $\lambda = 0$ must be bounded and that it emanates from the curve $(\lambda, 0)$ exclusively at $\lambda = 0$.

References

[1] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), 620–709.

[2] S. Cano-Casanova, *Compact Components of positive solutions for Superlinear Indefinite Elliptic Problems of Mixed Type* // Top. Meth. Non. Anal. **23** (2004), 45–72.

[3] S. Cano-Casanova, J. López-Gómez, and M. Molina-Meyer, *Isolas: compact solution components separated away from a given equilibrium curve*, Hiroshima Math. J. **34** (2004), 177-199.

[4] M. G. Crandall, and P. H. Rabinowitz, *Bifurcation from simple eigenvalues* // J. Funct. Anal. **8** (1971), 321–340.

[5] E. N. Dancer, *Global solution branches for positive mappings* // Arch. Rat. Mech. Anal. **52** (1973), 181–192.

[6] E. N. Dancer, *Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one* // Bull. London Math. Soc. **34** (2002), 533–538.

[7] J. Esquinas, *Optimal multiplicity in local bifurcation theory, II: General case* // J. Diff. Eqns. **75** (1988), 206–215.

[8] J. Esquinas, and J. López-Gómez, *Optimal multiplicity in local bifurcation theory, I: Generalized generic eigenvalues* // J. Diff. Eqns. **71** (1988), 72–92.

[9] P. Hess, and T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function* // Comm. Part. Diff. Eqns. **5** (1980), 99–1030.

[10] M. G. Krein, and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space* // Amer. Math. Soc. Transl. **10** (1962), 199–325.

[11] J. López-Gómez, [2001] *Spectral Theory and Nonlinear Functional Analysis*, Research Notes in Mathematics 426, CRC Press, Boca Raton 2001.

[12] J. López-Gómez, and M. Molina-Meyer, *The maximum principle for cooperative weakly elliptic systems and some applications* // Diff. Int. Eqns. **7** (1994), 383–398.

[13] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems* // **7** (1971), 487–513.

Contact information

Santiago Cano-Casanova
Departamento de Matemática Aplicada y Computación
Universidad Pontificia Comillas de Madrid
28015-Madrid,
Spain
E-Mail: `scano@dmc.icai.upco.es`

Julián López-Gómez
Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040-Madrid,
Spain
E-Mail: `Lopez_Gomez@mat.ucm.es`

Marcela Molina-Meyer
Departamento de Matemáticas
Universidad Carlos III de Madrid
28911-Leganés, Madrid,
Spain
E-Mail: `mmolinam@math.uc3m.es`