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Parametrization of extremals of Grötzsch's problem

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Abstract. We give parametric expressions for extremals of Grötzsch's problem.

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Introduction

Let $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}}$ be the Riemann sphere, $K := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and $T := \partial K$.

The formulation of Grötzsch's problem is the following:

Among all continua containing a fixed finite point collection (*i.e. collection of points*) *belonging to* K, *to find such one which has the minimal hyperbolic capacity.*

In [2] H. Grötzsch (in analogy with his results concerning Tchebotaröv's problem) stated the existence and uniqueness of the solution of his problem as well as his constructive property for it.

G. V. Kuz'mina gave more detailed treatment of this problem (see her book [3, Chapter 3]).

In the present work we establish some (1.1)-parametrization of extremals of Grötzsch's problem via proper class of simple rectilinear graphs, isomorphic and in a sense isometric to extremal continua (namely via the same class of graphs which participated in parametrization of extremals of Tchebotaröv's problem, see [7, 9]).

We shall consider Grötzsch's problem in the following completely equivalent form (see [3, Chapter 5]). Let $m \geq 1$ be an integer, and

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$\{a_j\} := \{a_j\}_{j=1}^{m+1}$ be a fixed unordered collection of distinct points $a_1, \ldots, a_{m+1} \in \overline{\mathbb{C}} \setminus \overline{K}$.

Grötzsch's problem. *Among all doubly connected domains $D \subset \overline{\mathbb{C}} \setminus \overline{K}$ embracing K and separating $\overline{K}$ from the whole point collection $\{a_j\}$, to find such one which has the maximal modulus (i.e. the maximal modulus of the family of closed curves contained in D, embracing K and separating the boundary components of D).*

For convenience sake we shall consider the following completely equivalent form of this problem:

Problem H. *To consider this problem under additional requirement that $a_{m+1} = \infty$.*

It is known (see for instance [3, Chapter 5]) that the mentioned maximum modulus in the problem is attained, and we denote it by $\mathrm{Mod}(\{a_j\})$. Moreover the doubly connected domain $D =: D^{\max}(\{a_j\})$ under question giving the maximal value for the mentioned modulus is uniquely defined by the point collection $\{a_j\}$, and its inner boundary component coincides with T.

The parametrization of Grötzsch's problem was considered in [8]. But the statement of that work about the extremality of its function (3) in Grötzsch's problem is wrong (cf. with [9]). Therefore theorems of the work [8] must be corrected in terms not related to its function (3) (cf. with [9]).

The author tried to consider one more representation of extremals of Grötzsch's (and Tchebotaröv's) problems, but Prof. G. V. Kuz'mina and Prof. E. G. Emelyanov have shown to me a counter example to it. I am greatly thankful to them.

1. Parametric formulation

The particular case $m = 1$ of Grötzsch's problem under consideration is a well-known Grötzsch's distortion theorem. Therefore for simplicity of formulations in the sequel we shall exclude this case and assume that $m \geq 2$.

As usual (cf. with [4, 6, 7]), we may omit all unessential points of the collection $\{a_j\}$ not influencing upon the extremal of the problem, and hence we shall consider only those collections $\{a_j\}_{j=1}^{m+1}$ for which *all* points a_j influence upon the extremal of the problem (that is equivalent to the requirement that all a_j are *end* points of the set $B := \overline{\mathbb{C}} \setminus (\overline{K} \cup D^{\max}(\{a_j\}))$, see [4] and cf. with [7], and therefore are simple poles of the quadratic differential $Q(w)dw^2$ of the problem (see [3, Chapter 5])). The class of all such collections will be denoted by M.

For any $r \in (1, \infty)$, let us denote $S_r := \{z \in \mathbb{C} : 1 < |z| < r\}$.

For the above given collection $\{a_j\}$, let $r(\{a_j\})$ denote that $r > 1$ for which the ring S_r is conformally equivalent to the domain $D^{\max}(\{a_j\})$.

Denote by $f_{\{a_j\}}$ the univalent conformal mapping of the ring $S_{r(\{a_j\})}$ onto the domain $D^{\max}(\{a_j\})$ for which $f_{\{a_j\}}(T) = T$ and $f_{\{a_j\}}(r(\{a_j\})) = \infty$.

Such a mapping exists, is unique and has the continuous (with respect to topology of $\overline{\mathbb{C}}$ in the image) extension $f^{\{a_j\}}$ onto $\overline{S}_{r(\{a_j\})}$ and $f^{\{a_j\}}(\overline{S}_{r(\{a_j\})}) = \overline{\mathbb{C}} \setminus K$ (for instance see [3, Chapter 5] and cf. with [8]).

The function $f_{\{a_j\}}$ will be called *the extremal* of the Problem H (corresponding to the element $\{a_j\} \in M$).

The quadratic differential $Q(w)dw^2$ of Problem H can be defined via its extremal function $f_{\{a_j\}}(z)$ as the left part of the functional-differential equation

$$-\Big(\frac{df_{\{a_j\}}(z)}{zf'_{\{a_j\}}(z)}\Big)^2 = -\Big(\frac{dz}{z}\Big)^2.$$

Because of our assumption, all points a_j belonging to the fixed point collection $\{a_j\} \in M$ are endpoints of the set B. So every other point of B is either a zero b_n of the order $\nu_n \geq 1$ of $Q(w)dw^2$ (and then it is a branch point of B), or this point is a regular, non-critical point of $Q(w)dw^2$ (and then it belongs to a critical trajectory of this quadratic differential). Thus exactly as in [6,7], we may consider B as an undirected, connected, simple, acyclic, plane graph on $\overline{\mathbb{C}} \setminus \overline{K}$ consisting of nodes of order one at all points a_j and only at them, nodes of orders $\nu_n + 2$ at all zeros b_n of degrees $\nu_n \geq 1$, and only at them, and of all critical analytic trajectories of $Q(w)dw^2$ (contained in B and ending at zeros or simple poles of $Q(w)dw^2$) as edges of the graph. This curvilinear geometric graph is connected and has no cycles. So it is a tree. And we shall denote it by $L(\{a_j\})$, assuming that $a_{m+1} = \infty$.

The total multiplicity of all zeros of $Q(w)dw^2$ on $\overline{\mathbb{C}} \setminus \overline{K}$ equals $m - 1$.

Let now k be the number of *different* zeros of $Q(w)dw^2$ on $\overline{\mathbb{C}} \setminus \overline{K}$. Then the number of edges of the graph under consideration equals $m + k$ (on the basis of the Euler's theorem).

Any elements $\{a_j\}$ and $\{a'_j\}$ of M are called *similar* if there is $t \in T$ such that $\{a'_j\} = t\{a_j\}$.

Let $\widetilde{M}$ be the factor set of M with respect to similarity.

One can show that for every $\{a_j\} \in M$ and any $t \in T$ there holds $r(t\{a_j\}) = r(\{a_j\})$.

Our main results in this problem can be expressed in the following way. We have defined a parametric set $\widetilde{G}$ consisting of simple, transparent geometric objects $\widetilde{\Gamma}$ (both of these notations are defined in Section 3 of [7], see also below Sections 3 and 4). Then for every such $\widetilde{\Gamma}$ and every $t \in T$ we have defined a point collection $\{a_j(\widetilde{\Gamma}, t)\} \in M$ and a univalent conformal mapping $f_{\widetilde{\Gamma},t} : S_{r(\{a_j(\widetilde{\Gamma},t)\})} \to \mathbb{C} \setminus \{a_j(\widetilde{\Gamma}, t)\}$ with $f_{\widetilde{\Gamma},t}(T) = T$, $f_{\widetilde{\Gamma},t}(r(\{a_j(\widetilde{\Gamma}, t)\})) = \infty$, $f_{\widetilde{\Gamma},t}(1) = t$, such that this $f_{\widetilde{\Gamma},t}$ is extremal in Grötzsch's problem for the mentioned point collection $\{a_j(\widetilde{\Gamma}, t)\}$. We have also

$$f_{\widetilde{\Gamma},t} = \bar{t} f_{\widetilde{\Gamma},1}.$$

The point collection $\{a_j(\widetilde{\Gamma}, t)\}$ is a function of data given by $\widetilde{\Gamma}$ and t.

And conversely, for every collection $\{a_j\} \in M$ there are (the unique) $t \in T$ and $\widetilde{\Gamma} \in \widetilde{G}$ such that the function $f_{\widetilde{\Gamma},t}$ is extremal in Grötzsch's problem for the mentioned point collection $\{a_j\}$ and $f_{\widetilde{\Gamma},t}(1) = t$.

Thus we have established the parametric (1,1)-correspondence between all couples $(\widetilde{\Gamma}, t)$ and extremals $f_{\{a_j\}}$ of the problem H for all collections $\{a_j\} \in M$.

Properties of the mentioned objects are described in Theorems 3.1–3.3.

2. Construction of extremals

Below we shall construct extremals of Problem H (normalized by the condition that ∞ is an *end* point of the boundary of the extremal image, or equivalently that ∞ is a simple pole of the quadratic differential of this problem). This class will be parametrized by means of geometric rectilinear graphs $\Gamma \in G$ defined in [6] and in Section 3 of the work [7]. For convenience sake in this Section we repeat some definitions and notations from [6, 7] being used below in construction of the extremals for Problem H.

Let G be the class of all finite, undirected, connected plane graphs Γ each of which satisfies the following conditions:

1) each edge γ of Γ is a rectilinear open interval in $\mathbb{C}$ of the length $|\gamma| > 0$, and these intervals mutually do not intersect each other, while nodes of the graph coincide with the endpoints of these intervals;

2) Γ does not contain nodes of order 2 and cycles;

3) the sum of lengths $|\gamma|$ of all intervals γ of the graph Γ equals π;

4) the point $\zeta = 0$ is a node of Γ of order 1, and the edge of Γ incident to this point is contained in the real half-axis $\operatorname{Re} \zeta > 0$.

Let $\operatorname{Supp} \Gamma$ denote the closure in $\mathbb{C}$ of the geometric union of all edges of the graph $\Gamma \in G$. Starting at the node 0, let us run along Γ in the direction in which the complementary to Γ domain $\mathbb{C} \setminus (\operatorname{Supp} \Gamma)$ remains on the left. Such a pass of Γ will be called *natural*. For every point ζ on an edge $\gamma \in \Gamma$, let $r_1(\Gamma, \zeta)$ and $r_2(\Gamma, \zeta)$ denote the length of the pass respectively to the first and to the second reaching the point ζ, while $r_1(\Gamma, 0) = 0$, $r_2(\Gamma, 0) = 2\pi$. Under a single such pass along an edge γ the growth of each of functions r_1 and r_2 equals $|\gamma|$. For every node v of the order $\tau(v)$, let $r_1(\Gamma, v), \ldots, r_{\tau(v)}(\Gamma, v)$ denote the length till the first, $\ldots, \tau(v)$th pass of v.

For every $\zeta \in \operatorname{Supp} \Gamma$ and all $j = 1, \ldots, \tau(\zeta)$ let us denote

$$\varepsilon_{\Gamma,j}(\zeta) := \exp\left(i r_j(\Gamma, \zeta)\right).$$

Let Γ' be another graph from G, and for every $\zeta' \in \operatorname{Supp} \Gamma'$ the objects $\tau'(\zeta')$ and $\varepsilon_{\Gamma',j'}(\zeta')$ be defined exactly as analogous objects were defined for Γ and $\zeta \in \operatorname{Supp} \Gamma$.

Then the graphs Γ and Γ' will be called *equivalent*, if there exists the isomorphism $\eta : \Gamma \to \Gamma'$ such that $\eta(0) = 0$ and for every node v of Γ we have

$$r_j(\Gamma', \eta(v)) = r_j(\Gamma, v) \quad \forall\, j = 1, \ldots, \tau(v).$$

If graphs $\Gamma, \Gamma' \in G$ are equivalent, then for every $\zeta \in \operatorname{Supp} \Gamma$ there corresponds a uniquely defined $\zeta' \in \operatorname{Supp} \Gamma'$ for which

$$\varepsilon_{\Gamma,j}(\zeta) = \varepsilon_{\Gamma',j}(\zeta') \quad \forall\, j = 1, \ldots, \tau(\zeta).$$

For a graph $\Gamma \in G$, let $V(\Gamma)$ be the set of all its nodes of order 1, and $W(\Gamma)$ be the set of all other its nodes (of orders ≥ 3). Let V be the set of all points $\varepsilon_{\Gamma,1}(p)$ ($\in T$), when p runs through the set $V(\Gamma)$. Denote by W_v the set of all points $\varepsilon_{\Gamma,j}(v)$ ($\in T$), when $v \in W(\Gamma)$ is fixed and j runs through the set of values $1, \ldots, \tau(v)$. Denote also

$$W := \bigcup_{v \in W(\Gamma)} W_v.$$

Clearly the point $z = 1$ is contained in V.

The cardinal numbers of the sets $V(\Gamma)$ and $W(\Gamma)$ are $m + 1$ and k, respectively, and Γ contains exactly $m + k$ edges.

For u and ζ in $\mathbb{C} \setminus \{0\}$, let us introduce the function

$$\rho(u, \zeta) := \frac{1}{|u|}(u - \zeta)\Big(\overline{u} - \frac{1}{\zeta}\Big).$$

Fix any $r > 1$.
Introduce the function $s : z \mapsto rz$.
Consider the ring $D_r := \{z \in \mathbb{C} : 1/r < |z| < r\}$.

3. Main results

For a fixed graph $\Gamma \in G,\ r \in (1, \infty),\ t \in T$ under the above notations and assumptions, we get the following result.

Theorem 3.1. *For every Γ, r, t under consideration, there is a function ϕ with the following properties. It is holomorhic and univalent in D_r, symmetric with respect to T, continuous on $\overline{S_r} \setminus \{r\}$, continuous in generalized sense (with respect to topology of $\overline{\mathbb{C}}$ in the image) on $\overline{S_r}$ and posessing the following properties. For every point $\zeta_0 \in \operatorname{Supp}\Gamma$ the function ϕ glues rational-analytically all points $r\varepsilon_{\Gamma,j}(\zeta_0)$ $(j = 1, \dots, \tau(\zeta_0))$ into one point denoted by $Y(\zeta_0)$, and ϕ is continuously and meromorphically extendable into a neighborhood of every point $z \in \overline{S_r} \setminus s(W)$ (holomorphically for every such $z \neq r$). Moreover $\phi(\overline{S_r}) = \overline{\mathbb{C}} \setminus K$, $\phi(r) = \infty$, while at the inner boundary of S_r there holds $\phi(T) = T, \phi(1) = t$, and the restriction ϕ_r of the function ϕ to S_r is extremal in Problem H for the collection of all points $A(p) := y(p)$ where p runs over the whole set $V(\Gamma)$. The quadratic differential of this problem is given by the formula:*

$$Q(w)dw^2 = -\frac{dw^2}{w^2}\frac{\prod\limits_{v\in W(\Gamma)} \rho(Y(v), w)^{\tau(v)-2}}{\prod\limits_{\alpha\in V\setminus\{1\}} \rho(\phi(r\alpha), w)}.$$

On the set $\overline{\mathbb{C}} \setminus \overline{K}$, the collection of all (simple) poles of $Q(w)dw^2$ belongs to M and coincides with the set of all $m+1$ points $A(p)$, while the set of all zeros of $Q(w)dw^2$ coincides with the set of all points $B(v) := Y(v)$, where v runs over all k points of the set $W(\Gamma)$, and the order of each zero $B(v)$ of $Q(w)dw^2$ equals $\tau(v)-2$. Each point $A(p)$ is an endpoint of some single critical trajectory of $Q(w)dw^2$. The boundary of the domain $\phi(S_r)$ with respect to $\overline{\mathbb{C}}\setminus\overline{K}$ is the union of $m+k$ critical trajectories of $Q(w)dw^2$, their $m+1$ endpoints $A(p)$ $(\forall\, p \in V(\Gamma))$ and k points $B(v)$ $(\forall\, v \in W(\Gamma))$. The functions ϕ and ϕ_r with the above properties are uniquely defined by the fixed Γ, r, t.

Let $\Gamma \in G$ be the fixed graph from Theorem 3.1 with all related to it objects and notations (in particular, $A(p)$ for all $p \in V(\Gamma)$). Using the notation $L(\{a_j\})$ from Section 1 defined for every point collection $\{a_j\} \in M$, let's define a particular collection $\{a_j\}$ as the set $\{A(p)\}_{p\in V}$ from Theorem 3.1 and denote $L(\{A(p)\}_{p\in V}) =: \Gamma_*$.

Then we get the following result.

Theorem 3.2. *The graph Γ_* is isomorphic to Γ, with the correspondence of the node $\zeta = 0$ of Γ to the node $w = A(0) = \infty$ of Γ_*, and the pass of Γ_* in the direction in which the domain $\overline{\mathbb{C}} \setminus (\Gamma_* \cup \overline{K})$ remains on the left, corresponds to the pass of Γ in the natural direction (in which the complementary to Γ domain $\mathbb{C} \setminus (\operatorname{Supp}\Gamma)$ remains on the left). Then the length of every pass along Γ_* in the metric $|Q^{1/2}dw|$ equals to the length of its pre-image on Γ with respect to the natural length measuring on Γ (see Section 2), and*

$$\int\limits_0^{2\pi} |Q^{1/2}(e^{i\theta})|\, d\theta = 2\pi.$$

Thus the graphs Γ and Γ_* are isomorphic, equally oriented relative to their complementary (with respect to $\overline{\mathbb{C}} \setminus \overline{K}$) domains and isometric in the sense of Theorem 3.2 (this isometry being consistent with the isomorphism and the direction of pass).

From the definitions we see that for every equivalent graphs $\Gamma', \Gamma'' \in G$ and related to them objects corresponding to each other in this equivalence (including objects of the form $p,\ v,\ \varepsilon_{\Gamma,1}(p),\ \varepsilon_{\Gamma,j}(v),\ V(\Gamma),\ W(\Gamma)$ for these graphs), the objects $V,\ W,\ \tau(v),\ W_v,\ A(p),\ B(v), \phi$ of similar form coincide.

Let $\tilde{G}$ denote the factor-set of G with respect to the equivalence, and $\tilde{\Gamma}$ denote its general element.

For a graph $\Gamma \in G$, let $\{\Gamma\}$ denote the class of all graphs from G equivalent to Γ.

Let $F : \tilde{G} \times (1, +\infty) \times T \to M$ be the mapping defined for each $\tilde{\Gamma} \in \tilde{G}, r \in (1, +\infty), t \in T$ as the collection $\{\phi(r\varepsilon_{\Gamma,1}(p)\}_{p \in V(\Gamma)}$, where ϕ is the function mentioned in Theorem 1 (with the requirement $\phi(1) = t$ for arbitrary $\Gamma \in \tilde{\Gamma}$ and the corresponding $V(\Gamma)$).

Theorem 3.3. *The class of all extremals of Problem H is parametrized by elements of the set $\tilde{G} \times (1, +\infty) \times T$, and this parametrization is one-to-one correspondence:*

1) *to every element $\tilde{\Gamma} \in \tilde{G}$, each finite $r > 1$ and any $t \in T$ there corresponds one (and only one) point collection $\{A_j(\tilde{\Gamma}, r, t)\} \in M$ for which the function ϕ_r mentioned in Theorem 3.1 (with any graph $\Gamma \in \tilde{\Gamma}$) is extremal in Problem H; this $\{A_j(\tilde{\Gamma}, r, t)\}$ is the collection of all points $A(p) := \phi(r\varepsilon_{\Gamma,1}(p))$ where p runs over the whole set $V(\Gamma)$, and in notation of Section 1 we have $r(\{A_j(\tilde{\Gamma}, r, t)\}) = r$ (and $\phi(1) = t,\ \phi(r) = \infty$).*

2) *and conversely, for every collection of points $\{a_j\} \in M$ there exists one and only one class $\tilde{\Gamma} \in \tilde{G}$, the single finite $r > 1$ and the single*

$t \in T$ such that the function ϕ_r with ϕ mentioned in Theorem 3.1 *(with arbitrary $\Gamma \in \tilde{\Gamma}$) is extremal in Problem H for $\{a_j\}$, and hence $\{a_j\} = F(\tilde{\Gamma}, r, t) = \{A_j(\tilde{\Gamma}, r, t)\}$, $r(\{a_j\}) = r$, $\phi(1) = t$ (and also $\phi(r) = \infty$).*

The proofs use arguments and facts established in Sections 5–10 of the work [7] with some essential modifications.

4. Couples of arcs and stars

Let Δ denote the class of all (unordered) couples $\{\delta^+, \delta^-\}$ consisting of non-intersecting open arcs δ^+, δ^- on the unit circle $T \subset \mathbb{C}$. For a couple $\{\delta^+, \delta^-\} =: \delta \in \Delta$ let us denote $\langle\delta\rangle := \delta^+ \cup \delta^-$. This $\langle\delta\rangle$ will be called *the support* of δ. Let also $|\delta^+|$ and $|\delta^-|$ be lengths of δ^+ and δ^-, respectively.

Let Δ_0 be the set of all $\delta := \{\delta^+, \delta^-\}$ for which the closure $\operatorname{Clos}\langle\delta\rangle$ of the set $\langle\delta\rangle$ is connected and does not coincide with T, and in this case let $P(\delta)$ denote the common endpoint of δ^+ and δ^-.

For any $n \in \mathbb{N}$, an unordered collection $\{\delta_1, \ldots, \delta_n\} =: A$ will be called *one-sheeted*, if $\langle\delta_1\rangle, \ldots, \langle\delta_n\rangle$ are mutually non-intersecting. We say that $\delta_1, \ldots, \delta_n$ *are members* of A. The set

$$\bigcup_{k=1}^{n} \langle\delta_k\rangle =: \operatorname{supp} A$$

will be called *the support of A*.

A one-sheeted unordered collection $A = \{\delta_1, \ldots, \delta_n\}$ will be called *a star,* if every member δ_j of A has a connected component $S(\delta_j, A)$ of the set $T \setminus \langle\delta_j\rangle$ which contains supports of all other members of A. In such a case the other connected component of $T \setminus \langle\delta_j\rangle$ will be denoted by $S_0(\delta_j, A)$ and called *the shadow of δ_j with respect to A*. The set

$$\bigcup_{k=1}^{n} S_0(\delta_j, A) =: S_0(A)$$

will be called *the shadow of A*.

A star A will be called *connectable*, if the set $T \setminus (S_0(A) \cup \operatorname{supp} A)$ consists of a finite number of points.

5. Constellations

A finite, non-empty collection of connectable stars will be called *a constellation.*

A constellation C will be called *irreducible*, if every its star contains at least three members.

Let A^1 and A^2 be different stars from C, and δ be a member of both of these stars. Then A^1 and A^2 will be called *neighbors*, and δ will be called *a link in* C (between these neighbors).

Let $k \geq 1$ be an integer. A sequence $\delta_1, \ldots, \delta_k$ of links in C is called *a chain* (*of the length* k) *in* C, if there exists a sequence $A^1, \ldots, A^{k+1}$ of stars in C with the following properties:

1) $A^j \neq A^{j+1}$ and δ_j is the link between A^j and A^{j+1} for each $j = 1, \ldots, k$;

2) if $k \geq 2$, then also $\delta_j \neq \delta_{j+1}$ for all $j = 1, \ldots, k-1$.

The stars $A^1, \ldots, A^{k+1}$ are called *vertices* of the chain $\delta_1, \ldots, \delta_k$.

A constellation C will be called *acyclic*, if for every chain $\delta_1, \ldots, \delta_k$ in it with the vertices $A^1, \ldots, A^{k+1}$ there holds $A^1 \neq A^{k+1}$, and if moreover each $\delta \in \Delta$ may be a member of at most two stars from C and $\langle \delta \rangle$ does not intersect supports of other stars from C.

Let C be an acyclic, irreducible constellation.

Every two neighbors from C have only one link between them in C.

It follows from our definitions and assumptions that for every chain of the length $k \geq 2$ all links of the chain are mutually different, and all vertices $A^1, \ldots, A^{k+1}$ of the chain are mutually different as well.

A chain $\delta_1, \ldots, \delta_k$ in C will be called *maximal,* if it is not contained in a longer chain in C (this means that there is no link δ in C such that some of the sequences $\delta, \delta_1, \ldots, \delta_k$ or $\delta_1, \ldots, \delta_k, \delta$ is a chain in C).

Every chain in C is contained in a maximal chain in C.

If a sequence $\delta_1, \ldots, \delta_k$ in C is a maximal chain in C and $A^1, \ldots, A^{k+1}$ are the vertices of this chain, then all members δ_0 of A^1 different from δ_1, and all members δ_{k+1} of A^{k+1} different from δ_k satisfy the following condition: their shadows $S_0(\delta_0, A_1)$ and $S_0(\delta_{k+1}, A^{k+1})$ do not intersect supports of other stars from C.

Every $A \in C$ containing at most one link will be called *a margin* star of C.

It is easily verified that C contains at least one margin star.

We see also that a margin star has at most one neighbor star.

A constellation C will be called *connected* if for each two different (if any) stars A', A'' from C there exists a chain $\delta_1, \ldots, \delta_k$ in C such that δ_1 is a member of A' and δ_k is a member of A''.

6. Proof of the main statement

Let $\Gamma \in G$.

Using the functions $\varepsilon_{\Gamma,j}(\zeta)$ (see Section 2), we shall define the following objects.

For every edge γ of Γ let δ_γ^+ and δ_γ^- be the images of γ in the maps $\zeta \mapsto \varepsilon_{\Gamma,1}(\zeta)$ and $\zeta \mapsto \varepsilon_{\Gamma,2}(\zeta)$, respectively, and in notation of Section 8 of [7] we have $(\delta_\gamma^+, \delta_\gamma^-) =: \delta_\gamma \in \Delta^1$.

For every $v \in W(\Gamma)$, let A_v be the collection of all couples δ_γ corresponding to all edges γ incident to v. Then A_v is a connectable star (see Section 4). Moreover, the collection

$$\{A_v\}_{v \in W(\Gamma)} =: C(\Gamma) =: C$$

is a constellation which is acyclic, irreducible and connected (see Section 5).

Consider the corresponding Riemann surface $\mathfrak{R}(C)$ built in Section 8 of [7] for the constellation $C := C(\Gamma)$. It is an oriented, compact, simply connected, schlichtartig Riemann surface conformally equivalent to a Riemann sphere (see Sections 8, 9 of [7]).

Let us denote

$$K_{1/r} := \{z \in \mathbb{C} : |z| < 1/r\},$$

$$T_{1/r} := \{z \in \mathbb{C} : |z| = 1/r\},$$

and

$$\mathfrak{R}^+ := \mathfrak{R}(C(\Gamma)) \setminus K_{1/r}.$$

The Riemann surface $\mathfrak{R}^+$ is a simply connected hyperbolic closed domain with the border $T_{1/r}$.

Let now $\mathfrak{R}$ denote the duplicate of $\mathfrak{R}^+$, and

$$\mathfrak{R}^- := \mathfrak{R} \setminus (\mathfrak{R}^+ \setminus T_{1/r}).$$

The surface $\mathfrak{R}$ is a Riemann surface topologically equivalent to $\overline{\mathbb{C}}$ and symmetric with respect to $T_{1/r}$. So it is conformally equivalent to $\overline{\mathbb{C}}$.

Let $\lambda_{\Gamma,t}$ denote the conformal homeomorphism of the Riemann surface $\mathfrak{R}$ onto $\overline{\mathbb{C}}$ normalized by the conditions $\lambda_{\Gamma,t}(T_{1/r}) = T$, $\lambda_{\Gamma,t}(1/r) = t$, $\lambda_{\Gamma,t}(1) = \infty$.

Denote by ϱ and σ the inversions of $\mathfrak{R}$ and $\overline{\mathbb{C}}$ with respect to $T_{1/r}$ and T, respectively.

We have

$$s^{-1}(T) = T_{1/r},$$

$$\varrho(T_{1/r}) = T_r.$$

Therefore all just the mentioned mappings admit reflections within respective surfaces with respect to mentioned circles, and for the functions extended by reflections we preserve the same notations.

Then because of the symmetries we have

$$\lambda_{\Gamma,t} \circ \varrho = \sigma \circ \lambda_{\Gamma,t}.$$

Let us consider the mapping $\eta_{C(\Gamma)} := \eta_C$ defined in Section 8 of [7] for the constellation $C = C(\Gamma)$, the corresponding superposition $\phi : \bar{S}_r \to \overline{\mathbb{C}} \setminus K$ defined by the formula

$$\phi := \phi_{\Gamma,r,t} := \lambda_{\Gamma,t} \circ \eta_{C(\Gamma)} \circ s^{-1},$$

and its inversion ϕ^{-1}.

Obviously

$$\eta_{C(\Gamma)}(T_{1/r}) = T_{1/r}.$$

Let

$$a_{t,j} := \phi(r\varepsilon_{\Gamma,1}(p_j)),$$

where p_j runs over the whole set V corresponding to Γ ($j = 1, \dots, m+1$).

Let us show that the restriction of the function ϕ to S_r is the extremal of Grötzsch's problem for the point collection $\{a_{t,j}\}$ introduced in this Section.

Let $h : S_r \to \overline{\mathbb{C}} \setminus (\overline{K} \cup \{a_{t,j}\})$ be any meromorphic univalent function with $h(T) = T$. Then for every edge γ of Γ, the corresponding $\delta_\gamma := (\delta_\gamma^+, \delta_\gamma^-)$ and any couple (z^+, z^-) of δ_γ-corresponding points (see Section 8 of [7]), the closure of the union of ϕ-images of radii of S_r passing through the points z^+ and z^- contains two non-intersecting arcs contained in $h(S_r)$ and connecting the boundary components of the set $h(S_r)$. Therefore on the basis of Theorems 5 and 6 from our work [5] one can show (cf. with [7]) that the restriction to S_r of the function ϕ is the extremal of Grötzsch's problem for $\{a_{t,j}\}$.

Therefore the function

$$\Big(\frac{\phi^{-1} \cdot (\phi' \circ \phi^{-1})}{\phi \circ \phi^{-1}}\Big)^2$$

is a rational function on $\overline{\mathbb{C}}$, and ϕ satisfies the equation (cf. with [3, Chapter 5]):

$$\Big(\frac{z\phi'(z)}{\phi(z)}\Big)^2 = \frac{p(\phi(z))}{q(\phi(z))},$$

where

$$p(w) := \prod_{j=1}^{m} \rho(a_{t,j}, w)$$

and

$$q(w) := \prod_{v \in W(\Gamma)} (\rho(b(v), w)^{\tau(v)-2}$$

with $b(v) := \phi(r\beta)\ \forall\, \beta \in W_v$.

For every $\gamma \in \Gamma$ and each $\zeta \in \langle\delta_\gamma\rangle$, let $\hat{\zeta}$ denote the point on $\langle\delta_\gamma\rangle$ which is $\delta_\gamma-$corresponding to ζ (see Section 8 of [7]).

Let a collection $\{a_j\} \in M$ of points be given. Let $f := f_{\{a_j\}}$ be the extremal of Grötzsch's problem for this collection. Lavrentiev [4] has established the result on the isometry (with respect to harmonic measure) of sewing of opposite sides of analytic arcs constituting the boundary of the extremal image in Tchebotaröv's problem (this result is generalized by Goluzin [1, p. 152–157]).

The following is a simple explanation of the isometry of the extremal f of Grötzsch's problem along critical trajectories. We have

$$Q(f)df^2 = -dz^2/z^2.$$

Hence

$$f'(z)^2 = -1/(Q(f(z))z^2),$$

and since Q is a rational function, therefore $|f'(z)|$ is the same for every couple of boundary points being sewed by f. So f is isometric along critical trajectories in the metric $|f'(z)dz|$ which coincides with isometry with respect to harmonic measure.

Let $L(\{a_j\})$ be the curvilinear graph (see above Section 1) generated by $f := f_{\{a_j\}}$. From here there follows the existence of a graph $\Gamma := \Gamma_{\{a_j\}} \in G$ which is isomorphic to $L(\{a_j\})$ (with the correspondence of the node $a_{m+1} \in L(\{a_j\})$ to the node $p_{m+1} \in V(\Gamma)$, see Theorem 3.1), isometric to it (with respect to natural measure in $\Gamma \in G$ and harmonic measure in $L(\{a_j\})$), and equally oriented relative to the complements. The function $f := f_{\{a_j\}}$ extended to the circle T_r gives one more realization of the Riemann surface $\mathfrak{R}_+$ (see above this Section) with similar normalizations at the circles T (for f) and $T_{1/r}$ (for ϕ) and at the preimage of ∞. Therefore the function $f/f(1)$ coincides with the function

$$\frac{\phi_{\Gamma_{\{a_j\}}}}{\phi_{\Gamma_{\{a_j\}}}(1)},$$

where $\phi =: \phi_{\Gamma_{\{a_j\}}}$ is the function constructed for this $\Gamma_{\{a_j\}}$. From here there follows that

$$a_j/f(1) = \frac{\phi_{\Gamma_{\{a_j\}}}(\alpha_j)}{\phi_{\Gamma_{\{a_j\}}}(1)} \qquad \forall\, \alpha_j \in V(\Gamma),$$

and the function

$$f(1)\frac{\phi_{\Gamma_{\{a_j\}}}}{\phi_{\Gamma_{\{a_j\}}}(1)}$$

is extremal in Grötzsch's problem for the given $\{a_j\}$.

Note that for every fixed $\widetilde{\Gamma} \in \widetilde{G}$ the collection $\{\phi(a_j)\}$ related to $\Gamma \in \widetilde{\Gamma}$ does not depend of the specific choice of the mentioned Γ from the fixed $\widetilde{\Gamma}$, and the corresponding function ϕ is extremal in Grötzsch's problem for the collection $\{\phi(a_j)\}$.

Conversely, for any collection $\{a_j\} \in M$ there exist (and are unique) $\widetilde{\Gamma}$ from $\widetilde{G}$ and $c \in \mathbb{C} \setminus \{0\}$ such that for every $\Gamma \in \widetilde{\Gamma}$ the function $c\phi$ is extremal for the collection $\{a_j\}$.

By the way, we can freely choose a Γ from G, and by this we can predetermine the topological structure of the curvilinear graph $L(\{a_j\})$ in the w-plane, and even its metric properties in the sense of Theorem 3.2.

From the above facts it is easy to derive all other statements of Theorems 3.1–3.3.

Remark 6.1. After receiving the referee report (November, 2010), I have to note that formerly this work is a part of another manuscript containing one more analytical statement and submitted for publication in 2008. But when G. V. Kuz'mina and E. G. Emelyanov shown me that the mentioned statement is wrong, I cancelled the submission. Recently the several new papers have been published: E. G. Emelyanov, *A sewing theorem for quadratic differentials* // Zapiski Nauchnuh Seminarov POMI, **371** (2009), 69–77; A. Yu. Solynin, *Quadratic differentials and weighted graphs on compact surfaces*, Analysis and Mathematical Physics. Trends in Mathematics, Birkhauser, Springer 473–505, 2009; P. Tamrazov, *Parametrization of extremals for some generalization of Chebotarev's problem* // Georgian Math. J., **17** (2010), 597–619. In Solynin's paper a general problem of constructing quadratic differentials starting from graphs is posed. The paper by Emel'yznov contains a different approach for the proof of the sewing theorems. Finally, in my paper it is established the parametrization of extremals for the problem which is one of the known generalizations of Chebotarev's problem. The parametrization of this problem essentially differs from that of Chebotarev's problem and Grötzsch's problem.

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