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On the Shapiro–Lopatinkii condition for elliptic problems

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(*Presented by M. M. Malamud*)

Abstract. This paper is concerned with elliptic problems including a small parameter multiplying higher order derivatives. We found algebraic conditions on the operator and boundary conditions which guarantee the Fredholm property, and prove an a priori estimate for the solution with a constant independent of the small parameter. These results are known for elliptic boundary value problems with small parameter in the half space $\mathbb{R}^n_+$. We extend them to the case of bounded domains with smooth boundary. The small parameter coercive conditions are formulated and two-sided estimate is proved.

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Introduction

This paper studies linear elliptic differential equations with a small parameter at the highest derivative when boundary conditions also contain the same parameter. We consider operators acting on functions in some domain D and depending on a small parameter $\varepsilon \geq 0$ in a special way. More precisely,

$$\varepsilon^{2m-2\mu} A_{2m}(x, D)u + \varepsilon^{2m-2\mu-1} A_{2m-1}(x, D)u + \cdots + A_{2\mu}(x, D)u = f,$$

supplemented by the boundary conditions

$$\varepsilon^{b_j-\beta_j} B_{j,b_j}(x', D)u + \varepsilon^{b_j-\beta_j-1} B_{j,b_{j-1}}(x', D)u + \cdots + B_{j,\beta_j}(x', D)u = u_j,$$

for $x' \in \partial D$, with $j = 1, \ldots, m$, where $A_{2m-i}(x, D)$ and $B_{j,b_{j-i}}(x', D)$ are differential operators of order $2m - i$ and b_{j-i}, respectively, with variable coefficients. This family of operators is assumed to be elliptic for each $\varepsilon \geq 0$.

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The subject of our interest is the behaviour of solutions $u_\varepsilon(x)$ when ε tends to zero. It is known (see, for example, [4]) that solutions u_ε defined on a manifold with boundary may have the so-called *boundary layer*. In this case the solution $u_\varepsilon(x)$ converges to $u_0(x)$, when $\varepsilon \to 0$, uniformly in each strictly inner bounded subdomain $\tilde{D} \subset D$, but need not converge at the boundary points $x' \in \partial D$. Essentially this concept was introduced by Prandtl in 1904 (for accurate historical background see e.g. [6]). He studied fluid flow with small viscosity over a surface and explained how insignificant friction forces influence on the main perfect fluid flow. His idea is based on splitting a solution into two parts, namely a solution near the boundary and a solution far away from the boundary, and stretching the coordinates in the normal direction of the boundary.

Lyusternik and Vishik in the paper [2] extended this idea to differential equations which depend on a small parameter polynomially. Some applications of the Lyusternik–Vishik method for partial differential equations (PDE) are discussed in [5]. This method suggests to look for a solution as the sum of a regular part, which depends uniformly on ε, and an additional function, which grows rapidly when $\varepsilon \to 0$. The regular part is found by using ordinary method of small parameter; the boundary layer is supposed to be a solution of some ordinary differential equation (ODE) in the direction normal to ∂D. This ODE is obtained using coordinate stretching in the direction normal to the boundary, as it was proposed by Prandtl.

In [2] the problem was considered in the case of Dirichlet boundary conditions B_j and strong uniform ellipticity of the operator A. Then the method was adapted to domains with conical points by Nazarov in [8]. In [9] it was extended to pseudodifferential operators and general elliptic boundary problems. In all these works uniform estimates in norms depending on ε for solutions were found under some generalised *coercivity condition*, and the estimates justify formal asymptotic series obtained by the Lyusternik-Vishik method. But the comprehensive theory of elliptic equations with small parameter was constructed by Volevich in [1]. For the problem (A, B) in the half-space $\mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ he introduced a *Shapiro–Lopatinskii condition with small parameter* and proved its necessity and sufficiency for the existence of two-sided uniform estimates for (A, B). Volevich used the norms proposed by Demidov in [7].

The paper by Volevich falls short of providing complete arguments in the case of arbitrary smooth bounded domains and the aim of the present paper is to extend the results of Volevich to the case of bounded domains D with smooth boundary ∂D using the local principle of elliptic theory (see, for example, [3]).

1. Asymptotic expansion

Now we apply the Vishik–Lyusternik method to the problem (A, B) to find an asymptotic expansion of solution u. The domain D is required to be bounded and have smooth boundary ∂D. Let us introduce new coordinates $(y_1, y_2, \dots, y_{n-1}, z)$ in D, such that $y \in \partial D$ is a variable on the surface ∂D and z is the distance to ∂D. By $A'(y, z, D_y, D_z, \varepsilon)$, $B'(y, D_y, D_z, \varepsilon)$ we denote operators A, B in the new variables.

We are looking for a solution of (A, B) in the form $u(x, \varepsilon) = U(x, \varepsilon) + V(y, z/\varepsilon, \varepsilon)$ where U is the regular part of u and V is the boundary layer. Suppose that the function $V(y, z/\varepsilon)$ satisfies the following three conditions:

1. $V(y, z/\varepsilon, \varepsilon)$ is a sufficiently smooth solution of the homogeneous equation $AV = 0$;
2. $V(y, z/\varepsilon, \varepsilon)$ depends on the "fast" variable $t = z/\varepsilon$;
3. $V(y, z/\varepsilon, \varepsilon)$ differs from zero only in a small strip near the boundary ∂D.

The regular part and the boundary layer are looked for as formal asymptotic series

$$U(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(x), \quad V(y, z/\varepsilon, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k v_k(y, z/\varepsilon).$$

The first series is called the *outer expansion* and the second one is called the *inner expansion*. The outer expansion is obtained using the standard procedure of small parameter method. We substitute the series for $U(x, \varepsilon)$ into the equation $A(x, D, \varepsilon)u = f$ and collect the terms with the same power of ε. It gives us the system

$$A_{2\mu} u_0 = f, \tag{1.1}$$

$$A_{2\mu} u_k = -\sum_{i=1}^{k} A_{2\mu+i} u_{k-i} \tag{1.2}$$

for unknowns u_k.

To determine the coefficients $v_k(y, z/\varepsilon)$ of the inner expansion we apply the operator $A'(y, z, D_y, D_z, \varepsilon)$ to $V(y, z/\varepsilon, \varepsilon)$. Condition 1 implies

$$\sum_{k=0}^{\infty} \varepsilon^k A'(y, z, D_y, D_z, \varepsilon)(v_k(y, z/\varepsilon)) = 0.$$

Let us rewrite the operator $A'(y,z,D_y,D_z,\varepsilon)$ in the variables $(y, z/\varepsilon = t)$. For the homogeneous part A'_k of degree k we have

$$A'_k(y,z,D_y,D_z,\varepsilon) = \varepsilon^{-2\mu+k} A'_k(y,\varepsilon t,\varepsilon D_y,D_t).$$

On expanding $A'_k(y,\varepsilon t,\varepsilon D_y,D_t)$ as Taylor series about the point $(y,0,0,D_t)$ we obtain

$$A'_k(y,z,D_y,D_z,\varepsilon) = \varepsilon^{-2\mu+k}\Bigl(A''_k(k,0,0,D_t) + \sum_{l=1}^{\infty} \varepsilon^l A_{k,l}(y,t,D_y,D_t)\Bigr),$$

where the operators $A_{k,l}$ have smooth coefficients.

Therefore,

$$A'(y,z,D_y,D_z,\varepsilon) = \varepsilon^{-2\mu}\Bigl(A''(y,0,0,D_t) + \sum_{l=1}^{\infty} \varepsilon^l A_l(y,t,D_y,D_t)\Bigr),$$

A_l depends on $A_{k,l}$ linearly.

So it defines equations for v_k

$$A''(\xi,0,0,D_t)v_k(y,t) = -\sum_{l=1}^{k} A_l(y,t,D_y,D_t)v_{k-l}.$$

Now we substitute the partial sums

$$U_n = \sum_{k=0}^{n} \varepsilon^k u_k(x),$$
$$V_n = \sum_{k=0}^{n} \varepsilon^k v_k(y,t)$$

into the original equation and boundary conditions and find the discrepancy. For $A(x,D,\varepsilon)$, it looks like

$$\begin{aligned} A(x,D,\varepsilon)&(u(x,\varepsilon) - U_n - V_n) \\ &= f - A_{2\mu}(x,D)u_0 - \bigl(A(x,D,\varepsilon)U_n - A_{2\mu}(x,D)u_0 \\ &\qquad + A'(y,z,D_y,D_z,\varepsilon)V_n\bigr) = O(\varepsilon^{n+1}). \end{aligned}$$

Hence, if we are able to find appropriate Banach spaces $\|u\|'$, $\|u\|'_{\partial D}$, such that the operator $(A(x,D,\varepsilon), B(x',D,\varepsilon))$ is bounded uniformly in ε then the difference $u(x,\varepsilon) - U_n - V_n$ is small and so the formal series approximates the solution $u(x,\varepsilon)$ indeed.

This problem was solved by Volevich [1] for the case where D is the half-space $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$. He used the norms for functions in D and their traces which had been introduced in [7]. They are of the form

$$\|u; H^{r,s}(\mathbb{R}^n)\| = \|(1+|\xi|^2)^{s/2}(1+\varepsilon^2|\xi|^2)^{(r-s)/2}\hat{u}\|_{L^2},$$
$$\|u; \mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})\| = \|u\|_{L^2(\mathbb{R}^{n-1})} + \||\eta|^{\sigma}(1+\varepsilon^2|\eta|^2)^{(\rho-\sigma)/2}u\|_{L^2(\mathbb{R}^{n-1})},$$

where $\sigma \geq 0$. In these norms there is an estimate

$$\|u; H^{r,s}(\mathbb{R}^n_+)\| \leq C \biggl(\|A(x,D)u; H^{r,s}(\mathbb{R}^n_+)\| + \sum_{j=1}^{m} \|B_j(D',\varepsilon)u; \mathcal{H}^{r-b_j-1/2, s-\beta_j-1/2}(\mathbb{R}^{n-1})\| + \|u; L^2(\mathbb{R}^n_+)\| \biggr), \quad (1.3)$$

where C does not depend on ε. A trace theorem for the norms $\|u; H^{r,s}(\mathbb{R}^n)\|$ and $\|u; \mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})\|$ is also proved in [1].

Theorem 1.1. *For $r > l + 1/2$ and $s \geq 0$, $s \neq l + 1/2$, we have*

$$\|D_n^l u(\cdot, 0); \mathcal{H}^{r-l-1/2, s-l-1/2}(\mathbb{R}^{n-1})\| \leq c\, \|u; H^{r,s}(\mathbb{R}^n_+)\|$$

with c a constant independent of ε.

Volevich [1] also proves that estimate (1.3) holds true if the operator (A, B) satisfies the *small parameter ellipticity condition*, the *Shapiro–Lopatinskii condition with small parameter* and the system of equations (1.1), (1.2) is correctly solvable. The problem (A, B) is called an elliptic problem with parameter if it satisfies all these conditions. The first two conditions are, of course, of greater interest than the last one. They read as follows:

Small parameter ellipticity condition: The operator $A(x, D, \varepsilon)$ is said to be small parameter elliptic at some point x_0 if its principal polynomial $A_0(x_0, \xi, \varepsilon)$ admits an estimate

$$|A_0(x_0, \xi, \varepsilon)| \geq c_{x_0} |\xi|^{2\mu} (1 + \varepsilon|\xi|)^{2m-2\mu}$$

from below.

Shapiro–Lopatinskii condition with small parameter: The problem (A, B) satisfies the Shapiro–Lopatinskii condition for every $\varepsilon \geq 0$.

As mentioned, this paper is aimed at extending the result of [1] to the case of bounded domains D with smooth boundary ∂D. To this end we develop the local principle that underlies elliptic theory (see for example [3]) in the case of problems with parameter.

2. The main spaces

Our first task is to introduce the main spaces. Hereinafter D stands for a bounded domain with smooth boundary in $\mathbb{R}^n$. The spaces $H^{r,s}(\mathbb{R}^n)$ and $H^{r,s}(\mathbb{R}^n_+)$ are exactly the same as those used in the works of Demidov (see for instance [7]). Namely, $H^{r,s}(\mathbb{R}^n)$ consists of all functions $u \in H^r(\mathbb{R}^n)$ which have finite norm $\|u\|_{r,s}$, and $H^{r,s}(\mathbb{R}^n_+)$ is the factor space $H^{r,s}(\mathbb{R}^n)/H^{r,s}_-(\mathbb{R}^n)$ where $H^{r,s}_-(\mathbb{R}^n)$ is the subspace of $H^{r,s}(\mathbb{R}^n)$ consisting of all functions with support in $\{x \in \mathbb{R}^n : x_n \le 0\}$. As usual, the factor space is endowed with the canonical norm

$$\|[u]; H^{r,s}(\mathbb{R}^n_+)\| = \inf_{u \in [u]} \|u\|_{r,s}.$$

When it does not cause misunderstanding we denote this norm simply by $\|u\|_{r,s}$. Analogously, we introduce the spaces of functions defined in some domain D. To wit,

$$H^{r,s}(D) := H^{r,s}(\mathbb{R}^n)/H^{r,s}_{\mathbb{R}^n \setminus D}(\mathbb{R}^n)$$

where functions of $H^{r,s}_{\mathbb{R}^n \setminus D}(\mathbb{R}^n)$ are supported outside the domain D. This space is also given the canonical norm $\|u; H^{r,s}(D)\|$, which we denote sometimes by $\|u\|_{r,s}$ for short.

Lemma 2.1. *Let f be a smooth function in $\mathbb{R}^n$, such that $f(x) = 1$ for $x \in D$. Then $\|u; H^{r,s}(D)\| = \|fu; H^{r,s}(D)\|$.*

Lemma 2.2. *If $u \in H^{r,s}(\mathbb{R}^n)$ and $\operatorname{supp} u \subset \overline{D}$, then $\|u; H^{r,s}(D)\| = \|u; H^{r,s}(\mathbb{R}^n)\|$.*

For positive integer numbers s and $r \ge s$ the space $H^{r,s}(D)$ proves to be the completion of $C^\infty(\bar{D})$ with respect to the norm $\|u; H^{r,s}(D)\|_{r,s}$. The elliptic technique used in this paper includes the "rectification" of the boundary. Therefore, the invariance of $\|\cdot\|_{r,s}$ with respect to a change of variables is one of the key points. For every fixed $\epsilon \ge 0$, the norms $\|\cdot\|_{r,s}$ are the ordinary Sobolev norms and the main question is what kind of coordinate transformations save the form of the dependence of $\|\cdot\|_{r,s}$ on ε. The following statement displays how ε enters into the norms $\|\cdot\|_{r,s}$.

Lemma 2.3. *For natural r and s satisfying $r \ge s$, the norm $\|u; H^{r,s}(D)\|^2$ has a representation*

$$\sum_{\substack{i=0 \\ i \text{ is even}}}^{r} a_{r,s,i}(\varepsilon)\|\Delta^{i/2}u\|^2_{L^2(D)} + \sum_{\substack{i=1 \\ i \text{ is odd}}}^{r} a_{r,s,i}(\varepsilon)\|\nabla^i u\|^2_{L^2(D)},$$

where $a_{r,s,i}(\varepsilon)$ are polynomials of degree $2i$ and $a_{r,s,0}(\varepsilon) \neq 0$.

Proof. Applying the binomial formula we get

$$(1+|\xi|^2)^s = \sum_{i=0}^{s} C_s^i |\xi|^{2i} \qquad \text{and} \qquad (1+\varepsilon^2|\xi|^2)^{r-s} = \sum_{i=0}^{r-s} C_{r-s}^i \varepsilon^{2i} |\xi|^{2i}.$$

Hence, on multiplying the left-hand sides of these equalities we obtain

$$(1+|\xi|^2)^s(1+\varepsilon^2|\xi|^2)^{r-s} = \sum_{i=0}^{r} a_{r,s,i}(\varepsilon)|\xi|^{2i}$$

where

$$a_{r,s,i}(\varepsilon) = \sum_{j=0}^{i} C_s^{i-j} C_{r-s}^j \varepsilon^{2j}. \tag{2.1}$$

Here, we assume $C_r^k = 0$ when $k > r$. If $\varepsilon = 0$ or $r = s$, then $a_{r,s,i} = C_s^i$. Therefore, $a_{r,s,i}(\varepsilon) \neq 0$ for all ε and $0 \le i \le r$. As a consequence we get

$$\|u\|_{r,s}^2 = \sum_{i=0}^{r} a_{r,s,i}(\varepsilon) \| |\xi|^i \hat{u} \|_{L^2}^2.$$

Furthermore,

$$\| |\xi|^i \hat{u} \|_{L^2}^2 = \begin{cases} \left\|\Delta^{i/2} u\right\|_{L^2}^2, & \text{if } i \text{ is even}, \\ \left\|\nabla^i u\right\|_{L^2}^2, & \text{if } i \text{ is odd}, \end{cases}$$

which establishes the lemma. □

Now everything is prepared for proving the invariance of the norm $\|\cdot\|_{r,s}$ with respect to local changes of variables $x = T(y)$.

Lemma 2.4. *Let $r, s \in \mathbb{Z}_{\ge 0}$ satisfy $r \ge s$. The norm $\|u; H^{r,s}(D)\|$ is invariant with respect to any local changes of variables in D of the form $x = T(y)$, such that*

1. *$T : U \to U'$ is a C^r-diffeomorphism of domains U and U' in $\mathbb{R}^n$, both U and U' intersecting D;*
2. *$T(U \cap \overline{D}) = U' \cap \overline{D}$;*
3. *$T(U \cap \partial D) = U' \cap \partial D$.*

Our task is to prove that there is a constant $C > 0$ independent of ε, with the property that

$$\|T^* u; H^{r,s}(D)\| \le C \, \|u; H^{r,s}(D)\| \tag{2.2}$$

for all smooth functions u in the closure of D supported in some compact set $K \subset U' \cap \bar{D}$). Here, by $T^* u(y) := u(T(y))$ is meant the pullback of u by the diffeomorphism T. If u is supported in K, then $T^* u$ is supported in $T^{-1}(K)$, which is a compact subset of $U' \cap \bar{D}$ by the properties of T. Since this applies to the inverse $T^{-1} : U' \to U$, it follows from (2.2) that the space $H^{r,s}(D)$ survives under the local C^r-diffeomerphisms of $\bar{D}$.

Proof. For the proof we make use of another norm in $H^{r,s}(D)$ which is obviously equivalent to $\|u; H^{r,s}(D)\|$ and more convenient here. To wit,

$$\|u; H^{r,s}(D)\| \cong \sum_{|\alpha| \le r} a_{r,s,|\alpha|}(\varepsilon) \|\partial^\alpha u; L^2(D)\| \tag{2.3}$$

(or

$$\|u; H^{r,s}(D)\| \cong \sum_{i=0}^{r} a_{r,s,i}(\varepsilon) \, \|u; H^i(D)\|,$$

as is easy to verify), where $a_{r,s,i}(\varepsilon)$ are the polynomials of Lemma 2.3. Fix a compact set K in $U' \cap \bar{D}$. As mentioned, if u is a smooth function in $\overline{D}$ with support in K, then $T^* u$ is a smooth function in $\overline{D}$ with support in $T^{-1}(K) \subset U \cap \overline{D}$. Obviously,

$$\begin{aligned} \|T^* u; H^{r,s}(D)\| &= \|u \circ T; H^{r,s}(U \cap D)\| \\ &= \sum_{|\alpha| \le r} a_{r,s,|\alpha|}(\varepsilon) \|\partial^\alpha (u \circ T); L^2(U \cap D)\|. \end{aligned}$$

By the chain rule,

$$\partial_y^\alpha \left(u(T(y)) \right) = \sum_{0 \ne \beta \le \alpha} c_{\alpha,\beta}(y) \left(\partial_x^\beta u \right) (T(y))$$

for any multiindex α with $|\alpha| \le r$. Here, the coefficients $c_{\alpha,\beta}(y)$ are polynomials of degree $|\beta|$ of partial derivatives of $T(y)$ up to order $|\alpha| - |\beta| + 1 \le r$. Since $T : U \to U'$ is a diffeomorphism of class C^r, all the $c_{\alpha,\beta}(y)$ are bounded on the compact set $T^{-1}(K)$ and the Jacobian $\det T'(y)$ does not vanish on $T^{-1}(K)$. This implies

$$\begin{aligned} \|T^* u; H^{r,s}(D)\| &\le c \sum_{|\alpha| \le r} a_{r,s,|\alpha|}(\varepsilon) \sum_{\beta \le \alpha} \|(\partial_x^\beta u) \circ T; L^2(T^{-1}(K))\| \\ &\le c \sum_{|\alpha| \le r} a_{r,s,|\alpha|}(\varepsilon) \sum_{\beta \le \alpha} \|\partial_x^\beta u; L^2(K)\|, \end{aligned}$$

where $c = c(T,r,K)$ is a constant independent of u and different in diverse applications. Interchanging the sums in α and β yields

$$\|T^*u; H^{r,s}(D)\| \leq c \sum_{|\beta|\leq r} \Bigg(\sum_{\substack{|\alpha|\leq r \\ \alpha\geq\beta}} a_{r,s,|\alpha|}(\varepsilon) \Bigg) \sum_{0\neq\beta\leq\alpha} \|\partial^\beta u; L^2(D)\|$$

for all smooth functions u in $\overline{D}$ with support in K.

Therefore, if there is a constant $C > 0$ such that

$$\sum_{\substack{|\alpha|\leq r \\ \alpha\geq\beta}} a_{r,s,|\alpha|}(\varepsilon) \leq C\, a_{r,s,|\beta|}(\varepsilon)$$

for each multiindex β of norm $|\beta| \leq r$, then the lemma follows. Since

$$\sum_{\substack{|\alpha|\leq r \\ \alpha\geq\beta}} a_{r,s,|\alpha|}(\varepsilon) \leq c \sum_{i=|\beta|}^{r} a_{r,s,i}(\varepsilon)$$

with c a constant dependent only on r and n, we are left with the task to show that there is a constant $C > 0$ independent of ε, such that

$$\sum_{i=i_0}^{r} a_{r,s,i}(\varepsilon) \leq C\, a_{r,s,i_0}(\varepsilon)$$

for all $i_0 = 0, 1, \ldots, r$. This latter estimate is in turn fulfilled if we show that

$$a_{r,s,i}(\varepsilon) \leq C\, a_{r,s,i-1}(\varepsilon) \tag{2.4}$$

for all $i = 1, \ldots, r$, where C is a constant independent of $\varepsilon \in [0,1]$. By formula (2.1),

$$a_{r,s,i}(\varepsilon) = \sum_{j=0}^{i-s-1} C_s^{i-j} C_{r-s}^{j} \varepsilon^{2j},$$

hence, estimate (2.4) is fulfilled for sufficiently small $\varepsilon > 0$ with any constant C greater than C_s^i/C_s^{i-1}. Since (2.4) is valid for all ε in any interval $[\varepsilon_0, 1]$ with $\varepsilon_0 > 0$, the proof is complete. □

Remark 2.1. The case of inner point is not singled out in the Lemma 2.4. Clearly, the problem is easier away from the boundary, for neither the condition 2 nor the condition 3 are no longer required.

Lemma 2.5. *The spaces $H^{r,s}(D)$ are invariant with respect to the local changes variables described in Lemma* 2.4*, when r, s are positive real numbers and $r \geq s$.*

Proof. This follows from Lemma 2.4 by using standard interpolation techniques (see e.g. [12]). □

To use local techniques it is convenient to define the spaces $\mathcal{H}^{\rho,\sigma}(\partial D)$ by locally rectifying the boundary surface. Since the boundary is compact, there is a finite covering $\{U_i\}_{i=1}^N$ of ∂D consisting of sufficiently small open subsets U_i of $\mathbb{R}^n$. Let $\{\phi_i\}$ be a partition of unity in a neighbourhood of ∂D subordinate to this covering. If U_i is small enough, there is a smooth diffeomorphism h_i of U_i onto an open set O_i in $\mathbb{R}^n$, such that $h_i(U_i \cap D) = O_i \cap \mathbb{R}^n_+$ and $h_i(U_i \cap \partial D) = O_i \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$. The transition mappings $T_{i,j} = h_i^{-1} \circ h_j$ prove to be local diffeomorphism of D, as explained in Lemma 2.4. For any smooth function u on the boundary the norm $\|(h_i^{-1})^*(\phi_i u); \mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})\|$ is obviously well defined and we set

$$\|u; \mathcal{H}^{\rho,\sigma}(\partial D)\| := \sum_{i=1}^{N} \|(h_i^{-1})^*(\phi_i u); \mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})\|, \tag{2.5}$$

where $(h_i^{-1})^*(\phi_i u) = (\phi_i u) \circ h_i^{-1}$. As usual, the space $\mathcal{H}^{\rho,\sigma}(\partial D)$ is introduced to be the completion of $C^\infty(\partial D)$ with respect to the norm (2.5).

When combined with the trace theorem for the spaces $H^{r,s}(\mathbb{R}^n_+)$ and $\mathcal{H}^{\rho,\sigma}(\mathbb{R}^{n-1})$ proved in [1], and Lemma 2.4, a familiar trick readily shows that the Banach spaces $\mathcal{H}^{\rho,\sigma}(\partial D)$ are actually independent of the particular choice of the covering of ∂D by coordinate patches $\{U_i\}$ in $\mathbb{R}^n$, the special coordinate system $h_i : U_i \to \mathbb{R}^n$ in U_i and the partition of unity $\{\phi_i\}$ in a neighbourhood of ∂D subordinate to the covering $\{U_i\}$. Any other choice of these data leads to an equivalent norm (2.5) in $C^\infty(\partial D)$.

Lemma 2.6. *As defined above, the spaces $\mathcal{H}^{\rho,\sigma}(\partial D)$ are invariant with respect to local diffeomerphisms of the boundary surface ∂D.*

The reader gives readily the concept of local diffeomorphisms of ∂D a sense similar to that of Lemma 2.4.

3. Auxiliary results

When compared to the usual local techniques of elliptic theory, the theory of elliptic boundary value problems with small parameter include only three additional estimates uniform in the parameter. To wit,

1. the invariance of the norm with respect to local changes of variables on the compact manifold $\overline{D}$;

2. estimates of the form $\varepsilon^k \|\partial^\alpha u\|_{r,s} \le c\,\|u\|_{r',s'}$ with c independent of ε;

3. inequalities like $\|u\|_{r,s} \le \delta\,\|u\|_{r',s'} + C(\delta)\,\|u\|_{L^2}$ with $r' \ge r$, $s' \ge s$ and $\delta > 0$ a fixed arbitrary small parameter.

As usual, we write α, β and γ for multiindices. By $\beta \le \alpha$ is meant that $\beta_i \le \alpha_i$ for all $i = 1, \dots, n$. We first recall several basic inequalities concerning Sobolev spaces. Directly from the multinomial theorem we obtain

$$|\xi^\alpha| \le \frac{1}{C_n^\alpha}\,|\xi|^{|\alpha|/2}, \tag{3.1}$$

where C_n^α is the multinomial coefficient. This inequality, if combined with the Plancherel theorem, yields

$$\|\partial^\alpha u\|_{L^2} \le \frac{1}{C_n^\alpha}\,\|\Delta^{|\alpha|/2} u\|_{L^2}$$

for all $u \in H^{|\alpha|} := H^{|\alpha|}(\mathbb{R}^n)$, where $\Delta^{|\alpha|/2}$ is a fractional power of the Laplace operator in $\mathbb{R}^n$.

Besides, we use the following consequence of the embedding theorem for Sobolev spaces (see e.g. [10]).

Theorem 3.1. *Suppose u is a square integrable function with compact support in $\mathbb{R}^n$ and $\alpha \in \mathbb{Z}^n_{\ge 0}$ is fixed. If, in addition, the weak derivatives $\partial^\beta u$ are square integrable for all $\beta \le \alpha$, then*

$$\|\partial^\beta u\|_{L^2} \le C\,\|\partial^\alpha u\|_{L^2},$$

where $C = \sup\{|x|^2 : x \in \operatorname{supp} u\}$.

We also need some basic inequalities for the norms $\|\cdot\|_{r,s}$.

Lemma 3.1. *Let $u \in H^{r,s}(\mathbb{R}^n)$ be a function with compact support, $k \ge 1$ an integer and α a multiindex. Then:*

1. *We have $\varepsilon\|u\|_{r,s} \le c\,\|u\|_{r+1,s}$, where c depends on the support of u but not on u and ε.*

2. *If $k > |\alpha|$, then $\varepsilon^k\|\partial^\alpha u\|_{r,s} \le c\,\|u\|_{r+k,s}$, the constant c being independent of u and ε.*

3. *If $k \le |\alpha|$, then $\varepsilon^k\,\|\partial^\alpha u\|_{r,s} \le c\,\|u\|_{r+|\alpha|,s+|\alpha|-k}$, where c is independent of u and ε.*

Proof. Using the expression for the norm in $H^{r,s}(\mathbb{R}^n)$ we get

$$\varepsilon \, \|\Delta^{1/2}u\|_{r,s} = \varepsilon \, \| |\xi| (1+|\xi|^2)^{s/2} (1+\varepsilon^2|\xi|^2)^{(r-s)/2} \hat{u} \|_{L^2} \le \|u\|_{r+1,s}.$$

As $\|u\|_{r,s} \le c \, \|\Delta^{1/2}u\|_{r,s}$, the part 1 is true.

The part 2 is proved in much the same way if one applies $k - |\alpha|$ times what has already been proved in the part 1.

To prove the part 3 we split the majorising factor as $\varepsilon^k |\xi|^{|\alpha|} = (\varepsilon|\xi|)^k \, |\xi|^{|\alpha|-k}$. The first factor contributes with order k to the terms with ε while the second one does $|\alpha| - k$ to the others. $\square$

The part 2 actually holds for all function in $H^{r+k,s}$ even if u fails to be of compact support.

Lemma 3.2. *Let δ be an arbitrary small positive number. Then there is a constant $C(\delta)$, such that*

$$\|u\|_{r-1,s-1} \le \delta \, \|u\|_{r,s} + C(\delta) \, \|u\|_{L^2}$$

for all $u \in H^{r,s}(\mathbb{R}^n)$.

Proof. Set $\langle \xi \rangle = \sqrt{1+|\xi|^2}$ for $\xi \in \mathbb{R}^n$. Given any $R > 0$, we obtain

$$\begin{aligned} \|u\|^2_{r-1,s-1} &= \int\limits_{|\xi|>R} \frac{\langle \xi \rangle^{2s}}{\langle \xi \rangle^2} \langle \varepsilon\xi \rangle^{2(r-s)} |\hat{u}|^2 d\xi + \int\limits_{|\xi|\le R} \langle \xi \rangle^{2(s-1)} \langle \varepsilon\xi \rangle^{2(r-s)} |\hat{u}|^2 d\xi \\ &\le \frac{1}{1+R^2} \, \|u\|^2_{r,s} + (1+R^2)^{s-1} (1+\varepsilon^2 R^2)^{r-s} \, \|u\|^2_{L^2}, \end{aligned}$$

Choosing $R > 0$ in such a way that $\delta^2 \le (1+R^2)^{-1}$, we establish the estimate, as is easy to check. $\square$

4. The main result

Now we are in a position to present the main result of this work. We impose two restrictions on the boundary value problem under study, namely, the condition of ellipticity and the Shapiro–Lopatinskii condition with small parameter. To formulate these denote by A_0 the principal part of the operator A which is understood here as

$$A_0(x, D, \varepsilon) := \varepsilon^{2m-2\mu} \, A_{2m,0}(x, D) + \cdots + \varepsilon \, A_{2\mu+1,0}(x, D) + A_{2\mu,0}(x, D),$$

where $A_{j,0}(x, \xi)$ stands for the principal homogeneous symbol of the differential operator $A_j(x, D)$ of order j, with $2\mu \le j \le 2m$. Recall that the differential operator $A(x, D, \varepsilon)$ is said to satisfy the small parameter ellipticity condition in the domain D if $n > 2$ and for every $x \in D$ the polynomial $A_0(x, \xi, \varepsilon)$ admits an estimate

$$|A_0(x, \xi, \varepsilon)| \geq c_x \, |\xi|^{2\mu}(1 + \varepsilon|\xi|)^{2m-2\mu}$$

for all $\xi \in \mathbb{R}^n$ and $\varepsilon \in [0, 1]$, where $c_x > 0$ is a constant which depends only on the point x.

In the case $n = 2$ the polynomial $A_0(x, \xi', \xi_n, \varepsilon)$ considered with respect to the variable ξ_n is assumed to possess exactly m roots in the upper complex half-plane and m roots in the lower half-plane, for every $x \in D$, $\varepsilon > 0$, $\xi' \in \mathbb{R}^{n-1}$.

As is well known in elliptic theory, the ellipticity condition guarantees in the case $n > 2$ that the polynomial $A_0(x, \xi', \xi_n, \varepsilon)$ has m roots in the upper half-plane and m roots in the lower one. So, this property can be taken as basis for the small parameter ellipticity definition.

By the Shapiro–Lopatinskii condition with a small parameter is just meant that the boundary value problem $(A(x, D, \varepsilon), B(x', D, \varepsilon))$ satisfies the usual Shapiro–Lopatinskii condition for each fixed $x' \in \partial D$ and $\varepsilon \in [0, 1]$. This latter condition means that the polynomials $B_j(x', \xi, \varepsilon)$ are linearly independent modulo $A(x', \xi, \varepsilon)$ for each point $x' \in \partial D$ and $\varepsilon \geq 0$.

Theorem 4.1. *Under the above conditions, if moreover $r \geq 2m$ and $s \geq 2\mu$, then there is an estimate*

$$\|u\|_{r,s} \leq C \Big(\|A(x, D, \varepsilon)u\|_{r-2m,s-2\mu} + \sum_{j=1}^{m} \|B_j(x', D, \varepsilon)u\|_{r-b_j-1/2, s-\beta_j-1/2} + \|u\|_{L^2(D)} \Big) \quad (4.1)$$

with C a constant independent of u and ε.

The proof exploits localisation techniques. *First*, using a finite covering $\{U_i\}$ of $\overline{D}$ by sufficiently small open sets (e.g. balls) in $\mathbb{R}^n$, we represent any function $u \in H^{r,s}(D)$ as the sum of functions $u_i \in H^{r,s}(D)$ compactly supported in $U_i \cap \overline{D}$, just setting $u_i = \phi_i u$ for a suitable partition of unity $\{\phi_i\}$ in $\overline{D}$ subordinate to the covering $\{U_i\}$. *Secondly*, for each summand u_i we formulate its own elliptic problem and find a priori estimates for its solutions. If U_i does not meet the boundary of D, then the support of u_i is a compact subset of D and the proof of (4.1) reduces to global analysis in $\mathbb{R}^n$ considered in [1]. For those U_i which intersect the boundary of D we choose a change of variables $x = h_i^{-1}(z)$ to rectify the boundary surface within U_i. To wit, $h_i(U_i \cap D) = O_i \cap \mathbb{R}^n_+$, where O_i is an open set in $\mathbb{R}^n$, and so in the coordinates y estimate (4.1) reduces to that in the case $D = \mathbb{R}^n_+$ treated in [1]. *Thirdly*, we glue together all a priori estimates for u_i thus obtaining a priori estimate (4.1) for u.

Perhaps the focus of local techniques is on the second and third steps. Taking for granted the estimates of the second step, we complete the proof of Theorem 4.1.

Proof. For each point $x_0 \in D$ we choose a neighbourhood U_{x_0} in D in which the estimate of Theorem 5.1 holds. And for each point $x_0 \in \partial D$ we choose a neighbourhood U_{x_0} in $\mathbb{R}^n$, such that the estimate of Theorem 5 is valid. Shrinking U_{x_0}, if necessary, one can assume that the surface $U_{x_0} \cap \partial D$ can be rectified by some diffeomorphism $h_i : U_{x_0} \to \mathbb{R}^n$, as explained above. The family $\{U_{x_0}\}_{x_0 \in \overline{D}}$ is an open covering of $\overline{D}$, hence it contains a finite family $\{U_i\}$ which covers $\overline{D}$. Fix a C^∞ partition of unity $\{\phi_i\}$ in a neighbourhood of $\overline{D}$ subordinate to the covering $\{U_i\}$.

Given any $u \in H^{r,s}(D)$, we get

$$u = \sum_i u_i$$

in D, where $u_i := \phi_i u$ belongs to $H^{s,r}(D)$ and $\operatorname{supp} u_i \subset U_i \cap \overline{D}$. By assumption, for any function u_i estimate (4.1) holds with a constant C depending on i. As the family $\{U_i\}$ is finite, there is no restriction of generality in assuming that C does not depend on i. Hence,

$$\begin{aligned}\|u\|_{r,s} \ \leq \ C \sum_i \Big(& \|A(x,D,\varepsilon)u_i\|_{r-2m,s-2\mu} \\ & + \sum_{j=1}^m \|B_j(x',D,\varepsilon)u_i\|_{r-b_j-1/2,s-\beta_j-1/2} + \|u_i\|_{L^2(D)} \Big).\end{aligned}$$

By the Leibniz formula,

$$\begin{aligned} A(x,D,\varepsilon)u_i &= \phi_i\, A(x,D,\varepsilon)u + [A,\phi_i]\, u, \\ B_j(x',D,\varepsilon)u_i &= \phi_i\, B_j(x,D,\varepsilon)u + [B_j,\phi_i]\, u, \end{aligned}$$

where $[A,\phi_i]u = A(\phi_i u) - \phi_i A u$ is the commutator of A and the operator of multiplication with ϕ_i, and similarly for $[B_j,\phi_i]$. The commutators are known to be differential operators of order less than that of A and B_j, respectively. From the structure of the operator $A(x,D,\varepsilon)$ we see that the summands of $[A,\phi_i]u$ are of the form

$$\varepsilon^{2m-2\mu-k} a_{k,\beta}(x) \partial^\beta u, \qquad (4.2)$$

where $k = 0,1,\ldots,2m-2\mu$, $|\beta| \leq 2m-k-1$ and $a_{k,\beta}$ are smooth functions in the closure of D independent of u.

To estimate the norm of (4.2) in $H^{r-2m,s-2\mu}$, we apply Lemma 3.1 and consider separately the cases

$$\begin{array}{rcl} 2m-2\mu-k & > & |\beta|, \\ 2m-2\mu-k & \leq & |\beta|. \end{array}$$

If e.g. $|\beta| \geq 2m-2\mu-k$, then

$$\varepsilon^{2m-2\mu-k}\|a_{k,\beta}\partial^\beta u\|_{r-2m,s-2\mu} \leq c\,\varepsilon^{2m-2\mu-k}\|a_{k,\beta}\partial^\beta u\|_{r-2m+|\beta|,s-2m+|\beta|+k},$$

where $|\beta|-2m+k \leq -1$. It follows that

$$\varepsilon^{2m-2\mu-k}\|a_{k,\beta}\partial^\beta u\|_{r-2m,s-2\mu} \leq c\,\|u\|_{r-1,s-1}$$

with c a constant independent of u and ε. Such terms are handled by Lemma 3.2. Analogously we estimate the summands (4.2) with $2m-2\mu-k > |\beta|$ and the commutators $[B_j, \phi_i]$, which establishes (4.1). □

5. Local estimates in the interior

Theorem 5.1. *For every $x_0 \in D$ there exists a neighbourhood U_{x_0} in D and a constant C independent of ε, such that*

$$\|u\|_{r,s} \leq C\left(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + \|u\|_{L^2}\right) \tag{5.1}$$

for all functions $u \in H^{r,s}(D)$ with compact support in U_{x_0}, where $r \geq 2m$, $s \geq 2\mu$ are integer.

This theorem is not contained in [1], for [1] focuses on differential operators with constant coefficients in $\mathbb{R}^n$.

Proof. If $u \in H^{r,s}(D)$ is compactly supported in D, it can be thought of as an element of $H^{r,s}(\mathbb{R}^n)$ as well. The norm of u in $H^{s,r}(D)$ just amounts to the norm of u in $H^{s,r}(\mathbb{R}^n)$. Hence, the paper [1] applies if $A(x,D,\varepsilon)$ has constant coefficients, as is the case e.g. for $A_0(x_0,D,\varepsilon)$, the principal part of $A(x,D,\varepsilon)$ with coefficients frozen at x_0. According to [1], there is a constant $C > 0$ independent of ε, such that

$$\|u\|_{r,s} \leq C\,\|A_0(x_0,D,\varepsilon)u\|_{r-2m,s-2\mu} \tag{5.2}$$

for all functions $u \in H^{r,s}(D)$ of compact support in D.

We are thus left with the task to majorise the right-hand side of (5.2) by that of (5.1) uniformly in $\varepsilon \in [0,1]$ on functions with compact support in U_{x_0}. To this end, we write

$$A_0(x_0, D, \varepsilon) = A(x, D, \varepsilon) - (A(x, D, \varepsilon) - A_0(x, D, \varepsilon)) \\ - (A_0(x, D, \varepsilon) - A_0(x_0, D, \varepsilon))$$

whence

$$\begin{aligned} &\|A_0(x_0, D, \varepsilon)u\|_{r-2m,s-2\mu} \\ &\leq \|A(x, D, \varepsilon)u\|_{r-2m,s-2\mu} + \|(A(x, D, \varepsilon) - A_0(x, D, \varepsilon))u\|_{r-2m,s-2\mu} \\ &\qquad + \|(A_0(x, D, \varepsilon) - A_0(x_0, D, \varepsilon))u\|_{r-2m,s-2\mu}. \end{aligned} \tag{5.3}$$

Our next concern will be to estimate the last two summands on the right-hand side of (5.3). We begin with the first of these two. By the very structure of the operator $A(x, D, \varepsilon)$, the difference $A(x, D, \varepsilon) - A_0(x, D, \varepsilon)$ is the sum of terms of the form

$$\varepsilon^{2m-2\mu-k} a_{k,\beta}(x) \partial^\beta u,$$

where $k = 0, 1, \ldots, 2m - 2\mu$, $|\beta| \leq 2m - k - 1$ and $a_{k,\beta}$ are smooth functions in the closure of D (cf. (4.2)). Hence, the reasoning used in the proof of Theorem 4.1 shows that the second summand on the right-hand side of (5.3) is dominated uniformly in $\varepsilon \in [0, 1]$ by the norm $\|u\|_{r-1,s-1}$. On applying Lemma 3.2 we conclude that

$$\|(A(x, D, \varepsilon) - A_0(x, D, \varepsilon))u\|_{r-2m,s-2\mu} \leq \delta \, \|u\|_{r,s} + C(\delta) \, \|u\|_{L^2}, \tag{5.4}$$

where $\delta > 0$ is an arbitrarily small parameter and $C(\delta)$ depends only on δ but not on u and ε.

It remains to estimate the last summand on the right-hand side of (5.3). Let us write

$$A_0(x, D, \varepsilon) = \sum_{2\mu \leq |\beta| \leq 2m} \varepsilon^{|\beta|-2\mu} A_{0,\beta}(x) \partial^\beta,$$

where $A_{0,\beta}$ are smooth functions on the closure of D. Then

$$\begin{aligned} &\|(A_0(x, D, \varepsilon) - A_0(x_0, D, \varepsilon))u\|_{r-2m,s-2\mu} \\ &\qquad \leq \sum_{2\mu \leq |\beta| \leq 2m} \varepsilon^{|\beta|-2\mu} \|(A_{0,\beta}(x) - A_{0,\beta}(x_0)) \partial^\beta u\|_{r-2m,s-2\mu}. \end{aligned}$$

To evaluate the summands we invoke the equivalent expression for the norm in $H^{r-2m,s-2\mu}(D)$ given by (2.3). The typical term is

$$a_{r-2m,s-2\mu,|\alpha|}(\varepsilon) \, \varepsilon^{|\beta|-2\mu} \left\| \partial^\alpha \big((A_{0,\beta}(x) - A_{0,\beta}(x_0)) \partial^\beta u \big) \right\|_{L^2(D)}$$

with $|\alpha| \le r-2m$ and $a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)$ are the polynomials defined in of (2.1). By the Leibniz formula,

$$\partial^\alpha\big((A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^\beta u\big)=(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^{\alpha+\beta}u+[\partial^\alpha, A_{0,\beta}]\,\partial^\beta u,$$

where the commutator $[\partial^\alpha, A_{0,\beta}]$ is a differential operator of order $|\alpha|-1$ with smooth coefficients in $\overline{D}$. Observe that $|\alpha|+|\beta| \le r$. Arguing as above we derive easily an estimate like (5.4) for the sum

$$\sum_{|\alpha|\le r-2m} a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)\,\varepsilon^{|\beta|-2\mu}\,\|[\partial^\alpha, A_{0,\beta}]u\|_{L^2(D)}$$

whenever $u \in H^{r,s}(D)$ is of compact support in D.

It is the term

$$a_{r-2m,s-2\mu,|\alpha|}(\varepsilon)\,\varepsilon^{|\beta|-2\mu}\,\|(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^{\alpha+\beta}u\|_{L^2(D)}$$

that admits a desired estimate only in the case if the support of u is small enough. (Recall that u is required to have compact support in U_{x_0}.) Since the coefficients $A_{0,\beta}(x)$ are Lipschitz continuous in $\overline{D}$, for any arbitrarily small $\delta' > 0$ there is a positive $\varrho = \varrho(\delta')$, such that

$$\|(A_{0,\beta}(x)-A_{0,\beta}(x_0))\partial^{\alpha+\beta}u\|_{L^2(D)} \le \delta'\,\|\partial^{\alpha+\beta}u\|_{L^2(D)}$$

for all functions $u \in H^{r,s}(D)$ with compact support in $B(x_0,\varrho)$, the ball of radius ϱ with center x_0.

Summarising we conclude that for each $\delta > 0$ there is a constant $C = C(\delta)$ independent of ε, such that

$$\|(A_0(x,D,\varepsilon)-A_0(x_0,D,\varepsilon))u\|_{r-2m,s-2\mu} \le \delta\,\|u\|_{r,s} + C(\delta)\,\|u\|_{L^2} \quad (5.5)$$

for all functions $u \in H^{r,s}(D)$ with compact support in $B(x_0,\varrho)$, provided that $\varrho = \varrho(\delta)$ is sufficiently small. Needless to say that $C(\delta)$ need not coincide with the similar constant of inequality (5.4), however, we may assume this without loss of generality.

On gathering estimates (5.3) and (5.4), (5.5) and substituting them into (5.1) we arrive at

$$(1-2C\delta)\,\|u\|_{r,s} \le C\,\big(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + 2C(\delta)\,\|u\|_{L^2}\big)$$

for all $u \in H^{r,s}(D)$ with compact support in $B(x_0,\varrho)$. Of course, this latter inequality does not yield any estimate for $\|u\|_{r,s}$ unless $1-2C\delta > 0$. Thus, choosing $\delta < 1/2C$ we get

$$\|u\|_{r,s} \le \frac{2CC(\delta)}{1-2C\delta}\,\big(\|A(x,D,\varepsilon)u\|_{r-2m,s-2\mu} + \|u\|_{L^2}\big),$$

if $C(\delta) \ge 1/2$. □

6. The case of boundary points

Localisation at a boundary point $x_0 \in \partial D$ requires not only small parameter ellipticity of the operator $A(x, D, \varepsilon)$ but also the Shapiro–Lopatinskii condition with small parameter.

Theorem 6.1. *For every point $x_0 \in \partial D$ there is a neighbourhood U_{x_0} in $\mathbb{R}^n$, such that*

$$\|u\|_{r,s} \leq C \Big(\|A(x, D, \varepsilon)u\|_{r-2m,s-2\mu} + \sum_{j=1}^{m} \|B_j(x', D, \varepsilon)u\|_{r-b_j-1/2,s-\beta_j-1/2} + \|u\|_{L^2(D)} \Big) \quad (6.1)$$

for all functions $u \in H^{r,s}(D)$ with compact support in $U_{x_0} \cap D$, where C is a constant independent of both u and $\varepsilon \in [0, 1]$.

Proof. Choose a neighbourhood U of x_0 in $\mathbb{R}^n$ and a diffeomorphism $z = h(x)$ of U onto an a neighbourhood O of the origin $0 = h(x_0)$ in $\mathbb{R}^n$ with the property that $h(U \cap D) = O \cap \mathbb{R}^n_+$ and $h(U \cap \partial D) = \{z \in O : z_n = 0\}$. If $u \in H^{r,s}(D)$ is a function with compact support in $U \cap \overline{D}$, then the pullback $\tilde{u} = (h^{-1})^* u$ belongs to $H^{r,s}(\mathbb{R}^n_+)$ and has compact support in $O \cap \overline{\mathbb{R}^n_+}$, which is due to Lemma 2.4. On setting

$$A^\sharp := (h^{-1})^* A h^*,$$
$$B_j^\sharp := (h^{-1})^* B_j h^*,$$

for $j = 1, \ldots, m$, we obtain the pullbacks of the operators A and B_j under the diffeomorphism $h : U \cap \overline{D} \to O \cap \overline{\mathbb{R}^n_+}$. It is easily seen that $A^\sharp$ and $B_j^\sharp$ are differential operators with small parameter $\varepsilon \in [0, 1]$ on $O \cap \overline{\mathbb{R}^n_+}$ in the sense explained above. We write $\tilde{A} := A^\sharp$ and $\tilde{B}_j := B_j^\sharp$ for short. Since the spaces $H^{r,s}(D)$ and $\mathcal{H}^{\rho,\sigma}(\partial D)$ are invariant under local diffeomorphisms of D, it follows that estimate (6.1) is equivalent to

$$\|\tilde{u}\|_{r,s} \leq C \Big(\|\tilde{A}(z, D, \varepsilon)\tilde{u}\|_{r-2m,s-2\mu} + \sum_{j=1}^{m} \|\tilde{B}_j(z', D, \varepsilon)\tilde{u}\|_{r-b_j-1/2,s-\beta_j-1/2} + \|\tilde{u}\|_{L^2(D)} \Big) \quad (6.2)$$

for all functions $\tilde{u} \in H^{r,s}(\mathbb{R}^n_+)$ with compact support in $O \cap \overline{\mathbb{R}^n_+}$, where C is a constant independent of $\tilde{u}$ and ε.

From the transformation formula for principal symbols of differential operators it follows that the problem

$$\begin{cases} \tilde{A}_0(0, D, \varepsilon)\tilde{u} = \tilde{f} & \text{for } z_n > 0, \\ \tilde{B}_{j,0}(0, D, \varepsilon)\tilde{u} = \tilde{u}_j & \text{for } z_n = 0, \end{cases}$$

where $j = 1, \ldots, m$, satisfies both the ellipticity condition and the Shapiro–Lopatinskii condition with small parameter in the half-space. We now apply the main result of [1] which says that there is a constant $C > 0$ independent of ε, such that the inequality

$$\|\tilde{u}\|_{r,s} \le C \Big(\|\tilde{A}_0(0, D, \varepsilon)\tilde{u}\|_{r-2m,s-2\mu} + \sum_{j=1}^{m} \|\tilde{B}_{j,0}(0, D, \varepsilon)\tilde{u}\|_{r-b_j-1/2,s-\beta_j-1/2} + \|\tilde{u}\|_{L^2(\mathbb{R}^n_+)} \Big)$$

holds true for all functions $\tilde{u} \in H^{r,s}(\mathbb{R}^n_+)$ with compact support in the closed half-space.

Estimate (6.2) follows from the latter estimate in much the same way as estimate (5.1) does from (5.2), see the proof of Theorem 5.1. The only difference consists in evaluating the boundary terms. However, estimates on the boundary are reduced readily to those in the half-space if one exploits the embedding theorem, see Theorem 1.1. Namely,

$$\begin{aligned} &\|(\tilde{B}_j(z', D, \varepsilon) - \tilde{B}_{j,0}(0, D, \varepsilon))\tilde{u};\, \mathcal{H}^{r-b_j-1/2,s-\beta_j-1/2}(\mathbb{R}^{n-1})\| \\ &\qquad \le c\, \|(\tilde{B}_j(z', D, \varepsilon) - \tilde{B}_{j,0}(0, D, \varepsilon))\tilde{u};\, H^{r-b_j,s-\beta_j}(\mathbb{R}^n_+)\| \end{aligned}$$

with c a constant independent of $\tilde{u}$ and ε. □

Conclusion

Theorem 4.1 answers the question about the Fredholm property of the elliptic problem (A, B) in the case where D is a bounded domain with smooth boundary. As is shown in Section 1, this allows one to apply the boundary layer method. However, the smoothness of boundary is a strong restriction, all arguments of this paper go through if the boundary is of mere class C^r. Still the solvability and regularity in smooth bounded domains lay the foundation for further researches and are a necessary step in constructing the theory of small parameter ellipticity in domains with singular points.

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