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Balleans of topological groups

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Abstract. A subset S of a topological group G is called bounded if, for every neighborhood U of the identity of G, there exists a finite subset F such that $S \subseteq FU$, $S \subseteq UF$. The family of all bounded subsets of G determines two structures on G, namely the left and right balleans $B_l(G)$ and $B_r(G)$, which are counterparts of the left and right uniformities of G. We study the relationships between the uniform and ballean structures on G, describe all topological groups admitting a metric compatible both with uniform and ballean structures, and construct a group analogue of Higson's compactification of a proper metric space.

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Introduction

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets and, for every $x \in X$ and $\alpha \in P, B(x, \alpha)$ is a subset of X which is called a *ball* of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is called the *support* of $\mathcal{B}, P$ is called the *set of radii.*

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\}, \quad B(A,\alpha) = \bigcup_{\alpha \in A} B(a,\alpha).$$

A ball structure $\mathcal{B}$ is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$, such that, for every $x \in X$,

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$$B^*(x,\alpha') \subseteq B(x,\alpha), \quad B(x,\beta') \subseteq B^*(x,\beta);$$

- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x,\alpha) \subseteq B^*(x,\alpha'), \quad B^*(x,\beta) \subseteq B(x,\beta');$$

- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x,\gamma),\gamma) \subseteq B(x,\alpha) \bigcap B(x,\beta),$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$.

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family

$$\left\{ \bigcup_{x \in X} B(x,\alpha) \times B(x,\alpha) : \alpha \in P \right\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x,U) = \{y \in X : (x,y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure $\mathcal{B}$ is a *ballean* if $\mathcal{B}$ is upper symmetric and upper multiplicative.

The balleans are coming from many different areas: group theory [4,5], coarse geometry [12] and asymptotic topology [2], combinatorics [8]. A ballean can also be defined in terms of entourages. In this case, it is called a coarse structure. In this paper we follow terminology from [9].

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. We say that a mapping $f : X_1 \to X_2$ is a *$\prec$-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta),$$

and note that $\prec$-mapping is a counterpart of a uniformly continuous mapping between the uniform topological spaces.

We say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *asymorphic* if there exists a bijection $f : X_1 \to X_2$ such that f and f^{-1} are $\prec$-mappings.

If $\mathcal{B}_1$, $\mathcal{B}_2$ are balleans with common support X and the identity mapping $id : X \to X$ is an asymorphism, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$, and write $\mathcal{B}_1 = \mathcal{B}_2$.

A ballean $\mathcal{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. We note that connectedness can be considered as a counterpart of Hausdorffness of a uniform topological space.

1. Balleans on groups

Let G be a group with the identity e, $\mathcal{F}_G$ be a family of all finite subsets of G, $\mathcal{I}$ be an ideal in the Boolean algebra of all subsets of G. We say that $\mathcal{I}$ is a *group ideal* if $\mathcal{F}_G \subseteq \mathcal{I}$ and $A, B \in \mathcal{I} \to AB^{-1} \in \mathcal{I}$. Every group ideal $\mathcal{I}$ determines two balleans (see [9, Chapter 6]) $\mathcal{B}_l(G, \mathcal{I})$ and $\mathcal{B}_r(G, \mathcal{I})$ on G, where $\mathcal{B}_l(G, \mathcal{I}){=}(G, \mathcal{I}, B_l)$, $\mathcal{B}_r(G, \mathcal{I}){=}(G, \mathcal{I}, B_r)$ and, for all $A \in \mathcal{I}$, $g \in G$,

$$\mathcal{B}_l(g, A) = g(A \bigcup \{e\}),\ \mathcal{B}_r(g, A) = (A \bigcup \{e\})g.$$

Now let G be finitely generated, S be a finite system of generators of G. The *left* (*right*) *Cayley graph* $Cay_l(G, S)$ ($Cay_r(G, S)$) is a graph with the set of vertices G and the set of edges $E_l = \{\{x, y\} : x^{-1}y \in S\}$ ($E_r = \{\{x, y\} : xy^{-1} \in S\}$). Clearly, these graphs are connected. Given any $x, y \in G$, we denote by $d_l(x, y)$ ($d_r(x, y)$) the length of a shortest path in $Cay_l(G, S)$ ($Cay_r(G, S)$) between x, y. The metric spaces $(G, d_l), (G, d_r)$ are an effective tool in geometrical group theory [4,5]. Every metric space can be considered as a ballean (see Section 2), and the balleans $\mathcal{B}_l(G, \mathcal{F}_G)$ $\mathcal{B}_r(G, \mathcal{F}_G)$ are asymorphic to the balleans determined by $(G, d_l), (G, d_r)$.

In what follows, all topological groups are supposed to be Hausdorff. A subset A of a topological group G is called *bounded* if, for every neighborhood U of the identity, there exists $F \in \mathcal{F}_G$ such that $A \subseteq FU$, $A \subseteq UF$. We note that A is bounded if and only if its closure in the completion of G by two-sided uniformity is compact.

A topological group G is said to be *totally bounded* (*σ-bounded, locally bounded*), if G is a bounded subset (G is a countable union of bounded subset, there is a bounded neighborhood of e).

Given a topological group (G, τ), the family $\mathcal{I}_\tau$ of all bounded subsets of G is a group ideal. A subject of this paper is the balleans $\mathcal{B}_l(G) = \mathcal{B}_l(G, \mathcal{I}_\tau)$, $\mathcal{B}_r(G) = \mathcal{B}_r(G, \mathcal{I}_\tau)$, which are called the left and right ballean of topological group G. For a locally compact group, these balleans were introduced and studied in [3].

Let G be a group with the identity e, $\mathcal{B} = (G, P, B)$ be a ballean on G. Following [9, Chapter 6], we say that $\mathcal{B}$ is

- *left* (*right*) *invariant* if all the shifts $x \mapsto gx$ ($x \mapsto xg$) are $\prec$-mappings;
- *uniformly left* (*right*) *invariant* if, for every $\alpha \in P$, there exists $\beta \in P$ such that $gB(x,\alpha) \subseteq B(gx,\beta)$ ($B(x,\alpha)g \subseteq B(xg,\beta)$) for all $x, g \in G$.

If $\mathcal{B}$ is uniformly left (right) invariant, then $\mathcal{B}$ is left (right) invariant, but the converse statement does not hold [9, Example 6.1.1].

Proposition 1.1. *For a connected ballean $\mathcal{B}$ on a group G, the following statements are equivalent*

(*i*) *$\mathcal{B}$ is uniformly left (right) invariant;*

(*ii*) *there exists a group ideal $\mathcal{I}$ on G such that $\mathcal{B} = \mathcal{B}_l(G,\mathcal{I})$ ($\mathcal{B} = \mathcal{B}_r(G,\mathcal{I})$).*

Proof. See [9, Section 6.1]. □

Given any $x \in G$, $A \subseteq G$, we put

$$x^G = \{g^{-1}xg : \ g \in G\}, \quad A^G = \bigcup_{a \in A} a^G.$$

We say that a group ideal $\mathcal{I}$ on G is *uniformly invariant* if $A^G \in \mathcal{I}$ for every $A \in \mathcal{I}$.

Proposition 1.2. *Let $\mathcal{I}$ be a group ideal on a group G. Then the following statements are equivalent*

(*i*) $\mathcal{B}_l(G,\mathcal{I}) = \mathcal{B}_r(G,\mathcal{I})$;

(*ii*) *$\mathcal{I}$ is uniformly invariant;*

(*iii*) *the mapping $x \mapsto x^{-1} : \mathcal{B}_l(G,\mathcal{I}) \to \mathcal{B}_l(G,\mathcal{I})$ is a $\prec$-mapping;*

(*iv*) *the mapping $(x,y) \to xy : \mathcal{B}_l(G,\mathcal{I}) \times \mathcal{B}_l(G,\mathcal{I}) \to \mathcal{B}_l(G,\mathcal{I})$ is a $\prec$-mapping.*

Proof. See [9, Section 6.1]. □

Proposition 1.3. *For a topological group G, the following statements are equivalent*

(*i*) $\mathcal{B}_l(G) = \mathcal{B}_r(G)$;

(*ii*) *a subset A^G is bounded for every bounded subset A;*

(*iii*) *the mapping $x \mapsto x^{-1} : \mathcal{B}_l(G) \to \mathcal{B}_l(G)$ is a $\prec$-mapping;*

(*iv*) *the mapping $(x, y) \mapsto xy : \mathcal{B}_l(G) \times \mathcal{B}_l(G) \to \mathcal{B}_l(G)$ is a $\prec$-mapping.*

Proof. Apply Proposition 1.2 to the group ideal $\mathcal{I}$ of all bounded subsets of G. □

Remark 1.1. By [13], for a locally compact group G, the condition (ii) of Proposition 1.3 is equivalent to the following one: x^G is bounded for every $x \in G$. We show that this statement does not hold for locally bounded groups. For each $n \in \omega$, we consider the semidirect product $A_n = B_n \leftthreetimes C_n$, where $B_n \simeq \mathbb{Z}_3$, $C_n \simeq \mathbb{Z}_2$ and put $G = \bigotimes_{n\in\omega} A_n$.

We endow G with the topology whose base at identity is formed by the subsets $\{\bigotimes_{m\geq n} C_n : m \in \omega\}$. Then G is a group with finite conjugated classes, the subset $C = \bigotimes_{n\in\omega} C_n$ is bounded, but C^G is unbounded.

2. Metrizability

A metric d on a set X determines the *metric ballean* $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$, where $\mathbb{R}^+ = \{r \in \mathbb{R} : r \geq 0\}$, $\mathbb{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}$. A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to some metric ballean. By [9, Theorem 2.1.1], a ballean $\mathcal{B} = (X, P, B)$ is metrizable if and only if $\mathcal{B}$ is connected and $cf\mathcal{B} \leqslant \aleph_0$, where *cofinality* $cf\mathcal{B}$ is the minimal cardinality of cofinal subsets of P. A subset $P' \subseteq P$ is cofinal if, for every $\alpha \in P$, there exists $\alpha' \in P$ such that $B(x, \alpha) \subseteq B(x, \alpha')$ for every $x \in X$.

Proposition 2.1. *Let d be a left invariant metric on a group G with the identity e, $V_r = \{x \in G : d(x, e) \leq r\}$, $r \in \mathbb{R}^+$. Then the family $\{V_r : r \in \mathbb{R}^+\}$ is a base for some group ideal $\mathcal{I}_d$ on G, and $\mathcal{B}(G, d) = \mathcal{B}_l(G, \mathcal{I}_\alpha)$.*

Proof. Given any $x, y \in G$, we have $d(x, e) = d(e, x^{-1})$ and $d(xy, e) = d(y, x^{-1}) \leq d(y, e) + d(x^{-1}, e) = d(y, e) + d(x, y)$, so $V_r = V_r^{-1}$ and $V_r V_s \subseteq V_{r+s}$ for all $r, s \in \mathbb{R}^+$. Clearly, every finite subset of G is contained in some ball V_r. Thus, $\mathcal{I}_d$ is a group ideal.

Since $d(x, y) \leq r$ if and only if $y \in xV_r$, $B(G, d) = B_l(G, \mathcal{I}_d)$. □

Proposition 2.2. *Let $\mathcal{I}$ be a group ideal with a countable base on a group G. Then there exists a left invariant metric d on G, taking integer values, such that $\mathcal{B}_l(G, \mathcal{I}) = \mathcal{B}(G, d)$.*

Proof. Since $\mathcal{I}$ has a countable base, we can choose a base $\{V_n : n \in \omega\}$ for $\mathcal{I}$ such that $V_0 = \{e\}$ and $V_n = V_n^{-1}$, $V_n V_n \subseteq V_{n+1}$ for each $n \in \omega$. Given any $x \in X$, we put

$$\|x\| = \min\{n \in \omega : x \in V_n\}.$$

By the choice of $\{V_n : n \in \omega\}$, we have

$$\|x\| = \| x^{-1} \|, \quad \|xy\| \leq \|x\| + \|y\|.$$

We define a metric d on G by the rule $d(x, y) = \|x^{-1}y\|$, and note that $\mathcal{B}(G, d) = \mathcal{B}_l(G, \mathcal{I})$. □

Now let G be a topological group. If G is first countable, by [6, Theorem 8.3], the left uniformity of G can be determined by some left invariant metric. If G is σ-bounded, by Proposition 2.2, the left ballean $\mathcal{B}_l(G)$ can also be determined by a left invariant metric. In the next theorem we stick together these two statements.

Theorem 2.1. *For every topological group G, the following statements are equivalent*

(i) *there is a left invariant metric d on G compatible both with left uniformity and left ballean structure of G;*

(ii) *G is first countable, locally bounded and σ-bounded.*

Proof. (ii) $\Rightarrow$ (i). If G is discrete, by Proposition 2.2, there exists a left invariant metric d on G taking integer values and determining left ballean structure of G. Clearly, d determines the discrete uniformity.

We assume that G is non-discrete and modify a construction of metric from [6, Theorem 8.3]. We fix a bounded symmetric neighborhood U_0 of the identity e of G and choose a family $\{U_n : n \in \mathbb{Z}\}$ of bounded symmetric neighborhoods of e such that

$$U_n U_n \subset U_{n+1}, \quad \bigcup_{n \in \mathbb{Z}} U_n = G,$$

and $\{U_n : n \in \mathbb{Z}\}$ is a base of neighborhoods of e. For each $n \in \mathbb{Z}$, we put $V_{2^n} = U_n$. Given any $r = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_n}$, $l_1 > l_2 > \cdots > l_n$, $l_i \in \mathbb{Z}$, we put

$$V_r = V_{2^{l_1}} \, V_{2^{l_2}} \, \cdots \, V_{2^{l_n}}.$$

Repeating the arguments proving Theorem 8.3 from [6], we conclude that

(1) $r < s \Rightarrow V_r \subset V_s$;

(2) $V_r V_{2^l} \subset V_{r+2^{l+2}}$.

Then we define a function $\varphi(x) = \inf\{r : x \in V_r\}$ and note that $\varphi(x) = 0$ if and only if $x = e$. We put

$$d(x, y) = \sup\{|\varphi(zx) - \varphi(zy)| : z \in G\},$$

and note that d is a left invariant metric on G.

By (1), (2) and [6, Theorem 8.3], d determines a left uniformity of G. If $d(x, e) < 2^l$ then $x \in V_2 l$. On the other hand, let $x \in V_2 l$. If $z \in V_r$, by (2), $zx \in V_{r+2} l$ so $\varphi(zx) \leq \varphi(z) + 2^{l+2}$. Analogously, if $zx \in V_r$ then $V_r V_{2^l}^{-1} e \subset V_{r+2}^{l+2}$ and $\varphi(z) \leq \varphi(zx) + 2^{l+2}$. It follows that $d(x, e) \leq 2^{l+2}$ so d determines the left ballean structure of G.

$(i) \Rightarrow (ii)$. Since the left uniformity of G is compactible with d , G is first countable. Since $B_l(G)$ is metrizable, by [5, Theorem 2.1.1], G is σ-bounded. Since $\mathcal{B}(G, d) = \mathcal{B}_l(G)$, each ball $B_d(x, r)$ is bounded, so G is locally bounded. □

Remark 2.1. In the discrete case, Theorem 2.1 guarantees a left invariant metric on a countable group G such that every ball $B(g, r)$ is finite. If G is finitely generated, then the word metric is appropriate. In the general case, we enumerate $G = \{g_n : n \in \omega\}$ so that $g_0 = l$ and if $g_n \neq g_n^{-1}$ then either $g_{n+1} = g_n^{-1}$ or $g_{n-1} = g_n^{-1}$. We define a weight function w on G inductively. Put $w(g_0) = 0$ and assume that we have defined $w(g_0), \dots, w(g_n)$. If $g_{n+1} = g_n^{-1}$ we put $w(g_{n+1}) = w(g_n)$, otherwise $w(g_{n+1}) = w(g_n) + 1$. Then, for every $g \in G$, we put

$$\|g\| = \min\{w(x_1) + \cdots + w(x_n) : g = x_1 \cdots x_n,\ x_1, \dots, x_n \in G,\ n \in \omega\}$$

The function $\|\cdot\|$ is an integer valued norm on G such that $\|x\| = \|x^{-1}\|$ for every $x \in G$, so we put $d(x, y) = \|x^{-1}y\|$.

A metric d on a set X is called an *ultrametric* if

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

for all $x, y, z \in X$. If G is a left invariant metric on a group G , then the set $\{x \in G : d(x, e) \leq r\}$ is a subgroup for every $r \in \mathbb{R}^+$.

Theorem 2.2. *For a topological group G, the following statements are equivalent*

(*i*) *there is a left invariant ultrametric d on G compatible both with left uniformity and left ballean structure of G;*

(*ii*) *there is a family $\{V_n : n \in \mathbb{Z}\}$ of open subgroups of G such that $V_n \subseteq V_{n+1}$, $|V_{n+1} : V_n| < \infty$, $\bigcup_{n=1}^{\infty} V_n = G$ and $\{V_n : n < 0\}$ is a base at the identity for the topology of G.*

Proof. $(i) \Rightarrow (ii)$. For every $n \in \mathbb{Z}$, we put

$$V_n = \{x \in G : d(x, e) \leq 2^n\}.$$

Since d determines $\mathcal{B}_l(G)$, $\bigcup_{n=1}^{\infty} V_n = G$ and each subgroup V_n is bounded, then $\mid V_{n+1} : V_n \mid < \infty$. Since d is compactible with the left uniformity of G, $\{V_n : n < 0\}$ is a base at the identity for the topology on G.

$(ii) \Rightarrow (i)$. Given any $x, y \in G$, we put

$$\|x\| = \min\{n : x \in V_n\}, \quad d(x, y) = \|x^{-1}y\|,$$

and note that d is a desired ultrametric on G. □

3. Determinability of topology by the balleans

It follows directly from the definitions that the balleans $\mathcal{B}_l(G)$ and $\mathcal{B}_r(G)$ of a topological group G are uniquely determined by the topology of G. In which respect the balleans $\mathcal{B}_l(G)$ and $\mathcal{B}_r(G)$ determine the topology of G? Let us try to specify this general question.

Let (G, τ) be a topological group, $\mathcal{I}_\tau$ be the ideal of bounded subsets of G. We denote by $\tau^{\#}$ the strongest group topology on G such that $\mathcal{I}_{\tau^{\#}} = \mathcal{I}_\tau$, and say that (G, τ) is b-determined if $\tau^{\#} = \tau$. Clearly, every discrete group is b-determined. A totally bounded group (G, τ) is b-determined if and only if τ is the maximal totally bounded topology on G.

Question 3.1. *Given a topological group G, how to detect whether G is b-determined?*

Question 3.2. *Let τ_1, τ_2 be group topologies on G such that $\mathcal{I}_{\tau_1} = \mathcal{I}_{\tau_2}$. Which topological properties* (*in particular, topological cardinal invariants*) *are common for* (G, τ_1) *and* (G, τ_2)*?*

We say that the topological groups G_1 and G_2 are *b-equivalent* if the balleans $\mathcal{B}_l(G)$ and $\mathcal{B}_r(G)$ are asymorphic.

Question 3.3. *Which properties of a topological group are invariant under b-equivalence?*

Question 3.4. *Given a group ideal $\mathcal{I}$ on G, how to detect whether there exists a group topology τ on G such that $\mathcal{I}$ is the ideal of all bounded subsets of (G, τ)?*

The following theorem is related to Question 3.1.

Theorem 3.1. *No b-determined topological Abelian group (G, τ) may contain a non-trivial convergent sequence. Every Abelian metrizable b-determined group is discrete.*

Proof. We denote by $\#$ the strongest totally bounded topology on G. Since G is Abelian, $(G, \#)$ is Hausdorff. By [1], $(G, \#)$ has no convergent sequences. Since $\mathcal{I}_{\tau \vee \#} = \mathcal{I}_\tau$, we have $\tau^\# \subseteq \#$. □

Remark 3.1. The Abelian condition is essential in Theorem 3.1. Indeed, let G be a semi-simple connected compact group Lie group. Clearly, G is metrizable, but G admits only one totally bounded (in fact, compact) group topology, so G is b-determined.

Remark 3.2. Let τ_1, τ_2 be a group topologies on a group G. Following [10], we say that τ_2 is totally bounded with respect to τ_1 if, for every neighbourhood U of e in τ_2, there exists a finite subset F such that FU is a neighbourhood of e in τ_1. Equivalently, every Cauchy ultrafilter in (G, τ_2) is a Cauchy ultrafilter in (G, τ_1). For every group topology τ on G, there exists the largest topology $\hat{\tau}$ totally bounded with respect to τ. Clearly, $\mathcal{I}_\tau = \mathcal{I}_{\hat{\tau}}$ so $\hat{\tau} \subseteq \tau^\#$. If (G, τ) is totally bounded, then $\hat{\tau} = \tau^\#$. But we cannot state that $\hat{\tau} = \tau^\#$ for every group topology τ. Indeed, let (G, τ) be a non-discrete topological group with only finite bounded subsets (see Example 3.1). Then $\tau^\#$ is discrete, but $\hat{\tau}$ is non-discrete. On the other hand, for every topological Abelian group (G, τ), we have $\# \subseteq \hat{\tau}$, so $(G, \hat{\tau})$ has no non-trivial convergent sequences.

Question 3.5. *Given a topological group (G, τ), how to detect whether $\hat{\tau} = \tau^\#$? $\tau = \hat{\tau}$?*

We construct a countable non-discrete topological group with only finite bounded subsets.

Example 3.1. Let $G = \bigotimes_{n \in \omega} G_n$ be the direct product of finite groups G_n, $|G_n| > 1$ with the identities e_n, $n \in \omega$. For every $g \in G$, we put

$$\mathrm{supp}(g) = \{g \in G : \; pr_n g \neq e_n\}.$$

We fix an arbitrary free ultrafilter φ on ω and, for every $\Phi \in \varphi$, put

$$[\Phi] = \{g \in G : \ \mathrm{supp}(g) \subset \Phi\}.$$

The family $\{[\Phi] : \Phi \in \varphi\}$ forms a base at the identity e for some non-discrete group topology τ on G.

We show that (G, τ) is complete. Let ψ be a ultrafilter Cauchy on G with respect to the left uniformity on (G, τ) (which coincides in this case with the right uniformity). To show that ψ converges in (G, τ), we endow each group G_n with the discrete topology, and consider G as a subgroup of the Cartesian product $H = \prod_{n\in\omega} G_n$. Since H is compact in the product topology, ψ converges in H to some element h. We put

$$X = \{n \in \omega : \ pr_n h \neq e_n\},$$

and consider two cases.

Case: X is infinite. We choose an infinite subset $Y \subset X$ such that $\omega \setminus Y \in \varphi$. Since ψ is an ultrafilter Cauchy in (G, τ), there exists $\Psi \in \psi$ such that $\mathrm{supp}(g^{-1}g') \subseteq Y \setminus \omega$ for all $g, g' \in \Psi$. We fix an arbitrary element $k \in Y$. Since ψ converges to h in H, there exists $\Psi' \in \psi$ such that $\Psi' \subseteq \Psi$ and $k \in \mathrm{supp}(g)$ for every $g \in \Psi'$. We fix an arbitrary element $x \in \Psi'$. Since Y is infinite, we can take an element $m \in Y \setminus \mathrm{supp}(x)$. Since ψ converges to h in H, there exists $\Psi'' \in \psi$ such that $\Psi'' \subset \Psi'$ and $m \in \mathrm{supp}(g)$ for every $g \in \Psi''$. We fix an arbitrary element $y \in \Psi''$. Then $m \in \mathrm{supp}(x^{-1}y)$, so $\mathrm{supp}(x^{-1}y) \nsubseteq \omega \setminus Y$, contradicting the choice of Ψ. Thus, this case is impossible.

Case: X is finite. Replacing ψ to $x^{-1}\psi$, we may suppose that $h = e$. We assume that ψ does not converge to e in τ, and choose an infinite subset $Y \subset \omega$ such that $\omega \setminus Y \in \varphi$. Repeating the arguments from the previous case, we get a contradiction, so ψ converges to k.

At last, we assume that (G, τ) contains an infinite closed bounded subset A. Since (G, τ) is complete, A is compact. Since A is countable, there exists an injective sequence $(a_n)_{n\in\omega}$ converging to some element a. We may suppose that $a = e$. Passing to a subsequence, we also suppose that $\max(a_n) < \min(a_{n+1})$ for every $n \in \omega$, where $\min(x)$ and $\max(x)$ are the first and the last non-zero coordinates of x. We put $M = \{\min(a_n) : n \in \omega\}$ and choose and infinite subset $Y \subset M$ such that $Y \notin \varphi$. Then $[\omega \setminus Y]$ is a neighbourhood of e in τ, but infinitely many members of $(a_n)_{n\in\omega}$ are outside of this neighbourhood, This contradiction shows that A is finite.

Question 3.6. *Let (G,τ) be a topological group such that τ is maximal in the class of all non-discrete group topologies on G. Is every bounded subset of (G,τ) finite?*

4. Slowly oscillating function

Every ballean $\mathcal{B} = (X, P, B)$ has a compact Hausdorff satellite, the *corona* $\check{\mathcal{B}}$. To describe $\check{\mathcal{B}}$, we endow X with the discrete topology and consider the Stone-Čech compactification βX of X. We take the points of βX to be the ultrafilters on X with the points of X identified with the principal ultrafilters. The topology of βX can be defined by stating that the sets of the form $\bar{A} = \{p \in \beta X : A \in p\}$, where A is a subset of X, form a base for the open sets.

We denote by $X^\sharp$ the set of all ultrafilters r on X such that every $R \in r$ is unbounded in $\mathcal{B}$. A subset V is called bounded in $\mathcal{B}$ if $V \subseteq B(x,\alpha)$ for some $x \in X$ and $\alpha \in P$. Clearly, $X^\sharp$ is a closed subset of βX.

Given any $r, q \in X^\sharp$, we say that r, q are *parallel* (and write $r \parallel q$) if there exists $\alpha \in P$ such that $B(R,\alpha) \in q$ for each $R \in r$. It is easy to see that $\parallel$ is an equivalence on $X^\sharp$. We denote by $\sim$ the minimal (by inclusion) closed (in $X^\sharp \times X^\sharp$) equivalence on $X^\sharp$ such that $\parallel \subseteq \sim$. The quotient $X^\sharp/\sim$ is a compact Hausdorff space. It is called a corona of $\mathcal{B}$ and is denoted by $\check{\mathcal{B}}$.

To clarify the virtual equivalence $\sim$ determining $\check{\mathcal{B}}$, we use the slowly oscillating functions.

A function $h : X \to \mathbb{R}$ is called *slowly oscillating* if , for every $\varepsilon > 0$ and every $\alpha \in P$, there exists a bounded subset V of X such that

$$\operatorname{diam} h(B(x,\alpha)) < \varepsilon$$

for every $x \in X \setminus V$, where $\operatorname{diam} A = \sup\{|a - b| : a, b \in A\}$.

Proposition 4.1. *Let $\mathcal{B} = (X, P, B)$ be a connected ballean, $q, r \in X^\sharp$. Then $q \sim r$ if and only if $h^\beta(q) = h^\beta(r)$ for every slowly oscillating function $h : X \to [0,1]$, where h^β is the extension of h to βX.*

Proof. See [11, Proposition 1]. □

Let X be a topological space. A pair (φ, Y) is called a compactification of X if Y is a compact space, $\varphi : X \to Y$ is a continuous mapping and $\varphi(X)$ is dense in Y. If in addition φ is an embedding, (φ, Y) is called a topological compactification. In this case we can identify X with $\varphi(X)$, $Y \backslash \varphi(X)$ is called the remainder of compactification.

Let X be a topological space and let A be a norm closed subalgebra of $C_{\mathbb{R}}(X)$ which contains all constant function. By [7, Lemma 21.39], there are a compact space Y and a continuous mapping $\varphi : X \to Y$ with the property that $\varphi(X)$ is dense in Y and $A = \{f \in C_{\mathbb{R}}(X) : f = g \circ \varphi$ for some $g \in C_{\mathbb{R}}(Y)\}$. The mapping φ is an embedding if, for every closed subset E of X and every $x \in X \setminus E$, there exists $f \in A$ such that $f(x) = 1$ and $f \mid_E \equiv 0$.

For a metric space (X, d), the set $S(X, d)$ of all bounded continuous slowly oscillating real functions on X is a norm closed subalgebra of $C_{\mathbb{R}}(X, d)$. Applying [7, Lemma 21.39], we get some compactification $(\chi, \chi(X, d))$ which is called the Higson's compactification and its reminder, the Higson's corona (see [12, Section 2.3]).

A metric space (X, d) is called *proper* if every closed ball in X is compact.

Proposition 4.2. *For a proper metric space (X, d), the following statements hold*

(*i*) $(\chi, \chi(X, d))$ *is a topological compactification;*

(*ii*) $(\chi(X, d) \setminus (X, d)$ *is homeomorphic to* $\check{\mathcal{B}}(X, d))$.

Proof. See [11, pp. 154–155]. □

For a topological group G, a function $f : G \to \mathbb{R}$ is said to be *left (right) slowly* oscillating if, for every $\varepsilon > 0$ and every bounded subset F of G, there exists a bounded subset V such that $\mid f(yx) - f(x) \mid < \varepsilon$ $(\mid f(xy) - f(x) \mid < \varepsilon)$ for all $x \in G \setminus V$, $y \in F$. Clearly, f is left (right) slowly oscillating if and only if f is slowly oscillating with respect to the ballean $\mathcal{B}_l(G)$ $(\mathcal{B}_r(G))$.

The families $S_l(G)$ and $S_r(G)$ of all bounded continuous left and right slowly oscillating functions on G are the norm closed subalgebras in $C_{\mathbb{R}}(G)$. Applying [7, Lemma 21.39], we get two compactifications $(\chi_l, \chi_l(G))$ and $(\chi_r, \chi_r(G))$ of G.

Proposition 4.3. *For a topological group G, the following statements hold*

(*i*) *if G is locally bounded, then $(\chi_l, \chi_l(G))$, and $(\chi_r, \chi_r(G))$ are topological compactifications;*

(*ii*) *if G is not locally bounded, then $\chi_l(G)$ and $\chi_r(G)$ are singletons.*

Proof. (*i*) In view of [7, Lemma 21.39], it suffices to show that any closed subset E of G and $x \in G \setminus E$ can be separated by left (right) bounded continuous slowly oscillating function. Since G is locally bounded, we can choose an open bounded neighborhood U of x such that $U \bigcap E = \emptyset$.

Since the space of G is completely regular, there is a continuous function $f : G \to [0,1]$ such that $f(x) = 1$ and $f\mid_{G\setminus U} \equiv 0$. Clearly, f is left and right slowly oscillating.

(ii) We show that every continuous left slowly oscillating function $f : G \to \mathbb{R}$ is constant. Let $a, b \in G$. Given any $\varepsilon > 0$, we choose a bounded subset V of G such that $\operatorname{diam} f(\{ba^{-1}, e\}x) < \varepsilon$ for each $x \in G \setminus V$. Since G is not locally bounded, for every neighbourhood U of a, there exists $x \in U \cap (G \setminus V)$. It follows that $\mid f(a) - f(b) \mid \leq \varepsilon$. $\square$

Remark 4.1. If G is locally compact, we can identify the remainders $\chi_l(G) \setminus G$ and $\chi_r(G) \setminus G$ with $\check{\mathcal{B}}_l(G)$ and $\check{\mathcal{B}}_r(G)$ respectively.

Remark 4.2. Let G be a countable non-discrete group G with finite bounded subsets. By Proposition 4.3 (ii), $\chi_l(G)$ is a singleton. On the other hand, by [11, Proposition 3], $|\check{\mathcal{B}}_l(G)| = 2^{2^{\aleph_0}}$.

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