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Adaptive scheme of discretization for one semiiterative method in solving ill-posed problems

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Abstract. In the paper we consider a new algorithm to solving linear ill-posed problem with operators of finite smoothness. The algorithm uses one semiiterative method for the regularization of original problem in combination with an adaptive strategy of discretization. For the operators the algorithm achieves the optimal order of accuracy. Moreover, it is more economic in the sense of amount of used discrete information compare with standard methods.

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1. Introduction

In a Hilbert space X with inner product $(\cdot,\cdot)$ and generated by it norm $\|x\| = \sqrt{(x,x)}$ consider an operator equation of the first kind

$$Ax = f. \tag{1.1}$$

Assume that A is a linear and compact operator with $\mathrm{Range}(A) \neq \overline{\mathrm{Range}(A)}$. We will construct a finite-dimensional approximations to normal solution of (1.1), i.e. to solution with minimal norm in X that satisfies the Holder-type source condition

$$x^\dagger \in M_{\mu,\rho}(A) = \{u : u = |A|^\mu v,\ \|v\| \le \rho\}, \quad \rho \ge 1, \quad 0 < \mu \le 1, \tag{1.2}$$

where $|A| = (A^*A)^{1/2}$, A^* is adjoint to A and parameter μ is supposed to be unknown.

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It is very often instead of exact right-hand side in (1.1) we know only some its perturbation f_δ: $\|f - f_\delta\| \le \delta$. Then the best accuracy of recovering the minimal-norm solutions of (1.1) that fill up the set $M_{\nu,\rho}(A)$ can be lower estimated by $\rho^{1/(\mu+1)}\delta^{\mu/(\mu+1)}$ (see, for example, [14, p. 14]).

Following [9] we introduce into consideration class $\mathcal{H}^r$, $r = 1, 2, \ldots$, of compact linear operators A, $\|A\| \le 1$, such that for any $m = 1, 2, \ldots$ following conditions

$$\| (I - P_m)A \| \le m^{-r}, \qquad \| A(I - P_m) \| \le m^{-r}$$

hold, where P_m is ortoprojector on linear span of first m elements of some orthonormal basis $E = \{e_i\}_{i=1}^{\infty}$ in space X. As the example of (1.1) with operator $A \in \mathcal{H}^r$ in space $X = L_2(0,1)$ one can consider Fredholm integral equation of the first kind

$$Ax(t) \equiv \int_0^1 k(t,\tau)x(\tau)\, d\tau = f(t),$$

where $\max_{0 \le t,\tau \le 1} |k(t,\tau)| \le 1$, operators A and A^* act from $L_2(0,1)$ to Sobolev space $W_2^r[0,1]$ and as a basis E can be chosen, for example, orthonormal in $[0,1]$ system of Legendre polynomials or (if $r = 1$) the orthonormal system of Haar functions. It is clearly that the class $\mathcal{H}^r$ includes Fredholm integral operator with the kernels from Sobolev class of smoothness.

To solve (1.1) we will consider projection methods which use Galerkin information as the discrete information about (1.1). Recall that by Galerkin information about equation (1.1) one usually mean a set of the inner products

$$(Ae_j, e_i), \qquad (f_\delta, e_i), \tag{1.3}$$

where the indexes (i,j) are selected from some bounded domain Ω of coordinate plane.

It possible to characterize economic properties of the corresponding projection method by the volume of inner products (1.3) required to construct approximate solution of (1.1).

In the first time the problem of construction of economic projection methods for solving (1.1) with operators from $\mathcal{H}^r$ and solutions $x^\dagger \in M_{\nu,\rho}(A)$ was investigated in [9] in the framework of traditional Galerkin discretization scheme with $\Omega = [1,m] \times [1,n]$. It is follows from [9] that to guarantee the optimal order of accuracy we need to choose $n \asymp m \asymp O(\delta^{-1/r})$, i.e. to compute at least $O(\delta^{-2/r})$ inner products (1.3).

Our aim is to construct an algorithm for solving (1.1) which uses adaptive choice of discretization level for some modified Galerkin scheme.

The algorithm on the same classes of equations will guarantes the optimal order of accuracy for solutions $x^\dagger$ (1.2) and is be more economic in the sense of using Galerkin information compare with methods considered in [9].

The idea of employment such adaptive discretization strategy to solve ill-posed problems was proposed in [6] and further was investigated in [11–13].

2. Semiiterative method

To construct stable approximations we need to regularize original problem (1.1). For this purpose we use one semiiterative method, so-called ν-method (see, for example, [3, Chapter 6.3]) for fixed parameter $\nu = 1$. The method is the procedure of the following type

$$x_0^\delta = 0; \quad x_k^\delta = x_{k-1}^\delta + \sigma_k (x_{k-1}^\delta - x_{k-2}^\delta) + \omega_k A^*(f_\delta - A x_{k-1}^\delta), \quad k = 1, 2, \dots, \tag{2.1}$$

where

$$\begin{gathered} \sigma_1 = 0, \qquad \omega_1 = 6/5, \\ \sigma_k = \frac{(k-1)(2k-3)(2k+1)}{(k+1)(2k+3)(2k-1)}, \quad \omega_k = 4\frac{(2k+1)k}{(k+1)(2k+3)}, \quad k > 1. \end{gathered} \tag{2.2}$$

In the case $\delta = 0$ in (2.1) we will use notation x_k instead of x_k^δ.

Remind that ν-methods were introduced by Brakhage in [1] to obtain theoretical estimations of the conjugate gradient method. Later they were studied as alternative of the method. In [8] was investigated 1/2-method, also known as Chebyshev method.

Rewrite (2.1) as following

$$x_k^\delta = g_k(A^*A)A^* f_\delta, \qquad x_k = g_k(A^*A)A^* f.$$

So-called generating function $g_k(\lambda)$ in the framework of ν-method is the polynomial of the exact degree k. It determines value of the error generated by perturbation in input data

$$x_k - x_k^\delta = g_k(A^*A)A^*(f - f_\delta).$$

Polynomial $r_k(\lambda)$ connected with generating function by the relation

$$r_k(l) = 1 - l g_k(l) \tag{2.3}$$

determines approximation error of the ν-method

$$x^\dagger - x_k = r_k(A^*A)x^\dagger.$$

For $\nu = 1$ the polynomial $r_k(\lambda)$ has the form (see [8])

$$r_k(\lambda) = \frac{1 - T_{k+1}(1 - 2\lambda)}{2(k+1)^2\lambda}, \tag{2.4}$$

where $T_k(\lambda) = \cos(k \arccos(\lambda))$ are the Chebyshev polynomials of the first kind.

The following estimates can be found in [3]:

[3, p. 167]

$$|r_k(\lambda)| \leq 1, \qquad \lambda \in [0,1], \quad k \in N, \tag{2.5}$$

[3, p. 163]

$$\sup_{0 \leq \lambda \leq 1} g_k(\lambda) = \left(|r_k'(\tilde{\lambda})|\right) = 2k^2, \qquad \tilde{\lambda} \in [0,1], \quad k \in N, \tag{2.6}$$

[3, Theorem 6.12]

$$|\lambda^\mu r_k(\lambda)| \leq \kappa_\mu k^{-2\mu}, \qquad \lambda \in [0,1], \quad k \in N, \quad 0 < \mu \leq 1, \tag{2.7}$$

where κ_μ is the some positive constant.

It is follows from (2.3), (2.5) and (2.6) that

$$\lambda^2 g_k(\lambda^2) = 1 - r_k(\lambda^2) \leq 1 + |r_k(\lambda^2)| \leq 2, \tag{2.8}$$

$$\lambda g_k^2(\lambda) = \lambda g_k(\lambda) g_k(\lambda) = (1 - r_k(\lambda)) g_k(\lambda) \leq 4k^2,$$

and hence

$$\sup_{0 \leq l \leq 1} \sqrt{\lambda} g_k(\lambda) \leq 2k. \tag{2.9}$$

Besides we will use Markov's inequality for the polynomials T_k of degree k defined on the interval $[a,b]$ with norm equal 1 in metric of space C (see, e.g., [7, Chapter 7]):

$$|T_k'(x)| \leq \frac{2k^2}{b-a}. \tag{2.10}$$

3. Auxiliary statements

Let λ_k be singular values of A and ϕ_k, ψ_k be the corresponding singular elements. Then operator A can be written as

$$A = \sum_i \lambda_i \phi_i(\cdot\,, \psi_i),$$

and herewith following relations

$$x^\dagger = |A|^\nu v = (A^*A)^{\nu/2} v = \sum_i |\lambda_i|^\nu \psi_i(\psi_i, v),$$
$$f := Ax^\dagger = A|A|^\nu v = \sum_i \lambda_i |\lambda_i|^\nu \phi_i(\psi_i, v) \tag{3.1}$$

are true. Hence we obtain the decompositions of x_k and Ax_k:

$$x_k = g_k(A^*A)A^*f = \sum_i g_k(|\lambda_i|^2)|\lambda_i|^{\nu+2}\psi_i(\psi_i, v),$$

$$\begin{aligned} Ax_k &= \sum_i \lambda_i \phi_i \Big(\psi_i, \sum_j g_k(|\lambda_j|^2)|\lambda_j|^{\nu+2}\psi_j(\psi_j, v)\Big) \\ &= \sum_i \lambda_i\phi_i \sum_j |\lambda_j|^{\nu+2}(\psi_j, v) g_k(|\lambda_j|^2)(\psi_i, \psi_j) \\ &= \sum_i \lambda_i |\lambda_i|^{\nu+2} g_k(|\lambda_i|^2)\phi_i(\psi_i, v). \end{aligned}$$

Let Ω be a bounded set of coordinate plane $[1;\infty] \times [1;\infty]$ which we use to discretize coefficients of the original problem (1.1). Then in the framework of the projection scheme one need to switch from A and f_δ to finite-dimensional coefficients A_Ω and $P_\Omega f_\delta$

$$A_\Omega = \sum_{(i,j)\in\Omega} (Ae_j, e_i)(\cdot, e_j)e_i, \qquad P_\Omega f_\delta = \sum_{i:(i,j)\in\Omega} (f_\delta, e_i)e_i.$$

Specific form of Ω and A_Ω we will indicate below (see (3.14) and (3.15)).

Error of the discretized version of the 1-method on k-th step can be written in the form

$$x^\dagger - g_k(A_\Omega^* A_\Omega)A_\Omega^* f_\delta = (x^\dagger - x_k) + g_k(A_\Omega^* A_\Omega)A_\Omega^*(f - f_\delta) + (x_k - g_k(A_\Omega^* A_\Omega)A_\Omega^* f). \tag{3.2}$$

We need to estimate all items in right-hand side of (3.2). Due to above the first item can be written as

$$\begin{aligned} x^\dagger - x_k &= (I - g_k(A^*A)A^*A)x^\dagger \\ &= \sum_i |\lambda_i|^\mu (1 - g_k(\lambda_i^2)\lambda_i^2)\psi_i(v, \psi_i) \\ &= \sum_i |\lambda_i|^\mu r_k(\lambda_i^2)\psi_i(v, \psi_i). \end{aligned}$$

Then

$$\|x^\dagger - x_k\|^2 = \sum_i \lambda_i^{2\mu} r_k^2(\lambda_i^2)(v,\psi_i)^2$$

or in other form

$$\|x^\dagger - x_k\|^2 = k^{-2\mu} c_{\mu,k}^2(v), \tag{3.3}$$

where $c_{\mu,k}^2(v) := k^{2\mu}\sum_i \lambda_i^{2\mu} r_k^2(\lambda_i^2)(v,\psi_i)^2$.

Now let us write following representation

$$\begin{aligned} Ax_k - f &= A(x_k - x^\dagger) \\ &= -\sum_j \lambda_j \phi_j \sum_i |\lambda_i|^\mu r_k(\lambda_i^2)(\psi_j,\psi_i)(v,\psi_i) \\ &= -\sum_i |\lambda_i|^\mu \lambda_i r_k(\lambda_i^2)\phi_i(v,\psi_i). \end{aligned}$$

Then

$$\|Ax_k - f\|^2 = \sum_i \lambda_i^{2(\mu+1)} r_k^2(\lambda_i^2)(v,\psi_i)^2,$$

or the same

$$\|Ax_k - f\|^2 = k^{-2(\mu+1)} d_{\mu,k}^2(v) \tag{3.4}$$

with $d_{\mu,k}^2(v) := k^{2(\mu+1)}\sum_i \lambda_i^{2(\mu+1)} r_k^2(\lambda_i^2)(v,\psi_i)^2$.

Lemma 3.1. *For the functions $c_{\mu,k}(v)$ and $d_{\mu,k}(v)$ following estimates*

$$|c_{\mu,k}(v)| \le |d_{\mu,k}(v)|^{\frac{\mu}{\mu+1}}\|v\|^{\frac{1}{\mu+1}}, \qquad |d_{\mu,k}(v)| \le \kappa_{\frac{\mu+1}{2}}\|v\|$$

are true.

Proof. Using Hölder's inequality with $p = (\mu+1)/\mu$, $q = \mu+1$ and (2.5) we have

$$\begin{aligned} |c_{\mu,k}(v)|^2 &= \sum_i \left(k^{2(\mu+1)}\lambda_i^{2(\mu+1)} r_k^2(\lambda_i^2)(v,\psi_i)^2\right)^{\frac{\mu}{\mu+1}} \left(r_k^2(\lambda_i^2)(v,\psi_i)^2\right)^{\frac{1}{\mu+1}} \\ &\le |d_{\mu,k}(v)|^{\frac{2\mu}{\mu+1}} \left(\sum_i (v,\psi_i)^2\right)^{\frac{1}{\mu+1}} = |d_{\mu,k}(v)|^{\frac{2\mu}{\mu+1}}\|v\|^{\frac{2}{\mu+1}}. \end{aligned}$$

Now taking into account (2.7) we obtain

$$d_{\mu,k}^2(v) = \sum_i k^{2(\mu+1)}\left((\lambda_i^2)^{\frac{\mu+1}{2}} r_k(\lambda_i^2)\right)^2 (v,\psi_i)^2$$

$$\leq k^{2(\mu+1)} \left(\kappa_{\frac{\mu+1}{2}} k^{-(\mu+1)}\right)^2 \sum_i (v, \psi_i)^2$$
$$= \kappa^2_{\frac{\mu+1}{2}} \sum_i (v, \psi_i)^2 = \kappa^2_{\frac{\mu+1}{2}} \|v\|^2.$$

Lemma is proved. □

To estimate the third item in right-hand side of (3.2) we need following lemma.

Lemma 3.2. *For the polynomials $r_k(\lambda)$ defined as* (2.4) *at any $\lambda, \mu \in [0,1]$ the estimates*

$$|r_k(\lambda) - r_k(\mu)| \leq 2k^2 |\lambda - \mu|, \tag{3.5}$$

$$|\lambda r_k(\lambda) - \mu r_k(\mu)| \leq |\lambda - \mu| \tag{3.6}$$

are true.

Proof. For the case of $\lambda = \mu$ the estimates are obvious. Now consider the case of $\lambda \neq \mu$. According to Mean Value Theorem there is a point $\lambda' \in [0,1]$ such that

$$\frac{r_k(\lambda) - r_k(\mu)}{\lambda - \mu} = r_k'(\lambda').$$

Using (2.6) we obtain

$$\frac{r_k(\lambda) - r_k(\mu)}{\lambda - \mu} \leq \sup_{0 \leq \lambda' \leq 1} |r_k'(\lambda')| \leq 2k^2.$$

Thus

$$|r_k(\lambda) - r_k(\mu)| \leq 2k^2 |\lambda - \mu|.$$

Now let us prove inequality (3.6). Due to definition (2.3) we have

$$\lambda r_k(\lambda) - \mu r_k(\mu) = \frac{T_{k+1}(1-2\mu) - T_{k+1}(1-2\lambda)}{2(k+1)^2}.$$

Again according to the Mean Value Theorem there is a point $\lambda'' \in [0,1]$ such that

$$\frac{T_{k+1}(1-2\mu) - T_{k+1}(1-2\lambda)}{2(\lambda - \mu)} = T_{k+1}'(\lambda'').$$

Since $T_{k+1}(1-2\lambda)$ is a polonomial of degree $k+1$ defined in interval $[-1,1]$ then using (2.10) we obtain

$$\frac{|T_{k+1}(1-2\mu) - T_{k+1}(1-2\lambda)|}{2|\lambda - \mu|} \leq \sup_{-1 \leq \lambda'' \leq 1} |T_{k+1}'(\lambda'')| \leq (k+1)^2.$$

Hence we have

$$|\lambda r_k(\lambda) - \mu r_k(\mu)| \le \frac{2(k+1)^2}{2(k+1)^2}|\lambda - \mu| = |\lambda - \mu|. \tag{3.7}$$

Lemma is proved. □

Corollary 3.1. *For any $l, \mu \in [0,1]$ the inequalities*

$$|\lambda(r_k(\lambda) - r_k(\mu))| \le 2|\lambda - \mu|, \tag{3.8}$$

$$\sqrt{\lambda}|r_k(\lambda) - r_k(\mu)| \le 2k|\lambda - \mu| \tag{3.9}$$

are true.

Proof. Due to (2.5) and (3.6) we have

$$|\lambda(r_k(\lambda) - r_k(\mu))| \le |\lambda r_k(\lambda) - \mu r_k(\mu)| + |\lambda - \mu||r_k(\mu)| \le 2|\lambda - \mu|$$

and estimate (3.8) is proved. From (3.5) and (3.8) it immediately follows that

$$\lambda|r_k(\lambda) - r_k(\mu)|^2 \le 4k^2|\lambda - \mu|^2,$$

and we have (3.9). □

Let us denote as $\hat{x}_k^\delta = g_k(A_\Omega^* A_\Omega)A_\Omega^* f_\delta$ an approximate solution obtained by discretized version of 1-method on k-th iteration step.

Lemma 3.3. *The error of the* 1*-method can be estimated by*

$$\|x^\dagger - \hat{x}_k^\delta\| \le k^{-\mu}|c_{\mu,k}(v)| + 2k\delta + 2k^2\|x^\dagger\|(\|A_\Omega^* A_\Omega - A^* A\| + \|A_\Omega^*(A_\Omega - A)\|). \tag{3.10}$$

Proof. Recall (see (3.2)) that we make use of following error representation

$$x^\dagger - \hat{x}_k^\delta = (x^\dagger - x_k) + g_k(A_\Omega^* A_\Omega)A_\Omega^*(f - f_\delta) + (x_k - g_k(A_\Omega^* A_\Omega)A_\Omega^* f).$$

We need to estimate the expression in right-hand side. For the first item due to (3.3) we have

$$\|x^\dagger - x_k\| \le k^{-\mu}|c_{\mu,k}(v)|.$$

To estimate the second item we use(2.9):

$$\|g_k(A_\Omega^* A_\Omega)A_\Omega^*(f - f_\delta)\| \le \|f - f_\delta\| \sup_{0 \le \lambda \le 1} \lambda^{1/2} g_k(\lambda) \le 2k\delta.$$

Rewrite the third item in the form

$$x_k - g_k(A_\Omega^* A_\Omega)A_\Omega^* f := g_k(A^*A)A^*Ax^\dagger - g_k(A_\Omega^* A_\Omega)A_\Omega^* Ax^\dagger = (T_1+T_2)x^\dagger, \tag{3.11}$$

where

$$T_1 = g_k(A^*A)A^*A - g_k(A_\Omega^* A_\Omega)A_\Omega^* A_\Omega,$$
$$T_2 = g_k(A_\Omega^* A_\Omega)A_\Omega^*(A_\Omega - A).$$

Taking into account (3.5) and (2.6) we have

$$\|T_1\| = \|r_k(A_\Omega^* A_\Omega) - r_k(A^*A)\| \le 2k^2\|A_\Omega^* A_\Omega - A^*A\|, \tag{3.12}$$

$$\begin{aligned}\|T_2\| \le \|g_k(A_\Omega^* A_\Omega)\|\|A_\Omega^*(A_\Omega - A)\| \\ \le \|A_\Omega^*(A_\Omega - A)\| \sup_{0\le\lambda\le 1} g_k(\lambda) \\ \le 2k^2\|A_\Omega^*(A_\Omega - A)\|.\end{aligned} \tag{3.13}$$

Hereby

$$\begin{aligned}\|x_k - g_k(A_\Omega^* A_\Omega)A_\Omega^* f\| &\le \|T_1 + T_2\|\|x^\dagger\| \\ &\le 2k^2\|x^\dagger\|(\|A_\Omega^* A_\Omega - A^*A\| + \|A_\Omega^*(A_\Omega - A)\|)\end{aligned}$$

and Lemma is proved. □

Let Γ_n be the domain

$$\Gamma_n := \bigcup_{i=1}^{2n(k)} (2^{i-1}, 2^i] \times [1, 2^{2n(k)-i}) \cup \{1\} \times [1, 2^{2n(k)}] \tag{3.14}$$

of coordinate plane connected with basis E which is used in formulation of the class $\mathcal{H}^r$. To construct discretized operators $A_{\Gamma_n} = A_{n(k)}$, $k = 1, 2, \ldots$, we will choose the indexes (i, j) of inner products (Ae_j, e_i) from domain Γ_n, i.e.

$$A_{n(k)} = A_k := \sum_{i=1}^{2n(k)} (P_{2^i} - P_{2^{i-1}})AP_{2^{2n(k)-i}} + P_1 AP_{2^{2n(k)}}. \tag{3.15}$$

Assume that this discretization satisfies the conditions

$$\|A^*A - A_k^* A_k\| \le \frac{\delta}{4\rho k}; \qquad \|(A^* - A_k^*)A\| \le \frac{\delta}{4\rho k}; \tag{3.16}$$

$$\|(A - A_k)A^*\| \leq \frac{\delta}{4\rho k}; \qquad \|A - A_k\| \leq \left(\frac{\delta}{4\rho k}\right)^{1/2};$$
$$\|(A - A_k)A_k^*\| \leq \frac{\delta}{4\rho k}. \tag{3.17}$$

Without lost of generality we will consider that

$$\delta k \leq 1. \tag{3.18}$$

It should be noted that in the first time the scheme (3.14)–(3.17) was considered in [5], where as the regularization was used the Tikhonov method.

Lemma 3.4. *For any $k > 0$ following inequality*

$$\|Ax_k - f\| \leq \|A_k \hat{x}_k^\delta - f\| + c_1\delta$$

holds with $c_1 = \frac{29}{4}$.

Proof. Let us represent expression $Ax_k - f$ in the form

$$Ax_k - f := Ag_k(A^*A)A^*f - f = Z_1 + Z_2 + Z_3 + Z_4 + Z_5, \tag{3.19}$$

where

$$Z_1 = Ag_k(A^*A)A^*(f - f_\delta); \qquad Z_2 = (A - A_k)A^*g_k(AA^*)f_\delta;$$
$$Z_3 = -(A - A_k)(g_k(A^*A)A^*f_\delta - \hat{x}_k^\delta); \quad Z_4 = A(g_k(A^*A)A^*f_\delta - \hat{x}_k^\delta);$$
$$Z_5 = A_k\hat{x}_k^\delta - f.$$

We need to estimate the elements $Z_1 - Z_4$. Taking into account (2.8) we obtain

$$\|Z_1\| \leq \|g_k(AA^*)AA^*\|\|f - f_\delta\| \leq \delta \sup_{0\leq\lambda\leq 1} g_k(\lambda^2)\lambda^2 \leq 2\delta.$$

Using (2.6), (2.9) and (3.16)–(3.17) we have

$$\begin{aligned}\|Z_2\| &\leq \|(A - A_k)A^*\|(\|g_k(A^*A)Ax^\dagger\| + \|g_k(A^*A)\|\|f - f_\delta\|)\\ &\leq \|(A - A_k)A^*\|(2k\|x^\dagger\| + 2k^2\delta)\\ &\leq 2k\|(A - A_k)A^*\|(\|x^\dagger\| + k\delta) \leq \frac{\rho + 1}{2\rho}\delta \leq \delta.\end{aligned}$$

Now

$$\|Z_3\| = \|(A - A_k)(g_k(A^*A)A^*f_\delta - g_k(A_k^*A_k)A_k^*f_\delta)\|$$

$$
\begin{aligned}
&= \|(A - A_k)(g_k(A^*A)A^*(f_\delta - f) - g_k(A_k^*A_k)A_k^*(f_\delta - f) \\
&\qquad + g_k(A^*A)A^*f - g_k(A_k^*A_k)A_k^*f)\| \\
&\le \|(A - A_k)(g_k(A^*A)A^*(f_\delta - f) - g_k(A_k^*A_k)A_k^*(f_\delta - f))\| \\
&\qquad + \|(A - A_k)(g_k(A^*A)A^*f - g_k(A_k^*A_k)A_k^*f)\|.
\end{aligned}
$$

Let us estimate right-hand side term by term. For the first item due to (2.6) and (3.17) we have

$$
\begin{aligned}
&\|(A - A_k)(g_k(A^*A)A^*(f_\delta - f) - g_k(A_k^*A_k)A_k^*(f_\delta - f))\| \\
&\quad \le (\|(A - A_k)A^*\|\|g_k(AA^*)\| + \|(A - A_k)A_k^*\|\|g_k(A_kA_k^*)\|)\|f_\delta - f\| \\
&\qquad \le \frac{k\delta}{\rho}\delta \le \delta.
\end{aligned}
$$

The second item one can estimate using (3.11)–(3.13) and (3.16)–(3.17):

$$
\begin{aligned}
&\|(A - A_k)(g_k(A^*A)A^*f - g_k(A_k^*A_k)A_k^*f)\| \\
&\qquad \le \|A - A_k\|\|x_k - g_k(A_k^*A_k)A_k^*f\| \\
&\qquad \le 2k^2\|A - A_k\|\|x^\dagger\| \left(\|A_k^*A_k - A^*A\| + \|A_k^*(A_k - A)\|\right) \\
&\qquad \le \left(\frac{\delta}{4\rho k}\right)^{1/2} k\delta \le \frac{\delta}{2}.
\end{aligned}
$$

Hence

$$\|Z_3\| \le \frac{3}{2}\delta.$$

At last we need to estimate Z_4. So,

$$\|Z_4\| = \|A(g_k(A^*A)A^*f_\delta - \hat{x}_k^\delta)\| \le \|F_1\| + \|F_2\|,$$

where

$$
\begin{aligned}
F_1 &= A(g_k(A^*A)A^* - g_k(A_k^*A_k)A_k^*)Ax^\dagger, \\
F_2 &= A(g_k(A^*A)A^* - g_k(A_k^*A_k)A_k^*)(f - f_\delta).
\end{aligned}
$$

Rewrite the element F_2 in the form

$$
\begin{aligned}
F_2 &= [Ag_k(A^*A)A^* - A_k^*g_k(A_k^*A_k)A_k](f - f_\delta) \\
&\qquad - (A - A_k)g_k(A_k^*A_k)A_k^*(f - f_\delta) =: G_1 + G_2.
\end{aligned}
$$

Taking into account (2.6), (2.9), (3.5) and (3.16)–(3.17) we obtain

$$\|G_1\| \le \|r_k(A^*A) - r_k(A_k^*A_k)\|\|f - f_\delta\| \le 2k^2\|A^*A - A_k^*A_k\|\delta \le \frac{\delta}{2}.$$

$$\|G_2\| \le \|A - A_k\|\|f - f_\delta\| \sup_{0\le\lambda\le 1} \sqrt{\lambda}g_k(\lambda) \le \delta.$$

F_1 can be represented as

$$\begin{aligned} F_1 &= A(g_k(A^*A)A^*A - g_k(A_k^*A_k)A_k^*A_k)x^\dagger - Ag_k(A_k^*A_k)A_k^*(A-A_k)x^\dagger \\ &= A(g_k(A^*A)A^*A - g_k(A_k^*A_k)A_k^*A_k)x^\dagger - A_kg_k(A_k^*A_k)A_k^*(A-A_k)x^\dagger \\ &\qquad - (A-A_k)g_k(A_k^*A_k)A_k^*(A-A_k)x^\dagger =: H_1+H_2+H_3. \end{aligned}$$

Using (2.6), (2.9), (3.9) it is easy to obtain

$$\|H_1\| \le |\sqrt{\lambda}(r_k(\lambda)-r_k(\mu))|\|x^\dagger\| \le 2\rho k\|A^*A - A_k^*A_k\| \le \frac{\delta}{2},$$

$$\|H_2\| \le |g_k(\lambda)\sqrt{\lambda}|\|A_k^*(A-A_k)\|\|x^\dagger\| \le 2\rho k\|A_k^*(A-A_k)\| \le \frac{\delta}{2}$$

$$\begin{aligned} \|H_3\| &\le |g_k(\lambda)|\|A-A_k\|\|A_k^*(A-A_k)\|\|x^\dagger\| \\ &\le 2\rho k^2\|A-A_k\|\|A_k^*(A-A_k)\| \le \frac{\delta}{4}. \end{aligned}$$

Collecting above estimates we have

$$\|F_1\| \le \frac{5}{4}\delta, \qquad \|F_2\| \le \frac{3}{2}\delta.$$

Hence

$$\|Z_4\| \le \frac{11}{4}\delta.$$

Substituting obtained estimates for the elements $Z_1 - Z_4$ in (3.19) we find the required estimate. □

4. Finite-dimensional algorithm

Proposed finite-dimensional algorithm of solving (1.1) with operators $A \in \mathcal{H}^r$ consist in combination of 1-method and adaptive discretization strategy (3.14)–(3.17).

Algorithm

1. Given data: $A \in \mathcal{H}^r, \delta, f_\delta, \rho$.

2. Iteration by $k = 1, 2, 3, \dots$

   - choosing discretization level n as minimal integer which satisfied

$$(1+2^{r+3})2^{-2rn}n \le \frac{\delta}{4\rho k}; \tag{4.1}$$

- if n is changed then $k := 1$ and Galerkin information is computed

$$\begin{aligned} &(f_\delta, e_i), \quad i \in (2^{2n(k-1)}, 2^{2n(k)}] \\ &(Ae_j, e_i), \quad (i,j) \in \Gamma_{n(k)} \setminus \Gamma_{n(k-1)}; \end{aligned} \tag{4.2}$$

- computation of k-th approximation

$$\hat{x}_k^\delta = \hat{x}_{k-1}^\delta + \sigma_k(\hat{x}_{k-1}^\delta - \hat{x}_{k-2}^\delta) + \omega_k A_{n(k)}^*(f_\delta - A_{n(k)}\hat{x}_{k-1}^\delta),$$

where σ_k, ω_k are calculated by (2.2).

3. Stop rule by discrepancy principle

$$\begin{aligned} &\| A_{n(K)}\hat{x}_K^\delta - P_{2^{2n(K)}} f_\delta \| \leq b\delta, \\ &\| A_{n(k)}\hat{x}_k^\delta - P_{2^{2n(k)}} f_\delta \| > b\delta, \qquad k < K, \end{aligned} \tag{4.3}$$

where $b > c_1 + 1 + \sqrt{2}$.

4. Approximate solution: $\hat{x}_K^\delta$.

Lemma 4.1. *If discretization level n is choosen from* (4.1) *then for operators* $A \in \mathcal{H}^r$ *and* A_k (3.15) *conditions* (3.16)–(3.17) *are satisfied.*

Proof. Inequalities (3.16) were proven in [5, Lemma 1]. The first two inequalities in (3.17) can be proven in the same way and the last one follows from [10, Lemma 3.3] if we take into consideration that $A^* \in \mathcal{H}^r$ and domain Γ is symmetrical with respect to the diagonal of coordinate plane. □

To estimate accuracy of the proposed algorithm we need following statement.

Lemma 4.2. *Let* K *be a number of iteration such that* (4.3) *is hold. Then there is a constant* $b_2 > 0$ *such that*

$$\|Ax_K - f\| \leq b_2\delta.$$

At the same time, if $K \geq c_2\delta^{-\frac{1}{\mu+1}}$, *where* $c_2 = \left(\rho(1+2^{\mu+1})\kappa_{\frac{\mu+1}{2}}\right)^{\frac{1}{\mu+1}}$, *there is a constant* b_1, $0 < b_1 < b_2$, *such that*

$$b_1\delta \leq \|Ax_K - f\|.$$

Proof. Taking into account (4.1) for any $k \le K$ and $f = Ax^\dagger$, $A \in \mathcal{H}^r$, $x^\dagger \in \mathcal{M}_{\mu,\rho}(A)$

$$\|(I - P_{2^{2n(k)}})f\| \le \delta.$$

Now we use representation

$$A_k \hat{x}_k^\delta - f = A_k \hat{x}_k^\delta - P_{2^{2n(k)}} f_\delta + P_{2^{2n(k)}}(f_\delta - f) + (P_{2^{2n(k)}} - I)f. \quad (4.4)$$

Due to orthogonality of $P_{2^{2n(k)}}(f_\delta - f)$ and $(P_{2^{2n(k)}} - I)f$ we have

$$\|P_{2^{2n(k)}}(f_\delta - f) + (P_{2^{2n(k)}} - I)f\|^2 \\ = \|P_{2^{2n(k)}}(f_\delta - f)\|^2 + \|(P_{2^{2n(k)}} - I)f\|^2 \le 2\delta^2.$$

Then for $k = K$ from (4.3) it follows that

$$\|A_K \hat{x}_K^\delta - f\| \le (b + \sqrt{2})\delta.$$

According to Lemma 3.4 we have

$$\|(Ax_K - f)\| \le b_2 \delta,$$

where $b_2 = b + c_1 + \sqrt{2}$.

Now we have to obtain lower bound. Taking into account representation

$$Ax_{k-1} - f = (Ax_k - f) - (Ax_k - Ax_{k-1})$$

we find

$$\|Ax_{k-1} - f\| \le \|Ax_k - f\| + \|A(x_k - x_{k-1})\|. \quad (4.5)$$

Using (2.3) we have

$$x_k - x_{k-1} = (g_k(A^*A)A^*A - g_{k-1}(A^*A)A^*A)x^\dagger \\ = -(r_k(A^*A) - r_{k-1}(A^*A))x^\dagger.$$

Then

$$\begin{aligned} \|A(x_k - x_{k-1})\| &= \|A(r_k(A^*A) - r_{k-1}(A^*A))|A|^\mu v\| \\ &\le \|v\| \sup_{0 \le l \le 1} |r_k(l) l^{\frac{\mu+1}{2}} - r_{k-1}(l) l^{\frac{\mu+1}{2}}| \\ &\le \rho \Big(\sup_{0 \le l \le 1} l^{\frac{\mu+1}{2}} r_k(l) + \sup_{0 \le l \le 1} l^{\frac{\mu+1}{2}} r_{k-1}(l) \Big) \\ &\le \rho \kappa_{\frac{\mu+1}{2}} \left(k^{-(\mu+1)} + (k-1)^{-(\mu+1)} \right) \\ &= \rho \kappa_{\frac{\mu+1}{2}} k^{-(\mu+1)} \Big(1 + \Big(\frac{k}{k-1} \Big)^{\mu+1} \Big) \\ &\le \rho \left(1 + 2^{\mu+1} \right) \kappa_{\frac{\mu+1}{2}} k^{-(\mu+1)}. \end{aligned}$$

Let $K \geq c_2\delta^{-\frac{1}{\mu+1}}$, where $c_2 = \left(\rho(1+2^{\mu+1})\kappa_{\frac{\mu+1}{2}}\right)^{\frac{1}{\mu+1}}$. Then

$$\|A(x_k - x_{k-1})\| \leq \delta$$

and due to (4.5)

$$\|Ax_K - f\| \geq \|Ax_{K-1} - f\| - \delta. \tag{4.6}$$

Using reverse triangle inequality to (3.19) for $k = K-1$ we obtain

$$\|Ax_{K-1} - f\| \geq \|A_{K-1}\hat{x}^\delta_{K-1} - f\| - c_1\delta. \tag{4.7}$$

Now we consider representation (4.4) for $k = K-1$. Applying triangle inequality to it we have

$$\|A_{K-1}\hat{x}^\delta_{K-1} - f\| \geq \|A_{K-1}\hat{x}^\delta_{K-1} - P_{2^{2n(k)}}f_\delta\| - \sqrt{2}\delta.$$

Than taking into consideration (4.6), (4.7) and (4.3) we obtain

$$\|Ax_K - f\| \geq b_1\delta,$$

where $b_1 = b - c_1 - (1+\sqrt{2})$. □

5. Optimality of the algorithm.

Theorem 5.1. *Algorithm* (4.1)–(4.3) *achieves the optimal order of accuracy* $O(\delta^{\mu/(\mu+1)})$ *on the class of equations* (1.1) *with operators* $A \in \mathcal{H}^r$ *and normal solutions* $x^\dagger \in M_{\mu,\rho}(A)$.

Proof. Due to (3.4), Lemma 3.1 and the first inequality in Lemma 4.1 we have

$$|c_{\mu,K}(v)|K^{-\mu} = |c_{\mu,K}(v)|\left(\frac{\| Ax_K - f \|}{|d_{\mu,K}(v)|}\right)^{\frac{\mu}{\mu+1}} \leq \rho^{\frac{1}{\mu+1}}(b_2\delta)^{\frac{\mu}{\mu+1}}.$$

It is follows from the second inequality in Lemma 4.1 (for $K \geq c_2\delta^{-\frac{1}{\mu+1}}$) that

$$\delta K = \delta\left(\frac{|d_{\mu,K}(v)|}{\|Ax_K - f\|}\right)^{\frac{1}{\mu+1}} \leq \delta\left(\frac{\rho\kappa_{(\mu+1)/2}}{b_1\delta}\right)^{\frac{1}{\mu+1}} = c_3\delta^{\frac{\mu}{\mu+1}}, \tag{5.1}$$

where $c_3 = \left(\frac{\rho\kappa_{(\mu+1)/2}}{b_1}\right)^{\frac{1}{\mu+1}}$. In other hand, for $K < c_2\delta^{-\frac{1}{\mu+1}}$ we immediately obtain

$$\delta K < c_2\delta^{\frac{\mu}{\mu+1}}.$$

Substituting the estimates in (3.10) and taking into consideration (3.16)–(3.17) we have

$$\|x^\dagger - \hat{x}_K \| \le \rho^{\frac{1}{\mu+1}} (b_2\delta)^{\frac{\mu}{\mu+1}} + 2\eta\delta^{\frac{\mu}{\mu+1}} + \eta\delta^{\frac{\mu}{\mu+1}} = \xi\delta^{\frac{\mu}{\mu+1}},$$

where $\xi = (\rho b_2^\mu)^{\frac{1}{\mu+1}} + 3\eta$, $\eta = \max\{c_2, c_3\}$. □

Corollary 5.1. *To achieve the optimal order of accuracy on the considered class of equations in the framework of algorithm* (3.15)–(4.3) *it is enough to calculate*

$$O(\delta^{-\frac{\nu+2}{(\nu+1)r}} \log^{1+1/r} \delta^{-1}) \tag{5.2}$$

of information functionals (4.2)*.*

Proof. To prove this statement it is sufficiently to estimate volume of the inner products that is equivalent to square of figure Γ_n, which is equal to $(n+1)2^{2n}$. Using (4.1) and (5.1) in this expression we have estimate (5.2). □

As we remind in Section 1 to achieve the optimal order of accuracy in traditional Galerkin discretization scheme it is necessary to calculate $O(\delta^{-2/r})$ inner products (4.2). Thus algorithm (4.1)–(4.3) is more economic compare with the methods proposed in [9] which use traditional Galerkin discretization scheme.

References

[1] H. Brakhage, *On ill-posed problems and the method of conjugate gradients*, [2], pp. 165–175.

[2] H. W. Engl, C. W. Groetsch, eds., *Inverse and Ill-Posed Problems*, Academic Press, Orlando, 1987.

[3] H. W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer academic publishers, 1996.

[4] M. Hanke, *Accelerated Landweber iterations for the solution of ill-posed equations* // Numer. Math., **60** (1991), 341–373.

[5] P. Maass, S. V. Pereverzev, R. Ramlau, S. G. Solodky, *An adaptive discretization for Tikhonov-Phillips regularization with a posteriori parameter selection* // Numer. Math., **87** (2001), N 1, 485–502.

[6] P. Maass, A. Rieder, *Wavelet-accelerated Tikhonov–Phillips regularization with applications*, Inverse Problems in Medical Imaging and Nondestructive Testing, Springer, Vienna, 1997, pp. 134–158.

[7] I. P. Natanson, *Constructive function theory*, Moscow, 1949.

[8] A. S. Nemirovskij, B. T. Polyak, *Iterative methods of solving linear ill-posed problems. I* // Technical cybernetics, (1984), N 2.

[9] R. Plato, G. Vainikko, *On the Regularization of Projection Methods for solving Ill-posed Problems* // Numer. Math., (1990), N 57, 63–79.

[10] S. G. Solodky, *Optimal scheme of discretization of operator equations*, Grand Ph. D. dissertation, Institute of Mathematics, National Academy of Science of Ukraine, Kiev, 2003, 300 p.

[11] S. G. Solodky, *An one scheme of discretization for Landweber method* // Zh. Vychisl. Mat. Mat. Fiz., **44** (2004), N 3, 387–396.

[12] S. G. Solodky, *Economic discretization scheme for the nonstationary iterated Tikhonov method*, J. Optim. Theory Appl., **132** (2007), N 1, 21–39.

[13] S. G. Solodky, *The optimal approximations for stationary iterated Tikhonov method* // East Journal on approximations, **10** (2004), N 1–2, 189–200.

[14] G. M. Vainikko, A. Yu. Veretennikov, *Iterative procedures in ill-posed problems*, Moskow, Nauka, 1986.

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