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Tietze Extension Theorem for Ordered Fuzzy G_δ-extremally Disconnected Spaces

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Abstract. In this paper, a new class of fuzzy topological spaces called ordered fuzzy G_δ-extremally disconnected spaces is introduced. Tietze extension theorem for ordered fuzzy G_δ-extremally disconnected spaces has been discussed as in [10] besides proving several other propositions and lemmas.

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Introduction

The fuzzy concept has invaded almost all branches of mathematics since the introduction of the concept by L. A. Zadeh [11]. Fuzzy sets have applications in many fields such as information [7] and control [8]. The theory of fuzzy topological space was introduced and developed by C. L. Chang [5] and since then various notions in classical topology have been extended to fuzzy topological space [3, 4]. A new class of fuzzy topological spaces called ordered fuzzy G_δ-extremally disconnected spaces is introduced in this paper by using the concepts of fuzzy extremally disconnected spaces [1], fuzzy G_δ-sets [2] and ordered fuzzy topology [6]. Some interesting properties and characterizations are studied. Tietze extension theorem for ordered fuzzy G_δ-extremally disconnected spaces has been discussed as in [10] besides proving several other propositions and lemmas.

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1. Preliminaries

Definition 1.1. *Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$ where each $\lambda_i \in T$* [2].

Definition 1.2. *Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. λ is called a fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty}\lambda_i$ where each $1 - \lambda_i \in T$.*

Definition 1.3. *Let (X, T) be any fuzzy topological space. For any fuzzy set λ in X we define the σ-closure of λ, denote by $\operatorname{cl}_\sigma \lambda$, to be the intersection of all fuzzy F_σ-sets containing λ. That is*

$$\operatorname{cl}_\sigma \lambda = \wedge\{\mu : \mu \text{ is a fuzzy } F_\sigma\text{-set and } \mu \geq \lambda\}.$$

Definition 1.4. *Let (X, T) be any fuzzy topological space. For any fuzzy set λ in X, we define the σ-interior of λ, denote by $\operatorname{int}_\sigma \lambda$, to be the union of all fuzzy G_δ-sets contained in λ. That is,*

$$\operatorname{int}_\sigma \lambda = \vee\{\mu : \mu \text{ is a fuzzy } G_\delta\text{-set and } \mu \leq \lambda\}.$$

Definition 1.5. *For each $t \in \mathbb{R}$, let $L_t, R_t : \mathbb{R}(I) \to I$ be given by $L_t(\lambda) = 1 - \lambda(t-)$ and $R_t(\lambda) = \lambda(t+)$. Define $\mathcal{L} = \{L_t : t \in \mathbb{R}\} \cup \{0, 1\}$ and $\mathcal{R} = \{R_t | t \in \mathbb{R}\} \cup \{0, 1\}$. Then $\mathcal{L}$ and $\mathcal{R}$ are called I-topologies on $\mathbb{R}(I)$* [9].

Definition 1.6. *Suppose (X, T) is a fuzzy topological space. X is said to be fuzzy extremally disconnected* [2] *if $\lambda \in T$ implies $\operatorname{cl} \lambda \in T$.*

Remark 1.1. The symbol $\langle t \rangle$ $(t \in \mathbb{R})$ stands for the member of $\mathbb{R}(L)$ containing λ such that $\lambda(t+) = \lambda(t-)' = 0$ [10].

2. Ordered Fuzzy G_δ-extremally Disconnected Spaces

In this section, the concept of ordered fuzzy G_δ-extremally disconnected spaces is introduced. Some interesting properties and characterizations are studied.

Definition 2.1. *Let $(X, T, \leq)$ be an ordered fuzzy topological space and let λ be any fuzzy set in $(X, T, \leq)$, λ is called fuzzy increasing G_δ/F_σ if $\lambda = \wedge_{i=1}^{\infty}\lambda_i$/if $\lambda = \vee_{i=1}^{\infty}\lambda_i$ where each λ_i is fuzzy increasing open/closed in $(X, T, \leq)$. The complement of fuzzy increasing G_δ/F_σ-set is fuzzy decreasing F_δ/G_σ.*

Definition 2.2. *Let λ be any fuzzy set in the ordered fuzzy topological space $(X, T, \leq)$. Then we define*

$$\begin{aligned} I_\sigma(\lambda) &= \text{fuzzy increasing } \sigma\text{-closure of } \lambda. \\ &= \text{the smallest fuzzy increasing } F_\sigma\text{-set containing } \lambda. \\ D_\sigma(\lambda) &= \text{fuzzy decreasing } \sigma\text{-closure of } \lambda. \\ &= \text{the smallest fuzzy decreasing } F_\sigma\text{-set containing } \lambda. \\ I^0_\sigma(\lambda) &= \text{fuzzy increasing } \sigma\text{-interior of } \lambda. \\ &= \text{the greatest fuzzy increasing } G_\delta\text{-set contained in } \lambda. \\ D^0_\sigma(\lambda) &= \text{fuzzy decreasing } \sigma\text{-interior of } \lambda. \\ &= \text{the greatest fuzzy decreasing } G_\delta\text{-set contained in } \lambda. \end{aligned}$$

Prorosition 2.1. *For any fuzzy set λ of an ordered fuzzy topological space $(X, T, \leq)$, the following equalities are valid.*

(*a*) $1 - I_\sigma(\lambda) = D^0_\sigma(1 - \lambda)$.

(*b*) $1 - D_\sigma(\lambda) = I^0_\sigma(1 - \lambda)$.

(*c*) $1 - I^0_\sigma(\lambda) = D_\sigma(1 - \lambda)$.

(*d*) $1 - D^0_\sigma(\lambda) = I_\sigma(1 - \lambda)$.

Proof. We shall prove (a) only, (b), (c), and (d) can be proved in a similar manner.

(a) Since $I_\sigma(\lambda)$ is a fuzzy increasing F_σ-set containing λ, $1 - I_\sigma(\lambda)$ is a fuzzy decreasing G_δ-set such that $1 - I_\sigma(\lambda) \leq 1 - \lambda$. Let μ be another fuzzy decreasing G_δ-set such that $\mu \leq 1 - \lambda$. Then $1 - \mu$ is a fuzzy increasing F_σ-set such that $1 - \mu \geq \lambda$. It follows that $I_\sigma(\lambda) \leq 1 - \mu$. That is, $\mu \leq 1 - I_\sigma(\lambda)$. Thus, $1 - I_\sigma(\lambda)$ is the largest fuzzy decreasing G_δ-set such that $1 - I_\sigma(\lambda) \leq 1 - \lambda$. That is, $1 - I_\sigma(\lambda) = 1 - D^0_\sigma(1 - \lambda)$. □

Definition 2.3. *Let $(X, T, \leq)$ be an ordered fuzzy topological space. Let λ be any fuzzy increasing G_δ-set in $(X, T, \leq)$. If $I_\sigma(\lambda)$ is fuzzy increasing G_δ-set in $(X, T, \leq)$, then $(X, T, \leq)$ is said to be upper fuzzy G_δ-extremally disconnected. Similarly we can define lower fuzzy G_δ-extremally disconnected space. $(X, T, \leq)$ is said to be ordered fuzzy G_δ-extremally disconnected if it is both upper and lower fuzzy G_δ-extremally disconnected.*

Example 2.1. Let $X = \{a, b, c\}$ and $T = \{0, 1, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ where

$\lambda_1 : X \to [0, 1]$ is such that $\lambda_1(a) = 0,\ \lambda_1(b) = 1/4,\ \lambda_1(c) = 3/4$,
$\lambda_2 : X \to [0, 1]$ is such that $\lambda_2(a) = 1,\ \lambda_2(b) = 3/4,\ \lambda_2(c) = 3/4$,

$\lambda_3 : X \to [0,1]$ is such that $\lambda_3(a) = 1,\ \lambda_3(b) = 3/4,\ \lambda_3(c) = 1/4,$

and

$\lambda_4 : X \to [0,1]$ is such that $\lambda_4(a) = 0,\ \lambda_4(b) = 1/4,\ \lambda_4(c) = 1/4.$

The partial order "$\leq$" is defined as $a \leq b,\ b \leq c$. Then $(X, T, \leq)$ is an ordered fuzzy topological space. It is clear that $(X, T, \leq)$ is an ordered fuzzy G_δ-extremally disconnected space.

Prorosition 2.2. *For an ordered fuzzy topological space $(X, T, \leq)$, the following statements are equivalent.*

- (a) *$(X, T, \leq)$ is upper fuzzy $G_d$elta-extremally disconnected.*
- (b) *For each fuzzy decreasing F_σ-set λ, $D_\sigma^0(\lambda)$ is a decreasing fuzzy F_σ-set.*
- (c) *For each fuzzy increasing G_δ-set λ, we have*

$$I_\sigma(\lambda) + D_\sigma(1 - I_\sigma(\lambda)) = 1.$$

- (d) *For each pair of fuzzy increasing G_δ-set λ and a fuzzy decreasing G_δ-set μ in $(X, T, \leq)$ with $I_\sigma(\lambda) + \mu = 1$, we have*

$$I_\sigma(\lambda) + D_\sigma(\mu) = 1.$$

Proof. $(a) \Rightarrow (b)$. Let λ be any fuzzy decreasing F_σ-set. We claim $D_\sigma^0(\lambda)$ is a fuzzy decreasing F_σ-set. Now $1 - \lambda$ is fuzzy increasing G_δ and so by assumption (a), $I_\sigma(1 - \lambda)$ is fuzzy increasing G_δ. That is, $D_\sigma^0(\lambda)$ is fuzzy decreasing F_σ.

$(b) \Rightarrow (c)$. Let λ be any fuzzy increasing G_δ-set. Then,

$$1 - I_\sigma(\lambda) = D_\sigma^0(1 - \lambda). \tag{2.1}$$

Consider $I_\sigma(\lambda) + D_\sigma(1 - I_\sigma(\lambda)) = I_\sigma(\lambda) + D_\sigma(D_\sigma^0(1 - \lambda))$. As λ is any fuzzy increasing G_δ-set, $1 - \lambda$ is fuzzy decreasing F_σ and by assumption (b), $D_\sigma^0(1 - \lambda)$ is fuzzy decreasing F_σ. Therefore,

$$D_\sigma(D_\sigma^0(1 - \lambda)) = D_\sigma^0(1 - \lambda).$$

Now,

$$I_\sigma(\lambda) + D_\sigma(D_\sigma^0(1 - \lambda)) = I_\sigma(\lambda) + D_\sigma^0(1 - \lambda) = 1.$$

That is,

$$I_\sigma(\lambda) + D_\sigma(1 - I_\sigma(\lambda)) = 1.$$

$(c) \Rightarrow (d)$. Let λ be any fuzzy increasing G_δ-set and μ be any fuzzy decreasing G_δ-set such that

$$I_\sigma(\lambda) + \mu = 1. \tag{2.2}$$

By assumption (c),

$$\begin{aligned} I_\sigma(\lambda) + D_\sigma(1 - I_\sigma(\lambda)) &= 1 \\ &= I_\sigma(\lambda) + \mu. \end{aligned} \tag{2.3}$$

That is, $\mu = D_\sigma(1 - I_\sigma(\lambda))$. Since $\mu = 1 - I_\sigma(\lambda)$,

$$D_\sigma(\mu) = D_\sigma(1 - I_\sigma(\lambda)). \tag{2.4}$$

From (2.3) and (2.4)

$$I_\sigma(\lambda) + D_\sigma(\mu) = 1.$$

$(d) \Rightarrow (a)$. Let λ be any fuzzy increasing G_δ-set. Put $\mu = 1 - I_\sigma(\lambda)$. Clearly, μ is fuzzy decreasing G_δ-set and from the construction of μ it follows that $I_\sigma(\lambda)+\mu = 1$. By assumption (d), we have $I_\sigma(\lambda)+D_\sigma(\mu) = 1$ and so $I_\sigma(\lambda) = 1 - D_\sigma(\mu)$ is fuzzy increasing G_δ. Therefore, $(X, T, \leq)$ is upper fuzzy G_δ-extremally disconnected. □

Prorosition 2.3. *Let $(X, T, \leq)$ be an ordered fuzzy topological space. Then $(X, T, \leq)$ is an upper fuzzy G_δ-extremally disconnected space $\Leftrightarrow$ for fuzzy decreasing G_δ-set λ and fuzzy decreasing F_σ-set μ such that $\lambda \leq \mu$, we have $D_\sigma(\lambda) \leq D^0_\sigma(\mu)$.*

Proof. Suppose $(X, T, \leq)$ is an upper fuzzy G_δ-xtremally disconnected space. Let λ be any fuzzy decreasing G_δ-set such that $\lambda \leq \mu$. Then by (b) of Proposition 2.2, $D^0_\sigma(\mu)$ is fuzzy decreasing F_σ. Also, since λ is fuzzy decreasing G_δ and $\lambda \leq \mu$, it follows that $\lambda \leq D^0_\sigma(\mu)$. Again, since $D^0_\sigma(\mu)$ is fuzzy decreasing F_σ, it follows that $D_\sigma(\lambda) \leq D^0_\sigma(\mu)$.

To prove the converse, let μ be any fuzzy decreasing F_σ-set. By Definition 2.2, $D^0_\sigma(\mu)$ is fuzzy decreasing G_δ and it is also clear that $D^0_\sigma(\mu) \leq \mu$. Therefore by assumption, it follows that $D_\sigma(D^0_\sigma(\mu)) \leq D^0_\sigma(\mu)$. This implies that $D^0_\sigma(\mu)$ is fuzzy decreasing F_σ. Hence by (b) of Proposition 2.2, it follows that $(X, T, \leq)$ is upper fuzzy G_δ-extremally disconnected. □

Remark 2.1. Let $(X, T, \leq)$ be an upper fuzzy G_δ-extremally disconnected space. Let $\{\lambda_i, 1 - \mu_i : i \in \mathbb{N}\}$ be a collection such that $\lambda_i,\ i \in \mathbb{N}$ are fuzzy decreasing G_δ-sets and $\mu_i,\ i \in \mathbb{N}$ are fuzzy decreasing F_σ-sets.

Let $\lambda,\ 1-\mu$ be fuzzy decreasing G_δ-set and fuzzy increasing G_δ-set respectively. If $\lambda_i \le \lambda \le \mu_j$ and $\lambda_i \le \mu \le \mu_j$ for all $i, j \in \mathbb{N}$, then there exists a fuzzy decreasing $G_\delta F_\sigma$-set γ such that

$$D_\sigma(\lambda_i) \le \gamma \le D^0_\sigma(\mu_j) \quad \text{for all } i, j \in \mathbb{N}.$$

By Proposition 2.3,

$$D_\sigma(\lambda_i) \le D_\sigma(\lambda) \wedge D^0_\sigma(\mu) \le D^0_\sigma(\mu_j) \qquad (i, j \in \mathbb{N}).$$

Put $\gamma = D_\sigma(\lambda) \wedge D^0_\sigma(\mu)$. Now γ satisfies our required condition.

Prorosition 2.4. *Let $(X, T, \le)$ be an ordered fuzzy G_δ-extremally disconnected space. Let $\{\lambda_q\}_{q\in\mathcal{Q}}$ and $\{\mu_q\}_{q\in\mathbb{Q}}$ be monotone increasing collections of fuzzy decreasing G_δ-sets and fuzzy decreasing F_σ-sets of $(X, T, \le)$ respectively and suppose that $\lambda_{q_1} \le \mu_{q_2}$ whenever $q_1 < q_2$ ($\mathbb{Q}$ is the set of rational numbers). Then there exists a monotone increasing collection $\{\gamma_q\}_{q\in\mathbb{Q}}$ of fuzzy decreasing $G_\delta F_\sigma$-sets of $(X, T, \le)$ such that $D_\sigma(\lambda_{q_1}) \le \gamma_{q_2}$ and $\gamma_{q_1} \le D^0_\sigma(\mu_{q_2})$ whenever $q_1 < q_2$.*

Proof. Let us arrange into sequence $\{q_n\}$ of rational numbers without repetitions. For every $n \ge 2$, we shall define inductively a collection $\{\gamma_{q_i} : 1 \le i \le n\} \subset I^X$ such that

$$\begin{aligned} &D_\sigma(\lambda_q) \le \gamma_{q_i} && \text{if } q < q_i, \\ &\gamma_{q_i} \le D^0_\sigma(\mu_q) && \text{if } q_i < q, \end{aligned} \qquad (S_n)$$

for all $i < n$.

By Proposition 2.3, the family $\{D_\sigma(\lambda_q)\}$ and $\{D^0_\sigma(\mu_q)\}$ satisfying $D_\sigma(\lambda_{q_1}) \le D^0_\sigma(\mu_{q_2})$ if $q_1 < q_2$. By Remark 2.1, there exists fuzzy decreasing $G_\delta F_\sigma$-set δ_1 such that

$$D_\sigma(\lambda_{q_1}) \le \delta_1 \le D^0_\sigma(\mu_{q_2}).$$

Setting $\gamma_{q_1} = \delta_1$ we get (S_2). Assume that fuzzy sets γ_{q_i} are already defined for $i < n$ and satisfy (S_n). Define

$$\Sigma = \vee\{\gamma_{q_i} : i < n,\ q_i < q_n\} \vee \lambda_{q_n}$$

and

$$\Phi = \wedge\{\gamma_{q_j} : j < n,\ q_j > q_n\} \wedge \mu_{q_n}.$$

Then we have that

$$D_\sigma(\gamma_{q_i}) \le D_\sigma(\Sigma) \le D^0_\sigma(\gamma_{q_j})$$

and

$$D_\sigma(\gamma_{q_i}) \leq D_\sigma(\Phi) \leq D^0_\sigma(\gamma_{q_j})$$

whenever $q_i < q_n < q_j$ $(i, j < n)$ as well as $\lambda_q \leq D_\sigma(\Sigma) \leq \mu_{q'}$ and $\lambda_q \leq D^0_\sigma(\Phi) \leq \mu_{q'}$ whenever $q < q_n < q'$. This shows that the countable collection $\{\gamma_{q_i} : i < n,\ q_i < q_n\} \cup \{\lambda_q : q < q_n\}$ and $\{\gamma_{q_i} : j < n,\ q_j > q_n\} \cup \{\mu_q : q > q_n\}$ together with Σ and Φ fulfill all conditions of the mentioned Remark 2.1. Hence, there exists a fuzzy decreasing $G_\delta F_s igma$-set δ_n such that

$$\begin{aligned} D_\sigma(\delta_n) &\leq \mu_q && \text{if } q_n < q, \\ \lambda_q &\leq D\sigma^0(\delta_n) && \text{if } q < q_n, \\ D_\sigma(\gamma_{q_i}) &\leq D^0_\sigma(\delta_n) && \text{if } q_i < q_n, \\ D_\sigma(\delta_n) &\leq D^0_\sigma(\gamma_{q_j}) && \text{if } q_n < q_j, \end{aligned}$$

where $1 \leq i, j \leq n-1$. Now, setting $\gamma_{q_n} = \delta_n$ we obtain the fuzzy sets $\gamma_{q_1}, \gamma_{q_2}, \ldots, \gamma_{q_n}$ that satisfy (S_{n+1}). Therefore, the collection $\{\gamma_{q_i} : i = 1, 2, \ldots\}$ has the required property. This completes the proof. □

Definition 2.4. *Let $(X, T, \leq)$ and $(Y, S, \leq)$ be ordered fuzzy topological spaces. A mapping $f : (X, T, \leq) \to (Y, S, \leq)$ is called fuzzy increasing/decreasing G_δ-continuous if $f^{-1}(\lambda)$ is fuzzy increasing/decreasing G_δ-set of $(X, T, \leq)$ for every fuzzy G_δ-set λ of $(Y, S, \leq)$. If f is both fuzzy increasing and fuzzy decreasing G_δ-continuous, then it is called ordered fuzzy G_δ-continuous.*

Definition 2.5. *Let $(X, T, \leq)$ be an ordered fuzzy topological space. A function $f : X \to \mathbb{R}(I)$ is called lower fuzzy G_δ-continuous if $f^{-1}(R_t)$ is fuzzy increasing of fuzzy decreasing G_δ for each $t \in \mathbb{R}$ and upper fuzzy G_δ-continuous if $f^{-1}(L_t)$ is fuzzy increasing of fuzzy decreasing G_δ for each $t \in \mathbb{R}$.*

Lemma 2.1. *Let $(X, T, \leq)$ be an ordered fuzzy topological space, let $\lambda \in I^X$, and let $f : X \to \mathbb{R}(I)$ be such that*

$$f(x)(t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda(x) & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1, \end{cases}$$

for all $x \in X$. Then f is lower/upper fuzzy G_δ-continuous iff λ is fuzzy increasing of decreasing G_δ/F_σ-set.

Proof. If suffices to observe that

$$f^{-1}(R_t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 1 \end{cases}$$

and

$$f^{-1}(L_{t'}) = \begin{cases} 1 & \text{if } t \leq 0, \\ \lambda & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Thus proved. □

Definition 2.6. *The characteristic function of* $\lambda \in I^X$ *is the map* $\chi_\lambda : X \to [0,1](I)$ *defined by* $\chi_\lambda(x) = (\lambda(x)),\ x \in X$ [10].

Lemma 2.2. *Let* $(X, T, \leq)$ *be an ordered fuzzy topological space, let* $\lambda \in I^X$. *Then* χ_λ *is lower/upper fuzzy* G_δ*-continuous iff* λ *is fuzzy increasing or decreasing* G_δ/F_σ*-set.*

Proof. Proof is similar to Lemma 2.1. □

Prorosition 2.5. *Let* $(X, T, \leq)$ *be an ordered fuzzy topological space. Then the following statements are equivalent.*

(*a*) $(X, T, \leq)$ *is upper fuzzy* G_δ*-extremally disconnected.*

(*b*) *If* $g, h : X \to \mathbb{R}(I)$, g *is lower fuzzy* G_δ*-continuous,* h *is upper fuzzy* G_δ*-continuous and* $g \leq h$, *then there exists an fuzzy increasing* G_δ*-continuous function* $f : (X, T, \leq) \to \mathbb{R}(I)$ *such that* $g \leq f \leq h$.

(*c*) *If* $1 - \lambda$ *is fuzzy increasing* G_δ*-set,* μ *is fuzzy decreasing* G_δ*-set and* $\mu \leq \lambda$, *then there exists fuzzy increasing* G_δ*-continuous function* $f : (X, T, \leq) \to [0,1](I)$ *such that* $\mu \leq (1 - L_1)f \leq R_0 f \leq \lambda$.

Proof. (*a*) ⇒ (*b*). Define $H_r = L_r h$ and $G_r = (1 - R_r)g$, $r \in \mathbb{Q}$. Thus we have two monotone increasing families of respectively fuzzy decreasing G_δ-sets and fuzzy decreasing F_σ-sets of $(X, T, \leq)$. Moreover, $H_r \leq G_s$ if $r < s$. By Proposition 2.4, there exists a monotone increasing family $\{F_r\}_{r \in \mathbb{Q}}$ of fuzzy decreasing $G_\delta F_\sigma$-sets of $(X, T, \leq)$ such that $D_\sigma(H_r) \leq F_s$ and $F_r \leq D_\sigma^0(G_s)$ whenever $r < s$. Letting $V_t = \wedge_{r<t}(1 - F_r)$ for all $t \in \mathbb{R}$, we define a monotone decreasing family $\{V_t : t \in \mathbb{R}\} \subset I^X$.

Moreover, we have $I_\sigma(V_t) \leq I^0_\sigma(V_s)$, whenever $s < t$. We have

$$\bigvee_{t\in\mathbb{R}} V_t = \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t}(1-F_r) \geq \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t}(1-G_r)$$
$$= \bigvee_{t\in\mathbb{R}} \bigwedge_{r<t} g^{-1}(R_r) = \bigvee_{t\in\mathbb{R}} g^{-1}(R_t) = g^{-1}\Big(\bigvee_{t\in\mathbb{R}} R_t\Big) = 1.$$

Similarly, $\wedge_{t\in\mathbb{R}} V_t = 0$.

We now define a function $f : (X, T, \leq) \to \mathbb{R}(I)$ satisfying the required properties. Let $f(x)(t) = V_t(x)$ for all $x \in X$ and $t \in \mathbb{R}$. By the above discussion, it follows that f is well defined. To prove f is fuzzy increasing G_δ-continuous, we observe that

$$\bigvee_{s>t} V_s = \bigvee_{s>t} I^0_\sigma(V_s), \qquad \bigwedge_{s<t} V_s = \bigwedge_{s<t} I_\sigma(V_s).$$

Then

$$f^{-1}(R_t) = \bigvee_{s>t} V_s = \bigvee_{s>t} I^0_\sigma(V_s)$$

is fuzzy increasing G_δ. Now

$$f^{-1}(1-L_t) = \bigwedge_{s<t} V_s = \bigwedge_{s<t} I_\sigma(V_s)$$

is fuzzy increasing F_σ so that f is fuzzy increasing G_δ-continuous. To conclude the proof it remains to show that $g \leq f \leq h$, that is $g^{-1}(1 - L_t) \leq f^{-1}(1-L_t) \leq h^{-1}(1-L_t)$ and $g^{-1}(R_t) \leq f^{-1}(R_t) \leq h^{-1}(R_t)$ for each $t \in \mathbb{R}$.

We have

$$g^{-1}(1-L_t) = \bigwedge_{s<t} g^{-1}(1-L_s) = \bigwedge_{s<t} \bigwedge_{r<s} g^{-1}(R_r)$$
$$= \bigwedge_{s<t} \bigwedge_{r<s}(1-G_r) \leq \bigwedge_{s<t} \bigwedge_{r<s}(1-F_r) = \bigwedge_{s<t} V_s = f^{-1}(1-L_t)$$

and

$$f^{-1}(1-L_t) = \bigwedge_{s<t} V_s = \bigwedge_{s<t} \bigwedge_{r<s}(1-F_r) \leq \bigwedge_{s<t} \bigwedge_{r<s}(1-H_r)$$
$$= \bigwedge_{s<t} \bigwedge_{r<s} h^{-1}(1-L_r) = \bigwedge_{s<t} h^{-1}(1-Ł_s) = h^{-1}(1-L_t).$$

Similarly, we obtain

$$g^{-1}(R_t) = \bigvee_{s>t} g^{-1}(R_s) = \bigvee_{s>t} \bigvee_{r>s} g^{-1}(R_r) = \bigvee_{s>t} \bigvee_{r>s} (1 - G_r)$$
$$\leq \bigvee_{s>t} \bigwedge_{r<s} (1 - F_r) = \bigvee_{s>t} V_s = f^{-1}(R_t)$$

and

$$f^{-1}(R_t) = \bigvee_{s>t} V_s = \bigvee_{s>t} \bigwedge_{r<s} (1 - F_r) \leq \bigvee_{s>t} \bigvee_{r>s} (1 - H_r)$$
$$= \bigvee_{s>t} \bigvee_{r>s} h^{-1}(1 - L_r) = \bigvee_{s>t} h^{-1}(R_s) = h^{-1}(R_t).$$

Thus, (b) is proved.

$(b) \Rightarrow (c)$. Suppose $1 - \lambda$ is fuzzy increasing G_δ and μ is fuzzy decreasing G_δ, such that $\mu \leq \lambda$. Then $\chi_\mu \leq \chi_\lambda$, χ_μ and χ_λ are lower and upper fuzzy G_δ-continuous functions respectively. Hence by (b), there exists fuzzy increasing G_δ continuous function $f : (X, T, \leq) \to \mathbb{R}(I)$ such that $\chi_\mu \leq f \leq \chi_\lambda$. Clearly, $f(x) \in [0,1](I)$ for all $x \in X$ and $\mu = (1 - L_1)\chi_\mu \leq (1 - L_1)f \leq R_0 f \leq R_0 \chi_\lambda = \lambda$.

$(c) \Rightarrow (a)$. This follows from Proposition 2.3, and the fact that $(1 - L_1)f$ and $R_0 f$ are fuzzy decreasing F_σ and fuzzy decreasing G_δ-sets respectively. Hence the result. □

Remark 2.2. Propositions 2.2–2.5 and Remark 2.1 can be discussed for other cases also.

3. Tietze Extension Theorem for Ordered Fuzzy G_δ-extremally Disconnected Spaces

In this section, Tietze extension theorem for ordered fuzzy G_δ-extremally disconnected space is studied.

Prorosition 3.1 (Tietze Extension Theorem). *Let $(X, T, \leq)$ be an upper fuzzy G_δ-extremally disconnected space and let $A \subset X$ be such that χ_A is fuzzy increasing G_δ in $(X, T, \leq)$. Let $f : (A, T/A) \to [0,1](I)$ [6] be an increasing fuzzy G_δ-continuous function. Then f has an increasing fuzzy G_δ-continuous extension over $(X, T, \leq)$.*

Proof. Let $g, h : X \to [0,1](I)$ be such that

$$g = f = h \text{ on } A \text{ and } g(x) = \langle 0 \rangle, \qquad h(x) = \langle 1 \rangle \text{ if } x \notin A.$$

We now have

$$R_t g = \begin{cases} \mu_t \wedge \chi_A & \text{if } t \geq 0 \\ 1 & \text{if } t < 0 \end{cases}$$

where μ_t is fuzzy increasing G_δ such that

$$\mu_t/A = R_t f$$

and

$$L_t h = \begin{cases} \lambda_t \wedge \chi_A & \text{if } t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}$$

where λ_t is increasing fuzzy G_δ such that

$$\lambda_t/A = L_t f.$$

Thus, g is lower fuzzy G_δ-continuous, h is upper fuzzy G_δ-continuous and $g \leq h$. By Proposition 2.5, there exists an increasing fuzzy G_δ-continuous function $F : X \to [0,1](I)$ such that $g \leq F \leq h$; hence $F \equiv f$ on A. □

Remark 3.1. The above proposition can be discussed for other cases also.

References

[1] G. Balasubramanian, *Fuzzy disconnectedness and its stronger forms* // Indian J. Pure Appl. Math., **24** (1993), 27–30.

[2] G. Balasubramanian, *Maximal fuzzy topologies* // Kybernetika, **31** (1995), 459–464.

[3] G. Balasubramanian, *On fuzzy β-compact spaces and fuzzy β-extremally disconnected spaces* // Kybernetica, **33** (1997), 271–277.

[4] G. Balasubramanian and P. Sundaram, *On some generalizations of fuzzy continuous functions* // Fuzzy Sets and Systems, **86** (1997), 93–100.

[5] C. L. Chang, *Fuzzy topological spaces* // J. Math. Anal. Appl., **24** (1968), 182–190.

[6] A. K. Katsaras, *Ordered fuzzy topological spaces* // J. Math. Anal. Appl., **84** (1981), 44–58.

[7] P. Smets, *The degree of belief in a fuzzy event* // Information Sciences, **25** (1981), 1–19.

[8] M. Sugeno, *An intoductory survey of fuzzy control* // Information Sciences, **36** (1985), 59–83.

[9] G. Thangaraj and G. Balasubramanian, *On fuzzy basically disconnected spaces* // The Journal of Fuzzy Mathematics, **9** (2001), 103–110.

[10] Tomasz Kubiak, *Extending continuous L-real functions* // Math. Japonica, **6** (1986), 875–887.

[11] L. A. Zadeh, *Fuzzy sets* // Information and Control, **8** (1965), 338–353.

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