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Vanishing Viscosity Method and Diffusion Processes

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Abstract. We construct diffusion processes associated with nonlinear parabolic equations and study their behavior as the viscosity (diffusion) coefficients go to zero. It allows to construct regularizations for solutions to hyperbolic equations and systems and study their vnishing viscosity limits.

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1. Introduction

The vanishing viscosity method (v.v.m) is a very popular approach to construct the solution to the Cauchy problem for nonlinear hyperbolic equations and systems. This method was proved to be especially effective for the investigation of the solution to the Cauchy problem for the Burgers equation

$$u_t + uu_x = \frac{\varepsilon^2}{2} u_{xx}, \quad u(0,x) = u_0(x), \quad x \in R^1 \tag{1.1}$$

and systems associated with it. In this case one can give the explicit formula for the solution to (1.1)

$$u(t,x) = -\varepsilon^2 \frac{\phi_x(t,x)}{\phi(t,x)}, \quad t \ge 0, \quad x \in R^1 \tag{1.2}$$

where ϕ is the solution to the Cauchy problem for the heat equation

$$\phi_t = \frac{\varepsilon^2}{2} \phi_{xx}, \quad \phi(0,x) = \phi_0(x)$$

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and use it for further investigation and in particular for studying the vanishing viscosity limit as $\varepsilon \to 0$. The transformation (1.2) discovered by Hopf [11] is called the Cole-Hopf transformation. It was and intensively used by many authors (see references in [14]).

Another direction to use the vanishing viscosity method (v.v.m) to study solutions of the Cauchy problem for nonlinear hyperbolic equations and systems (started by Gelfand [10]) is based on the investigation of a special Cauchy problem. It allows to consider rather general equations and systems and construct some special solutions for them (simple or plane waves or self-similar solutions). This approach is based on the possibility to reduce this special Cauchy problem for a nonlinear parabolic equation or system to a boundary problem for an ordinary second order differential equation or system. Namely, it works for example if one considers the Cauchy problem

$$u_t^\varepsilon + \operatorname{div} F(u^\varepsilon) = \frac{\varepsilon^2}{2}\Delta u^\varepsilon$$

with special initial data

$$u^\varepsilon(0,x) = u_0(x) = \begin{cases} u_-, & \text{if (x,h)<0,} \\ u_+, & \text{if (x,h)>0} \end{cases} \tag{1.3}$$

(called the Riemann problem) for some $h \in R^d$ and looks for a plane wave solution $u^\varepsilon(t,x) = v(\frac{(x,h)-\sigma t}{\varepsilon})$. In this case the function $v(\zeta)$ where $\zeta = \frac{(x,h)-\sigma t}{\varepsilon}$ solves the boundary problem

$$[-\sigma + D_v F(v)]v_\zeta = \frac{1}{2}v_{\zeta\zeta} \tag{1.4}$$

$$v(-\infty) = u_-, \quad v(\infty) = u_+, \quad \lim_{z\to\pm\infty} v_z = 0$$

where the constant σ should be defined. Integrating (1.4) we get

$$v_\zeta = F(v) - \sigma v + C$$

for some constant vector C and

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = \begin{cases} u_-, & \text{if } (x,h) - \sigma t < 0, \\ u_+, & \text{if } (x,h) - \sigma t > 0, \end{cases}$$

keping in mind boundary conditions. Finally, we derive

$$F(u_-) - F(u_+) = \sigma[u_- - u_+]$$

to define σ.

In this paper we suggest an alternative interpretation of the v.v.m based on the probabilistic interpretation of the solution to the Cauchy problem for parabolic equations and systems. It allows to study a Cauchy problem with rather general initial data for nonlinear parabolic equations and systems and to obtain more information about the vanishing viscosity limit of its solution. This approach allows to consider various diffusion perturbations for systems of conservation laws and balance laws which correspond to physical nature of these systems.

A probabilistic approach to the Cauchy problem for a class of linear parabolic systems having diagonal higher order terms (the corresponding coefficient matrix is proportional to the unity matrix) and non-diagonal lower order terms was developed by Yu. Dalecky [7], [8] and D. Stroock [13]. The approach was based on the notion of a multiplicative operator functional of a diffusion process. The investigation of nonlinear scalar parabolic equations with the help of solutions to stochastic differential equations was suggested by H. McKean [12] and M. Freidlin [9] and extended to nonlinear systems by Yu. Dalecky and Ya. Belopolskaya [2]–[4].

Note that the probabilistic interpretation of v.v.m. perfectly works if one considers a quasilinear parabolic equation. But the situation drastically changes if one considers a system of such equations.

In general due to the structure of the probabilistic representation of the solution to the Cauchy problem of the nonlinear parabolic system, the first order non-diagonal terms should degenerate together with the second order terms. An attempt to preserve first order non-diagonal terms leads to consideration of a singular stochastic equation and seems to be hopeless. Recently it was discovered in [5] that there exists a class of parabolic systems that allows to construct the probabilistic counterpart of the vanishing viscosity method.

For the simplicity we restrict ourselves with the investigation of the Cauchy problem for nonlinear parabolic and hyperbolic systems with one dimensional spatial variable though in general the method does not require this restriction. The multidimensional systems will be considered elsewhere.

Note that the probabilistic approach of [2] is quite different from the approach by Kruzkov based on the introduction of additional space variable while studying the Cauchy problem for the scalar equation $u_t + F(u_x) = 0$.

2. Nonlinear Parabolic Equations and Diffusion Processes

We recall the construction of diffusion processes and multiplicative functionals of these processes associated with nonlinear parabolic equations and systems. More details can be found in [2], [3].

Consider the Cauchy problem for a scalar nonlinear parabolic equation

$$\frac{\partial f}{\partial t} + (a(x,f),\nabla)f = \frac{1}{2}TrA^*(x,f)f''A(x,f), \quad f(0,x) = f_0(x). \quad (2.1)$$

Here $a(x,f) \in R^d,\ A(x,f) \in L(R^d),\ x \in R^d,\ f \in R^1$.

In the sequel we use C, K to denote absolute constants (if a constant depend on a real parameter f we denote it by K_f). We say that condition **C 2.1** holds if

$$\|a(x,f)\|^2 + \sigma^2(A(x,f))\| \leq C[1 + \|x\|^2 + K_f\|f\|^p],$$

$$\|a(x,f)-a(y,f_1)\|^2+\sigma^2(A(x,f)-A(y,f_1)) \leq L\|x-y\|^2+C_{f,f_1}\|f-f_1\|^2$$

where $f, f_1 \in R^1$, $\sigma^2(A) = \sum_{k=1}^n \|Ae_k\|^2$, $\{e_k\}_{k=1}^d$ is the orthonormal basis in R^d and

$$\sup_x \|f(x)\|^2 \leq K_0, \quad \sup_x \|\nabla f(x)\|^2 \leq K_0.$$

Besides we assume that v_0 is a C^2 smooth function.

We use below the following notations $a_f(\xi(\tau)) = a(\xi(\tau), f(t-\tau, \xi(\tau)))$ for $0 \leq \tau \leq t$.

Let $w(t) \in R^d$ be a Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$. For any random variable $\xi \in R^d$ we denote by

$$E\xi = \int_\Omega \xi(\omega)P(d\omega)$$

the expectation of ξ. Let $\mathcal{F}_t \subset \mathcal{F}$ be a set of $w(t)$-adapted σ-subalgebras then for a function $f : R^d \to R^{d_1}$ $E_s f(w(t)) = E\{f(w(t))|\mathcal{F}_s\}$ denotes the conditional expectation.

To construct the solutions to (2.1) we reduce it to the stochastic system

$$d\xi = -a_f(\xi(\tau))\,d\tau + A_f(\xi(\tau))\,dw(\tau), \quad \xi(0) = x, \quad (2.2)$$

$$f(t,x) = Ef_0(\xi_x(t)). \quad (2.3)$$

and construct the solution to it by the successive approximation method.

To this end we consider stochastic equations

$$d\xi^k(\tau) = -a_{f^k}(\xi^k(\tau))\,d\tau + A_{f^k}(\xi^k(\tau))\,dw(\tau), \quad \xi^k(0) = x, \tag{2.4}$$

where

$$f^0(t,x) = f_0(x), \quad f^{k+1}(t,x) = Ef_0(\xi^k(t)). \tag{2.5}$$

To prove the convergence of $(\xi^k(t), f^k(t,x))$ satisfying (2.4), (2.5) to a limit $(\xi(t), f(t,x))$ as $k \to \infty$ we need a number of auxiliary estimates.

Let $\mathcal{L}$ be the subspace of the space $C_0(R^1 \times R^d, R^1)$ consisting of Lipschitz continuous functions f such that

$$|f(t,x) - f(t,y)|^2 \le L_f(t)\|x-y\|^2, \quad t \in [0,T] \quad x, y \in R^d.$$

Denote by $[K_f]^{\frac{1}{2}} = \|f\|_{\mathcal{L}} = \sup_{x\in R^d} |f(x)|$ the norm of the element $f \in \mathcal{L}$.

Let $v(s,x)$ be a scalar function such that $\|v(s,x)\|^2_{\mathcal{L}} \le K_v(s) < \infty$, $|v(s,x) - v(s,y)|^2 \le L_v(s)\|x-y\|^2$, $L_v(s), K_v(s) < \infty$ for $s \in [0,T]$.

Consider the stochastic equation

$$\xi(t) = x + \int_s^t a_v(\xi(\tau))\,dt + \int_0^t A_v(\xi(\tau))\,dw. \tag{2.6}$$

We use the notation $\xi_{s,x,v}(t)$ for the solution of this equation if we are interested in the particular dependence of the process $\xi(t)$ on these parameters.

Lemma 2.1. *Assume that* **C 2.1** *holds. Then the solution $\xi_{x,v}(t)$ of* (2.6) *satisfies the following estimates*

$$E\|\xi(t)\|^2 \le \left[\|x\|^2 + \int_0^t K_v^p(\tau)\,d\tau\right]e^{Ct},$$

$$E\|\xi_{x,v}(t) - \xi_{y,v}(t)\|^2 \le \|x-y\|^2 e^{\int_s^t [L+L_v(\tau)]d\tau}. \tag{2.7}$$

$$E\|\xi_{x,v}(t) - \xi_{x,v}(t)\|^2 \le \int_s^t [v(\tau) - v_1(\tau)\|_{\mathcal{L}}\,d\tau e^{\int_s^t L_v(\tau)d\tau}. \tag{2.8}$$

In addition for $f(t-s,x) = Ef_0(\xi_{s,x}(t))$ the estimates

$$\|f(t-s)\|_{\mathcal{L}} \le K_0$$

and

$$|f(t-s,x)-f(t-s,y)|^2 \leq L_f(t-s)\|x-y\|^2 \exp\left[L(t-s)+\int\limits_s^t L_f(\tau)\,d\tau\right] \tag{2.9}$$

hold.

The proof of these estimates is standard and based on the properties of stochastic integrals (see [3]).

Let $L_f(t)$ and $K_f(t)$ be minimal constants such that $\|f(t,x)-f(t,y)\|^2 \leq L_f(t)\|x-y\|^2$ and $\|f(t)\|^2_{\mathcal{L}} \leq K_f(t)$ hold.

Lemma 2.2. *Let* **C 2.1** *hold. Then there exists an interval* $[0,T]$ *and functions* $\alpha(t)$*,* $\beta(t)$ *bounded over this interval and such that*

$$\|f(t)\|^2_{\mathcal{L}} \leq \alpha(t), \quad |f(t,x)-f(t,y)|^2 \leq \beta(t)\|x-y\|^2. \tag{2.10}$$

if $\|v(t)\|^2_{\mathcal{L}} \leq \alpha(t)$ *and* $|v(t,x)-v(t,y)|^2 \leq \beta(t)\|x-y\|^2$.

Proof. Under **C 2.1** we can choose $\alpha(t) = K_0$. To prove the second estimate in (2.10) we notice that

$$L_f(t-s) \leq K_0^1 e^{\int_s^t [L+L_f(\tau)]d\tau} \tag{2.11}$$

results from (2.9).

We choose for β the solution to the equation

$$\beta(t-s) = K_0^1 \exp\left[L(t-s)+\int\limits_s^t \beta(\tau)\,d\tau\right]. \tag{2.12}$$

and notice that β solves the Cauchy problem

$$\frac{d\beta(t-s)}{ds} = [L+\beta(t-s)]\beta(t-s), \quad \beta(0) = K_0^1$$

and admits the explicit representation

$$\beta(t-s) = \frac{K_0^1 L e^{L(t-s)}}{L+K_0^1-K_0^1 e^{L(t-s)}}. \tag{2.13}$$

Hence for $t,s \in [0,T]$, $t \leq s$ with T satisfying the estimate

$$T < \frac{1}{L} \ln \frac{L+K_0^1}{K_0^1} \tag{2.14}$$

the function β is bounded and meets the demands of the lemma. $\square$

Coming back to the system (2.5) we can prove the following statement.

Theorem 2.1. *Assume that* **C 2.1** *holds. Then the functions* $f^k(t)$ *determined by* (2.5) *uniformly converge as* $k \to \infty$ *to a limiting function* $f(t)$ *for all* $t \in [0,T]$ *with* T *satisfying* (2.13).

Proof. By Lemma 2.2 we know that the mapping $\Phi(s,x,v) = Eu_0(\xi_{s,x}(t))$ acts in the space $\mathcal{L}$. Let

$$\kappa^k(t-s,x) = |f^{k+1}(t-s,x) - f^k(t-s,x)| \quad \text{and} \quad \zeta^k(t-s) = \sup_x \kappa^k(t-s,x).$$

By the estimates of Lemma 2.2 we have

$$\kappa^k(t-s,x) \le L_{f0}^2 \int_s^t \|f^k(t-\tau) - f^{k-1}(t-\tau)\|\, d\tau e^{L_f(t-s)}$$

and hence

$$\zeta^k(t-s) \le \delta^k \int_s^t \dots \int_s^{t_2} \|f^1(t-\tau_1) - f^0\|^2\, d\tau_1 \dots d\tau_k$$

holds with $\delta = K_0 L_{f0}^2 \exp[L_f(t-s)]$.

Since f^k are uniformly bounded and

$$\|f^1(t-s,\cdot) - f^0(\cdot)\|_{\mathcal{L}} \le \text{const} < \infty,$$

we get

$$\|f^k(t,\cdot) - f^{k-1}(t,\cdot)\|_{\mathcal{L}} \le \frac{N^k}{k!}\,\text{const}$$

where $N = \delta(t-s)$. It results that for each $t \in [0,T]$ the family $f^k(t,\cdot)$ uniformly converges to a limiting function $f(t,\cdot)$. In addition it is easy to check that $f(t,x)$ is Lipschitz continuous in x. In fact by Lemma 2.2 for each $t \in [0,T]$ we have

$$|f^k(t,x) - f^k(t,y)| \le \beta(t)\|x-y\|$$

where $\beta(t)$ is determined by (2.12) and the estimate is uniform in k.

To prove that the above constructed solution is unique we assume on the contrary that there exist two solutions $f_1(t,x)$, $f_2(t,x)$ to (2.2), (2.3) possessing the same initial data $f_1(0,x) = f_2(0,x) = f_0(x)$. It results from Lemma 2.1 that

$$\|f_1(t,\cdot) - f_2(t,\cdot)\|_{\mathcal{L}} \le \int_0^t \|f_1(\tau,\cdot) - f_2(\tau,\cdot)\|_{\mathcal{L}}\, d\tau$$

and hence $\|f_1(t,\cdot) - f_2(t,\cdot)\|_{\mathcal{L}} = 0$. Finally, we know that stochastic equations with Lipschitz coefficients have unique solutions to the Cauchy problem that yields the uniqueness of the solution to the system (2.2), (2.3).

In addition notice that the uniqueness of the Markov process $\xi(t)$ which solves (2.2) leads to the equality $f_1(t,x) = Ef_0(\xi(t)) = f_2(t,x)$. $\square$

As the result we get the following assertion.

Theorem 2.2. *Let* **C 2.1** *holds. Then there exists the interval* $[0,T]$ *determined by* (2.13) *and the function* $f(t)$ *given by* (2.3) *is the unique in* $\mathcal{L}$ *weak solution to the Cauchy problem* (2.1) *defined on* $[0,T]$.

Remark 2.1. We call f a weak solution to (2.1) because if we succeed to prove that $f(t)$ belongs to $C^2(R^d)$ then we can prove that (2.3) gives a unique classic solution to (2.1) defined on $[0,T]$.

Consider next the Cauchy problem for the parabolic system

$$\frac{\partial u_k}{\partial t} + B^i_{kl}(u)\nabla_i u_l + c_{kl}(u)u_l = \frac{1}{2}TrA^*(u)u''_k A(u),$$

$$u_k(0,x) = u_{0k}(x), \quad k = 1,2,\ldots,d_1, \quad x \in R^d. \tag{2.15}$$

Notice that if $B^i_{kl}(u) = u^i\delta_{kl}$, $c(u) = 0$, $A = \sigma I$ and $d = d_1 = 1$ then (2.15) coincides with the Burgers equation (1.1) and the approach to construct its solution developed in this section is an alternative to one connected with the Cole-Hopf transformation mentioned in the introduction. Note that in the multidimensional case it allows to consider the Cauchy problem for the Burgers equation without assumption that the initial data should be a potential vector field which is necessary due to (1.2) in the application of the Cole-Hopf transformation.

Note that one can treat the system (2.15) as a scalar equation of the form

$$\frac{\partial \Phi}{\partial s} + \nabla_{M(y)}\Phi = \frac{1}{2}D^{ik}(y)\frac{\partial \Phi}{\partial y_i \partial y_k}, \quad \Phi(T,y) = \Phi_0(y) = \langle h, u_0(x)\rangle. \tag{2.16}$$

with respect to the scalar function $\Phi(s,y) = \langle h, u(s,x)\rangle$, $y = (x,h) \in R^q$, where $\langle\cdot,\cdot\rangle$ is the inner product in R^{d_1}. Here $M = (0,c) \in R^q$, $D = \begin{pmatrix} F & B \\ B & 0 \end{pmatrix}$ is a $q \times q$ matrix and $F = A^*A$.

Below we state the conditions which guarantee that the coefficients M, D of this auxiliary scalar equation obey the estimates that ensure the existence and uniqueness of the solution to the Cauchy problem (2.16) and hence to the above system (2.15) as a consequence of a fixed point theorem.

Namely we construct a solution $\Phi(t,z)$, $z=(x,h)\in Z$ which belongs to the set Θ of continuous functions $\Phi(x,h)$ linear in the argument $h\in R^{d_1}$ and bounded for a fixed h. By this reason we meet additional difficulties due to the necessity to prove that successive approximations are uniformly bounded that was granted before by the bounded initial data.

Note that this idea slightly resembles the idea to introduce additional space variables used by Kruzkov to study quasi-linear scalar parabolic equations.

Consider the set $\Theta=\{\Phi:\Phi(z)=\langle h,\phi(x)\rangle$, where $\phi\in C_b(R^d,R^{d_1})$ are Lipschitz continuous functions. It is a linear space with the norm

$$\|\Phi\|_\Theta=\sup_{\|h\|=1}\sup_{x\in R^d}|\langle h,\phi(x)\rangle|.$$

Here and below we identify R^{d_1} with its dual $R^{d_1}=(R^{d_1})^*$. Denote by Θ_1 the linear space $\Theta_1=C_b(R^d,R^{d_1})$ with the norm $\|\phi\|_{\Theta_1}=\sup_{x\in R^d}\|\phi(x)\|$. It should be mentioned that $\|\Phi\|_\Theta=\|\phi\|_{\Theta_1}$.

Let $c(u)\in R^{d_1},\ C(u)y\in L(R^{d_1}),\ x,y\in R^d,\ u\in R^{d_1}$. To prove the convergence of the successive approximations we need some restrictions on the coefficients of (2.16).

We say that **C 2.2** holds if **C 2.1** holds, $B=CA$ and besides the following estimates are valid

$$\langle c(f)h,h\rangle+\sigma^2\|C(f)h\|\le\left[C_0+C_1\|f\|^p\right]\|h\|^2,$$

$$\left\|[c(f)-c(f_1)]h\right\|^2+\sigma^2\big([C(f)-C(f_1)]h\big)\le\left[M_{f,f_1}\|f-f_1\|^2\right]\|h\|^2$$

where $C_1>0,\ C_0$ are absolute constants and M_{f,f_1} is a positive constant depending on the maximum of norms of functions f and f_1. In addition we assume that all coefficients are smooth and their derivatives satisfy the above estimates.

Consider the system of stochastic equations

$$d\xi=A(u(t-\tau,\xi(\tau))\,dw(\tau),\quad \xi(0)=x, \tag{2.17}$$

$$\begin{aligned} d\eta(\tau)=&-c(u(t-\tau,\xi(\tau))\eta(\tau)\,d\tau\\ &-C(u(t-\tau,\xi(\tau))(\eta(\tau),dw(\tau)),\quad \eta(0)=h, \end{aligned}\tag{2.18}$$

$$\langle h,u(t,x)\rangle=E\langle\eta(t),u_0(\xi(t)\rangle. \tag{2.19}$$

We construct the solution to (2.17)–(2.19) by a successive approximation method.

Let

$$u^0(t,x) = u_0(x), \quad \xi^0(t) = x, \tag{2.20}$$

$$d\xi^k = A(u^k(t-\tau, \xi^k(\tau))\, dw(\tau), \quad \xi^k(0) = x, \tag{2.21}$$

$$\begin{aligned} d\eta^k(t) = c(u^k(t-\tau, \xi^k(\tau))\eta^k(\tau)\, d\tau \\ + C(u^k(t-\tau, \xi^k(\tau))(\eta^k(\tau), dw(\tau)), \quad \eta^k(0) = h, \end{aligned} \tag{2.22}$$

$$\langle h, u^{k+1}(t,x)\rangle = E\langle \eta^k(t), u_0(\xi^k(t))\rangle. \tag{2.23}$$

To prove that the family $u^k(t)$ is uniformly bounded or what is the same to prove the corresponding property for $\Phi^k(t,z) = \langle h, u^k(t,x)\rangle$ for all h such that $\|h\| = 1$ we introduce the two component Wiener process $W(t) = (w(t), w(t))$ with identical components $w(t) \in R^d$ and prove the following assertion.

Lemma 2.3. *Let* **C 2.2** *hold,* $u_0 \in \Theta_1$ *and* $\Psi(t,z)$ *be determined by*

$$\Psi(t,z) = E\Phi_0(\zeta_{0,z}(t)), \tag{2.24}$$

where $\zeta(t)$ *solves the Cauchy problem for the stochastic differential equation*

$$d\zeta = M_\Phi(\zeta(t))\, dt + D_\Phi(\zeta(t))\, dW, \quad \zeta(0) = z. \tag{2.25}$$

Then there exists an interval Δ_1 *such that* $\Psi(t) \in \Theta$ *for all* $t \in \Delta_1$ *if* $\Phi \in \Theta$*. In other words the function* $g(t,x)$ *determined by*

$$\langle h, g(t,x)\rangle = E\langle \eta(t), u_0(\xi(t))\rangle, \tag{2.26}$$

where the processes $(\xi(t), \eta(t))$ *solve the Cauchy problems*

$$d\xi = A_v(\xi(\tau))\, dw, \quad \xi(s) = x,$$

$$\begin{aligned} d\eta = -c(v(t-\tau, \xi(\tau))\eta(\tau)\, d\tau \\ + C(v(t-\tau, \xi(\tau))(\eta(\tau), dw(\tau)), \quad \eta(s) = h \end{aligned} \tag{2.27}$$

belongs to Θ_1 *if* $v \in \Theta_1$ *for all* $t \in \Delta_1$*.*

To prove that $u(t)$ has a uniformly bounded Lipschitz constant if $v(t)$ possesses this property we consider the system obtained from ()() by formal differentiation in x variable. The resulting system has the same structure as (2.15) and hence we can apply the above approach to prove the uniform boundedness of $\nabla u(t,x)$ on a certain interval $[0, T_2]$ that yields the following statement.

Theorem 2.3. *Assume that* **C 2.2** *holds. Then there exists an interval* $[0,\tau]$ *with the length bounded*

$$|\tau| < \frac{1}{2C_0+3C_1}\ln\Big[1+\frac{2C_0+3C_1}{3C_1K_{u_0}}\Big]$$

such that for all $t \in [0,\tau]$ *there exists a unique bounded solution to the Cauchy problem* (2.15). *If in addition* $2C_0+3C_1=0$*, then* $|\tau| < [3C_1K_{u_0}]^{-1}$. *If in addition the solution to* (2.15) *constructed in this way is smooth enough it represents a unique classical solution defined on a (possibly smaller) interval* $[0,\tau_1]$.

3. Vanishing Viscosity Limit

Unfortunately the approach of the previous section does not allow to deal with the vanishing viscosity limit immediately. Actually the **C 2.2** says that $B = CA$ and hence if A goes to zero then B goes to zero too. To avoid this situation we can take $A^\varepsilon = \varepsilon A$ and $C^\varepsilon = \varepsilon^{-1}C$ but in this case coefficients in (2.18) come to be singular. To overcome this obstacle we need more assumptions about $B^i_{lm},\ l,m = 1,\dots,d_1,\ i = 1,\dots,d$ in (2.15). For the simplicity we restrict ourselves to the case $d = 1$.

Consider the Cauchy problem

$$u_t + a(u)u_x + B(u)u_x = \frac{\varepsilon^2}{2}A^2(u)u_{xx}, \quad u(0,x) = v(x) \in R^{d_1}. \tag{3.1}$$

We say that the condition **C 3** holds if $A(u)$ and $B(u)$ satisfy **C 2.2**, $A(u) > 0$ and the matrix $B_{kl}(u),\ k,l = 1,\dots d_1$ has eigenvectors l_α, $\alpha = 1,\dots,d_1$ corresponding to distinct real eigenvalues $\lambda_\alpha(u)$, $Cl_\alpha = \lambda_\alpha l_\alpha$ $(\lambda_\alpha \neq \lambda_\beta)$, $\alpha,\beta = 1,\dots,d_1$ if $\alpha \neq \beta)$ $\langle l_\alpha, l_\beta\rangle = \delta_{\alpha\beta}$, where $\delta_{\alpha\beta}$ is the Kronecker symbol. In particular to ensure that **C 3** holds it is enough to assume B to be symmetric.

Theorem 3.1. *Let* **C 3** *holds. Then the solution* u^ε *to the Cauchy problem* (3.1) *admits the representation*

$$u^\varepsilon(s,x) = \sum_{\beta=1}^{d_1} \tilde u^\varepsilon_\beta(t,x) l_\beta,$$

where

$$\tilde u^\varepsilon_\beta(t,x) = E\tilde v_\beta(\gamma^\varepsilon_\beta(t)) = \int_{-\infty}^{\infty} v_\beta(y)\mu^{\gamma^\varepsilon}(t,dy),$$

the diffusion process $\gamma^\varepsilon_\beta(t) \in R^d$ satisfies the stochastic equation

$$\gamma^\varepsilon_\beta(t) = x - \int\limits_s^t \big[a(\gamma^\varepsilon(\tau), u^\varepsilon(t-\tau, \gamma^\varepsilon_\beta(\tau)) - \lambda_\beta(u^\varepsilon(t-\tau, \gamma^\varepsilon_\beta(\tau))\big]\, d\tau \\ + \varepsilon[w(t) - w(s)]. \quad (3.2)$$

and $\mu^{\gamma^\varepsilon}(t, dy) = P\{\gamma^\varepsilon(t) \in dy\}$ is a probability distribution of the process $\gamma^\varepsilon(t)$.

Proof. The process $\eta^\varepsilon(t)$ governed by

$$d\eta(\tau) = \varepsilon^{-1} C(u(t-\tau, \xi(\tau))(\eta(\tau), dw(\tau)), \qquad \eta(0) = h, \quad (3.3)$$

gives rise to a multiplicative operator functional $G^\varepsilon(t; \xi(\cdot))$ which can be represented in the form $\eta^\varepsilon(t) = G^\varepsilon(t; \xi^\varepsilon(\cdot))h = \exp[G^\varepsilon(t; \xi^\varepsilon(\cdot))]h$, where

$$G^\varepsilon(t) = \frac{1}{\varepsilon} \int\limits_0^t \sum_{i=1}^n C^i\, dw^i(\tau) - \frac{1}{2\varepsilon^2} \int\limits_0^t \sum_{i=1}^n [C^i]^2\, d\tau. \quad (3.4)$$

Choosing $h = l_\alpha$ in (2.2), we obtain

$$\langle l_\alpha, u(t,x)\rangle = E\langle \eta^\varepsilon_{l_\alpha}(t), u_0(\xi^\varepsilon(t))\rangle \\ = \Big\langle l_\alpha, E\Big\{\exp\Big[\int\limits_0^t \varepsilon^{-1}(\lambda_\alpha(\xi^\varepsilon(\tau)), dw(\tau)) \\ - \frac{\varepsilon^{-2}}{2} \int\limits_0^t \|\lambda_\alpha(\xi^\varepsilon(\tau))\|^2\, d\tau\Big] u_0(\xi^\varepsilon(t))\Big\}\Big\rangle. \quad (3.5)$$

By the Girsanov formula and condition **C 2.2** we derive from (3.5) that

$$E\langle \eta^\varepsilon_{l_\alpha}(t), v_(\xi^\varepsilon(t))\rangle = \Big\langle l_\alpha, E\sum_{\beta=1}^{d_1} \tilde v_\beta(\gamma^\varepsilon_\beta(t)) l_\beta\Big\rangle = E\tilde v_\alpha(\gamma^\varepsilon_\alpha(t)), \quad (3.6)$$

where $\gamma^\varepsilon_\alpha(t)$ satisfies (3.2). It is easy to derive from (3.6) that $\tilde u^\varepsilon_\alpha(s,x) = E\tilde v_\alpha(\gamma^\varepsilon_\alpha(t))$ and finally, $u(s,x) = \sum_{\alpha=1}^{d_1} E\, \tilde v_\alpha(\gamma^\varepsilon_\alpha(t)) l_\alpha$. □

Corollary 3.1. *The process $\gamma^\varepsilon_\alpha(t)$, governed by* (3.2) *continuously depends on ε and the limit $\gamma_\alpha(t) = P - \lim_{\varepsilon\to 0} \gamma^\varepsilon_\alpha(t)$ exists. In addition $\gamma_\alpha(t)$ coincides with the characteristic curve of the hyperbolic equation*

$$\frac{\partial u_\alpha}{\partial s} + \nabla_{a(x)+\lambda_\alpha} u_\alpha = 0 \quad (3.7)$$

and the solution u to the Cauchy problem

$$u_t + a(u)u_x + B(u) \circ \nabla u = 0, \quad u(0,x) = v(x) \tag{3.8}$$

has the form $u(t,x) = \sum_{\alpha=1}^{d_1} \tilde{v}_\alpha(\gamma_\alpha(t)) r_\alpha$.

The proof of the Corollary statements follows immediately from (3.2) and other results of Theorem 3.1

Final remarks. As a conclusion we discuss some possible developments and applications of the above approach.

1. The approach discussed in Sections 2, 3 allows to construct local (in time) continuous solutions of parabolic and hyperbolic systems. The approach might be used as well to construct more complicated solutions of hyperbolic systems by v.v.m. including both continuous and singular parts (shock waves and even δ-shock waves) which evolve along a discontinuity line. To construct the equation for the discontinuity line by v.v.m. one can use special properties of averaged trajectories of diffusion processes (see[1]).

2. The above approach is based on the so called backward Kolmogorov equations for diffusion processes. One can consider the similar approach based on forward Kolmogorov equations which might be more corresponding to real physical systems. In this case one can use the duality between forward and backward Kolmogorov equations as a background for comparing the probabilistic weak solutions of the Cauchy problem for nonlinear parabolic equations and systems and weak solutions that satisfy the integral identity.

3. Based on the results from [5] one can use the above approach to construct the probabilistic representation for the solution of the Cauchy problem for parabolic systems with a more complicate higher order part. Namely, under some additional assumptions one can consider the system with nondiagonal higher order part and use it for studying the corresponding hyperbolic system by v.v.m. See for example [6].

4. One can see as an example of a system that satisfies the above assumptions the system of gas dynamic equations in the phase space R^d. In the case $d = 1$ the above approach was applied to this system in [1]. The case $d = 3$ will be considered elsewhere.

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