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The fourth order of accuracy sequential type rational splitting of inhomogeneous evolution problem

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Abstract. In the present work symmetrized sequential type decomposition scheme of the fourth degree precision for the solution of inhomogeneous evolution problem is constructed. The fourth degree precision is reached by introducing the complex parameter $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ and by the approximation of the semigroup through the rational approximation. For the considered scheme the explicit a priori estimation is obtained.

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Introduction

One of the most effective methods to solve multi-dimensional evolution problems is a decomposition method. Decomposition schemes with first and second order accuracy were constructed in the sixties of the XX century (see [8, 11] and references therein). Q. Sheng has proved that in the real number field there do not exist automatically stable decomposition schemes with an accuracy order higher than two (see [12]). Decomposition schemes are called automatically stable if a sum of the absolute values of its split coefficients (coefficients of exponentials' products) equals to one, and the real parts of exponential powers are positive. In the work [1] there is constructed decomposition schemes with the higher order accuracy, but their corresponding decomposition formulas are not automatically stable. In the works [2–5] introducing the complex parameter,

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we have constructed automatically stable decomposition schemes with third order accuracy for two- and multi-dimensional evolution problems and with fourth order accuracy for two-dimensional evolution problem (evolution problem with the operator A is called m-dimensional, if it can be represented as a sum of m summands $A = A_1 + \cdots + A_m$). The new idea is an introduction of a complex parameter, which allows us to break the order 2 barrier.

Decomposition formulas constructed in the above mentioned works represent formulas of exponential splitting. Exponential splitting is called a splitting which approximates a semigroup by a combination of semigroups generated by the summands of the operator generating the given semigroup. In view of numerical computations, it is important a rational splitting of the multi-dimensional problem (We call rational splitting such a splitting of the evolution problem that is obtained from the exponential splitting by replacing the semigroups generated by the summands of its main operators with the corresponding rational approximations). Hence, if we have an exponential splitting with some order precision and the same order rational approximation of a semigroup, we can construct a rational splitting of the evolution problem. In the work [7] we have constructed the rational splitting with the third order precision.

In the present work, we have constructed the fourth order precision rational splitting for inhomogeneous evolution problem. We say that the rational approximation of the semigroup used in the work is of Kranc–Nickolson type, as if we replace the parameter α with 1, we obtain the classic Kranc–Nickolson approximation. In addition, let us note that in the scalar case, the considered rational approximation represents a Pade classic approximation (see [14]). For the rational approximation constructed in the work, there is obtained the explicit *a priori* estimate.

1. Statement of the problem and main result

Let us consider the Cauchy abstract problem in the Banach space X :

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi, \tag{1.1}$$

where A is a closed linear operator with the definition domain $D\,[A]$, which is everywhere dense in X, φ is a given element from X, $f(t) \in C^1\left([0;\infty)\,;X\right)$.

Let the operator $(-A)$ generate the strongly continuous semigroup $\{\exp(-tA)\}_{t\geq 0}$, then the solution of the problem (1.1) is given by the following formula ([9, 10]):

$$u(t) = U(t, A)\varphi + \int_0^t U(t-s, A) f(s)\, ds, \tag{1.2}$$

where $U(t, A) = exp\,(-tA)$.

Let $A = A_1 + A_2$, where A_j $(j = 1, 2)$ are densely defined, closed, linear operators in X.

As it is well-known, the essence of decomposition method consists in splitting the semigroup $U\,(t, A)$ by means of the semigroups $U\,(t, A_j)$ $(j = 1, 2)$. In [6] there is constructed the following decomposition formula with the local precision of Fifth order:

$$\begin{aligned} T\,(\tau) = {} & U\Big(\tau, \frac{\overline{\alpha}}{4}A_1\Big) U\Big(\tau, \frac{\overline{\alpha}}{2}A_2\Big) U\Big(\tau, \frac{1}{4}A_1\Big) U\Big(\tau, \frac{\alpha}{2}A_2\Big) U\Big(\tau, \frac{\alpha}{2}A_1\Big) \\ & \times U\Big(\tau, \frac{\alpha}{2}A_2\Big) U\Big(\tau, \frac{1}{4}A_1\Big) U\Big(\tau, \frac{\overline{\alpha}}{2}A_2\Big) U\Big(\tau, \frac{\overline{\alpha}}{4}A_1\Big). \end{aligned} \tag{1.3}$$

where $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ $\left(i = \sqrt{-1}\right)$.

In the above-mentioned work it is shown that:

$$U\,(\tau, A) - T(\tau) = O_p\left(\tau^5\right),$$

where $O_p\left(\tau^5\right)$ is the operator, norm of which is of the fifth order with respect to τ (more precisely, in the case of the unbounded operator $\left\|O_p\left(\tau^5\right)\varphi\right\| = O\left(\tau^5\right)$ for any φ from the definition domain of $O_p\left(\tau^5\right)$). In the present work (see Section 2) we construct the semigroup approximations with the local precision of the fifth order using the following rational approximation:

$$W\,(\tau, A) = \Big(I - \frac{\alpha}{2}\tau A\Big)\Big(I + \frac{\overline{\alpha}}{2}\tau A\Big)^{-1}\Big(I - \frac{\overline{\alpha}}{2}\tau A\Big)\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1}. \tag{1.4}$$

The approximation defined by formula (1.4) in the scalar case represent the Pade approximations for exponential functions [14].

On the basis of formulas (1.3) and (1.4) we can construct the following decomposition formula:

$$\begin{aligned} V(\tau) = {} & W\Big(\tau, \frac{\overline{\alpha}}{4}A_1\Big) W\Big(\tau, \frac{\overline{\alpha}}{2}A_2\Big) W\Big(\tau, \frac{1}{4}A_1\Big) W\Big(\tau, \frac{\alpha}{2}A_2\Big) W\Big(\tau, \frac{\alpha}{2}A_1\Big) \\ & \times W\Big(\tau, \frac{\alpha}{2}A_2\Big) W\Big(\tau, \frac{1}{4}A_1\Big) W\Big(\tau, \frac{\overline{\alpha}}{2}A_2\Big) W\Big(\tau, \frac{\overline{\alpha}}{4}A_1\Big). \end{aligned} \tag{1.5}$$

Below we shall show that this formula has the precision of the fifth order:

$$U(\tau, A) - V(\tau) = O_p\left(\tau^5\right).$$

In the present work, on the basis of formula (1.5), a decomposition scheme with the fourth order precision will be constructed for the solution of problem (1.1).

Let us introduce the following net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau,\ k = 0, 1, \ldots,\ \tau > 0\}.$$

According to formula (1.2), we have:

$$\begin{aligned} u(t_k) &= U(t_k, A)\,\varphi + \int_0^{t_k} U(t_k - s, A)\, f(s)\, ds \\ &= U(\tau, A)\, U(t_{k-1}, A)\,\varphi + \int_0^{t_{k-1}} U(\tau, A)\, U(t_{k-1} - s, A)\, f(s)\, ds \\ &\qquad + \int_{t_{k-1}}^{t_k} U(t_k - s, A)\, f(s)\, ds \\ &= U(\tau, A) \left[U(t_{k-1}, A)\,\varphi + \int_0^{t_{k-1}} U(t_{k-1} - s, A)\, f(s)\, ds \right] \\ &\qquad + \int_{t_{k-1}}^{t_k} U(t_k - s, A)\, f(s)\,. \end{aligned}$$

From here we have

$$u(t_k) = U(\tau, A)\, u(t_{k-1}) + \int_{t_{k-1}}^{t_k} U(t_k - s, A)\, f(s)\, ds.$$

Let us use Simpson's formula and rewrite this formula in the following form:

$$\begin{aligned} u(t_k) = U(\tau, A)\, u(t_{k-1}) + \frac{\tau}{6}\Big(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) \\ + U(\tau, A)\, f(t_{k-1}) \Big) + R_{5,k}(\tau), \qquad (1.6) \end{aligned}$$

$$u(t_0) = \varphi, \quad k = 1, 2, \ldots\ .$$

For the sufficiently smooth function f the following estimate is true (see. Lemma 2.3):

$$\|R_{k,5}(\tau)\| = O\left(\tau^5\right). \qquad (1.7)$$

On the basis of formula (1.6) let us construct the following scheme:

$$u_k = V(\tau)\, u_{k-1} + \frac{\tau}{6}\Big(f(t_k) + 4V\left(\frac{\tau}{2}\right) f\left(t_{k-1/2}\right) + V(\tau)\, f(t_{k-1})\Big),$$
$$u_0 = \varphi, \quad k = 1, 2, \dots \,. \tag{1.8}$$

Let us perform the computation of the scheme (1.8) by the following algorithm:

$$u_k = u_k^{(0)} + \frac{2\tau}{3} u_k^{(1)} + \frac{\tau}{6} f(t_k),$$

where $u_{k,0}$ is calculated by the scheme:

$$\begin{aligned}
&u_{k-8/9}^{(0)} = W\Big(\tau, \frac{\overline{\alpha}}{4}A_1\Big)\Big(u_{k-1} + \frac{\tau}{6} f(t_{k-1})\Big),\\
&u_{k-7/9}^{(0)} = W\Big(\tau, \frac{\overline{\alpha}}{2}A_2\Big)u_{k-8/9}^{(0)}, \quad u_{k-6/9}^{(0)} = W\Big(\tau, \frac{1}{4}A_1\Big)u_{k-7/9}^{(0)},\\
&u_{k-5/9}^{(0)} = W\left(\tau, \frac{\alpha}{2}A_2\right)u_{k-6/9}^{(0)}, \quad u_{k-4/9}^{(0)} = W\left(\tau, \frac{\alpha}{2}A_1\right)u_{k-5/9}^{(0)},\\
&u_{k-3/9}^{(0)} = W\Big(\tau, \frac{\alpha}{2}A_2\Big)u_{k-4/9}^{(0)}, \quad u_{k-2/9}^{(0)} = W\Big(\tau, \frac{1}{4}A_1\Big)u_{k-3/9}^{(0)},\\
&v_{k-1/9}^{(0)} = W\Big(\tau, \frac{\overline{\alpha}}{2}A_2\Big)u_{k-2/9}^{(0)},\\
&\quad u_k^{(0)} = W\Big(\tau, \frac{\alpha}{2}A_2\Big)u_{k-1/9}^{(0)}, \quad u_0^{(0)} = \varphi + \frac{\tau}{6} f(0),
\end{aligned} \tag{1.9}$$

and $u_{k,1}$ by the scheme:

$$\begin{aligned}
&u_{k-8/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{\overline{\alpha}}{4}A_1\Big) f\left(t_{k-1/2}\right), \quad u_{k-7/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{\overline{\alpha}}{2}A_2\Big)u_{k-8/9}^{(1)},\\
&u_{k-6/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{1}{4}A_1\Big)u_{k-7/9}^{(1)}, \quad u_{k-5/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{\alpha}{2}A_2\Big)u_{k-6/9}^{(1)},\\
&u_{k-4/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{\alpha}{2}A_1\Big)u_{k-5/9}^{(1)}, \quad u_{k-3/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{\alpha}{2}A_2\Big)u_{k-4/9}^{(1)},\\
&u_{k-2/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{1}{4}A_1\Big)u_{k-3/9}^{(1)}, \quad v_{k-1/9}^{(1)} = W\Big(\frac{\tau}{2}, \frac{\overline{\alpha}}{2}A_2\Big)u_{k-2/9}^{(1)},\\
&\quad u_k^{(1)} = W\Big(\frac{\tau}{2}, \frac{\alpha}{2}A_2\Big)u_{k-1/9}^{(1)}.
\end{aligned} \tag{1.10}$$

To estimate an error of approximate solution we need the natural powers ($A^s,\ s = 2, 3, 4, 5$) of the operator $A = A_1 + A_2$. They are usually defined as follows:

$$\begin{aligned}
&A^2 = \left(A_1^2 + A_2^2\right) + \left(A_1A_2 + A_2A_1\right),\\
&A^3 = \left(A_1^3 + A_2^3\right) + \left(A_1^2A_2 + \dots + A_2^2A_1\right) + \left(A_1A_2A_1 + A_2A_1A_2\right),
\end{aligned}$$

Analogously are defined $A^s,\ s = 4, 5$.

It is obvious that the definition domain $D\left(A^s\right)$ of the operator A^s represents an intersection of definition domains of its addends.

Let us introduce the following notations:

$$\|\varphi\|_A = \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D\left(A\right);$$
$$\|\varphi\|_{A^2} = \left\|A_1^2\varphi\right\| + \left\|A_2^2\varphi\right\| + \|A_1A_2\varphi\| + \|A_2A_1\varphi\|, \quad \varphi \in D\left(A^2\right),$$

where $\|\cdot\|$ is a norm in X. $\|\varphi\|_{A^s}$, $(s = 3, 4, 5)$ is defined analogously.

The following theorem takes place:

Theorem 1.1. *Let the following conditions be satisfied:*

(a) *There exists such $\tau_0 > 0$ that for any $0 < \tau \leq \tau_0$ there exist operators $\left(I + \tau\lambda\gamma A_j\right)^{-1}$, $j = 1, 2,$ $\gamma = 1, \alpha, \overline{\alpha},$ $\lambda = \alpha, \overline{\alpha}$ and they are bounded. Besides, the following inequalities are true:*

$$\|W\left(\tau, \gamma A_j\right)\| \leq e^{\omega\tau}, \quad \omega = \text{const} > 0;$$

(b) *The operator $(-A)$ generates the strongly continuous semigroup $U\left(t, A\right) = \exp\left(-tA\right)$, for which the following inequality is true:*

$$\|U(t, A)\| \leq Me^{\omega t}, \quad M, \omega = \text{const} > 0;$$

(c) *$U\left(s, A\right)\varphi \in D\left(A^5\right)$ for any $s \geq 0$;*

(d) *$f(t) \in C^4([0, \infty); X)$; $f'(t) \in D\left(A^3\right)$, $f''(t) \in D\left(A^2\right)$, $f'''(t) \in D\left(A\right)$ and $U\left(s, A\right)f\left(t\right) \in D\left(A^5\right)$ for any fixed t and s $(t, s \geq 0)$.*

Then the following estimate holds:

$$\begin{aligned}\|u(t_k) - u_k\| \leq ce^{\omega_0 t_k} t_k \tau^4 \Big(& \sup_{s\in[0,t_k]} \|U(s, A)\varphi\|_{A^5} \\ & + t_k \sup_{s,t\in[0,t_k]} \|U(s, A)f(t)\|_{A^5} + \sup_{t\in[0,t_k]} \|f(t)\|_{A^4} \\ & + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f''(t)\|_{A^2} \\ & + \sup_{t\in[0,t_k]} \|f'''(t)\|_A + \sup_{t\in[0,t_k]} \|f^{(IV)}(t)\|\Big),\end{aligned}$$

where c and ω_0 are positive constants.

Remark 1.1. In the case when operators $A_1,$ A_2 are matrices, it is obvious that conditions (a) and (b) of the Theorem 1.1 are automatically satisfied. Also these conditions are satisfied, if A_1, A_2 and A are self-adjoint, positive definite operators, even more in this case $\|W\left(\tau, \gamma A_j\right)\| \leq$

1 and $\|U(t,A)\| \le 1$. The requirement (a) of the theorem puts the condition for the spectrum of A_j. Namely, the spectrum of A_j must be placed within sector with the angle less than 60 degrees, because in case of turning of spectrum by ± 60 degrees (this is caused by multiplying of A_j on $\alpha^2 = 1/3\left(\cos 60^0 + i\sin 60^0\right)$ and $\overline{\alpha}^2$ parameters) the spectrum area will stay in the positive (right) half-plane.

2. Auxiliary lemmas

Let us prove the auxiliary lemmas on which the proof of the Theorem 1.1 is based.

Lemma 2.1. *If the condition* (a) *of the Theorem* 1.1 *is satisfied, then for the operator* $W(t,A)$ *the following decomposition is true:*

$$W(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_{W,k}(t,A), \quad k = 1,\dots,5, \tag{2.1}$$

where, for the residual member, the following estimate holds:

$$\|R_{W,k}(t,A)\varphi\| \le c_0 e^{\omega_0 t} t^k \|A^k\varphi\|, \quad \varphi \in D(A^k), \tag{2.2}$$

where c_0 *and* ω_0 *are positive constants.*

Proof. We obviously have:

$$\begin{aligned}(I+\gamma A)^{-1} &= I - I + (I+\gamma A)^{-1} \\ &= I - (I+\gamma A)^{-1}(I+\gamma A - I) = I - \gamma A (I+A)^{-1}.\end{aligned}$$

From this for any natural k we can get the following expansion:

$$(I+\gamma A)^{-1} = \sum_{i=0}^{k-1} (-1)^i \gamma^i A^i + \gamma^k A^k (I+\gamma A)^{-1}. \tag{2.3}$$

Let us rewrite $W(\tau,A)$ in the following form:

$$W(\tau,A) = S(\tau,A) - \frac{1}{2}\tau A S(\tau,A) + \frac{1}{12}\tau^2 A^2 S(\tau,A)$$

where

$$S(\tau,A) = \left(I + \frac{\overline{\alpha}}{2}\tau A\right)^{-1}\left(I + \frac{\alpha}{2}\tau A\right)^{-1}.$$

Let us decompose $S(\tau,A)$ by means of the formula (2.3)), we obtain the following recurrent relation:

$$S(\tau,A) = I - \frac{\alpha}{2}\tau A\left(I + \frac{\alpha}{2}\tau A\right)^{-1} - \frac{\overline{\alpha}}{2}\tau A S(\tau,A). \tag{2.4}$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.4) up to the first order, we obtain:

$$W\left(\tau, A\right) = I - R_{W,1}\left(\tau, A\right), \tag{2.5}$$

where

$$R_{W,1}\left(\tau, A\right) = \tau A \left(\frac{\alpha}{2}\left(I + \frac{\alpha}{2}\tau A\right)^{-1} - \frac{\overline{\alpha}+1}{2} S\left(\tau, A\right)\right) + \frac{1}{12}\tau^2 A^2 S\left(\tau, A\right).$$

Since $(I + \lambda\tau A)^{-1}$ is bounded according to the condition *(a)* of the Theorem 1.1, therefore:

$$\|R_{W,1}(\tau, A)\varphi\| \le c_0 e^{\omega_0 \tau}\tau \|A\varphi\|, \quad \varphi \in D\left(A\right). \tag{2.6}$$

Let us decompose the rational approximation $W\left(\tau, A\right)$ according to the formula (2.4) up to the second order:

$$\begin{aligned} W\left(\tau, A\right) = I - \tau A\Big(\frac{\alpha}{2}I - \frac{\alpha^2}{4}\tau A\left(I + \frac{\alpha}{2}\tau A\right)^{-1} + \frac{1+\overline{\alpha}}{2}I \\ - \frac{\alpha + \alpha\overline{\alpha}}{4}\tau A\left(I + \frac{\alpha}{2}\tau A\right)^{-1} - \frac{\overline{\alpha}+\overline{\alpha}^2}{4}\tau A S\left(\tau, A\right)\Big) \\ + \frac{1}{12}\tau^2 A^2 S\left(\tau, A\right) = I - \tau A + R_{W,2}(\tau, A) \end{aligned}$$

where

$$\begin{aligned} R_{W,2}\left(\tau, A\right) = \frac{\alpha^2 + \alpha + \alpha\overline{\alpha}}{4}\tau A\left(I + \frac{\alpha}{2}\tau A\right)^{-1} + \frac{3\overline{\alpha} + 3\overline{\alpha}^2 + 1}{12} S\left(\tau, A\right) \\ = \frac{\alpha - \frac{1}{3} + \alpha + \frac{1}{3}}{4}\tau A\left(I + \frac{\alpha}{2}\tau A\right)^{-1} + \frac{3\overline{\alpha} + 3\overline{\alpha} - 1 + 1}{12} S\left(\tau, A\right) \\ = \tau^2 A^2\left(\frac{\alpha}{2}\left(I + \frac{\alpha}{2}\tau A\right)^{-1} + \frac{\overline{\alpha}}{2} S\left(\tau, A\right)\right). \end{aligned}$$

According to the condition (a) of the Theorem 1.1 we have:

$$\|R_{W,2}(\tau, A)\varphi\| \le c_0 e^{\omega_0 \tau}\tau^2 \left\|A^2\varphi\right\|, \quad \varphi \in D\left(A^2\right). \tag{2.7}$$

Let us decompose the rational approximation $W\left(\tau, A\right)$ according to the formula (2.4) up to the third order:

$$\begin{aligned} W\left(\tau, A\right) = I - \tau A + \tau^2 A^2\Big(\frac{\alpha}{2}I - \frac{\alpha^2}{4}\tau A\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} \\ + \frac{\overline{\alpha}}{2}\Big(I - \frac{\alpha}{2}\tau A\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} - \frac{\overline{\alpha}}{2}\tau A S\left(\tau, A\right)\Big)\Big) \end{aligned}$$

$$= I - \tau A + \frac{1}{2}\tau^2 A^2 + R_{W,3}(\tau, A), \quad (2.8)$$

where

$$R_{W,3}(\tau, A) = -\tau^3 A^3 \Big(\frac{1+3\alpha^2}{12}\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} + \frac{\overline{\alpha}^2}{4} R(\tau, A)\Big)$$
$$= -\tau^3 A^3 \Big(\frac{\alpha}{4}\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} + \frac{\overline{\alpha}^2}{4} R(\tau, A)\Big).$$

According to the condition (a) of the Theorem 1.1 we have:

$$\|R_{W,3}(\tau, A)\varphi\| \leq c_0 e^{\omega_0 \tau} \tau^3 \left\|A^3 \varphi\right\|, \quad \varphi \in D\left(A^3\right). \quad (2.9)$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.4) up to the fourth order:

$$W(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \tau^3 A^3 \Big(\frac{\alpha}{4} I - \frac{\alpha^2}{8}\tau A\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1}$$
$$+ \frac{\overline{\alpha}^2}{4}\Big(I - \frac{\alpha}{2}\tau A\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} - \frac{\overline{\alpha}}{2}\tau A S(\tau, A)\Big)\Big)$$
$$= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_{W,4}(\tau, A), \quad (2.10)$$

where

$$R_{W,4}(\tau, A) = \tau^4 A^4 \Big(\frac{\alpha^2 + \alpha\overline{\alpha}^2}{8}\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} + \frac{\overline{\alpha}^3}{8} S(\tau, A)\Big)$$
$$= \tau^4 A^4 \Big(\frac{\alpha}{12}\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} + \frac{\overline{\alpha}^3}{8} S(\tau, A)\Big)$$

According to the condition (a) of the Theorem 1.1 we have:

$$\|R_{W,4}(\tau, A)\varphi\| \leq c_0 e^{\omega_0 \tau} \tau^4 \left\|A^4 \varphi\right\|, \quad \varphi \in D\left(A^4\right). \quad (2.11)$$

Let us decompose the rational approximation $W(\tau, A)$ according to the formula (2.4) up to the fifth order:

$$W(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3$$
$$- \tau^4 A^4 \Big(\frac{\alpha}{12} - \frac{\alpha^2}{24}\tau A\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1}$$
$$+ \frac{\overline{\alpha}^3}{8}\Big(I - \frac{\alpha}{2}\tau A\Big(I + \frac{\alpha}{2}\tau A\Big)^{-1} - \frac{\overline{\alpha}}{2}\tau A S(\tau, A)\Big)\Big)$$
$$= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 - \frac{1}{24}\tau^4 A^4 + R_{W,5}(\tau, A), \quad (2.12)$$

where

$$R_{W,5}(\tau, A) = \tau^5 A^5 \Big(\frac{2\alpha^2 + 3\overline{\alpha}^3 \alpha}{48} \Big(I + \frac{\alpha}{2} \tau A \Big)^{-1} + \frac{\overline{\alpha}^4}{16} S(\tau, A) \Big)$$
$$= \tau^5 A^5 \Big(\frac{\alpha}{24} \Big(I + \frac{\alpha}{2} \tau A \Big)^{-1} + \frac{\overline{\alpha}^4}{16} S(\tau, A) \Big)$$

According to the condition (a) of the Theorem 1.1 we have:

$$\|R_{W,5}(\tau, A)\varphi\| \leq c_0 e^{\omega_0 \tau} \tau^5 \left\|A^5 \varphi\right\|, \quad \varphi \in D\left(A^5\right) \tag{2.13}$$

□

Lemma 2.2. *If the conditions* (a), (b) *and* (c) *of the Theorem* 1.1 *are satisfied, then the following estimate holds:*

$$\left\| \left[U^k(\tau, A) - V^k(\tau) \right] \varphi \right\| \leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [0, t_k]} \|U(s, A)\varphi\|_{A^5}, \tag{2.14}$$

where c *and* ω_0 *are positive constants.*

Proof. The following formula is true (see T. Kato [9, p. 603]):

$$A \int_r^t U(s, A)\, ds = U(r, A) - U(t, A), \quad 0 \leq r \leq t. \tag{2.15}$$

Hence we get the following expansion:

$$U(t, A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t, A), \tag{2.16}$$

where

$$R_k(t, A) = (-A)^k \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} U(s, A)\, ds\, ds_{k-1} \cdots ds_1. \tag{2.17}$$

Let us decompose W operators in the expression of $V(\tau)$ according to the formula (2.1) from right to left, so that each residual member be of the fifth order. We shall have:

$$V(\tau) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_{V,5}(\tau), \tag{2.18}$$

where for the residual member according to the condition (a) of the Theorem 1.1 we have the following estimate:

$$\|R_{V,5}(\tau)\varphi\| \leq c e^{\omega_0 \tau} \tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \tag{2.19}$$

From the (2.15) and (2.17) it follows:

$$U\left(\tau,A\right)-V\left(\tau\right)=R_5\left(\tau,A\right)-R_{V,5}\left(\tau\right).$$

From here according to inequalities (2.16) and (2.18) we obtain the following estimate:

$$\left\|\left[U\left(\tau,A\right)-V\left(\tau\right)\right]\varphi\right\|\leq ce^{\omega_0\tau}\tau^5\left\|\varphi\right\|_{A^5}, \quad \varphi\in D\left(A^5\right). \tag{2.20}$$

The following representation is obvious:

$$[U^k(\tau,A)-V^k(\tau)]\varphi=\sum_{i=1}^{k}V^{k-i}(\tau)[U(\tau,A)-V(\tau)]U^{i-1}(\tau,A)\varphi.$$

Hence, according to the conditions (a), (b), (c) of the Theorem 1.1 and inequality (2.19), we have the sought estimate □

Lemma 2.3. *Let the following conditions be satisfied:*

(a) *The operator A satisfies the conditions of the Theorem* 1.1*;*

(b) *$f(t)\in C^4([0,\infty);X)$, and $f(t)\in D\left(A^4\right)$, $f^{(k)}(t)\in D\left(A^{4-k}\right)$ $(k=1,2,3)$ for every fixed $t\geq 0$.*

Then the following estimate holds

$$\left\|R_{5,k}\left(\tau\right)\right\|\leq ce^{\omega_0\tau}\tau^5\sum_{i=0}^{4}\max_{s\in[t_{k-1},t_k]}\left\|f^{(i)}\left(s\right)\right\|_{A^{4-i}}, \tag{2.21}$$

where $R_{5,k}\left(\tau\right)$ is a residual member of simpson formula,

$$R_{5,k}\left(\tau\right)=\int_{t_{k-1}}^{t_k}U\left(t_k-s,A\right)f\left(s\right)\,ds$$
$$-\frac{\tau}{6}\Big(f\left(t_k\right)+4U\left(\frac{\tau}{2},A\right)f\left(t_{k-1/2}\right)+U\left(\tau,A\right)f\left(t_{k-1}\right)\Big) \tag{2.22}$$

and where c and ω_0 are positive constants, and $f^{(0)}\left(s\right)=f\left(s\right)$.

Proof. By means of changing variables, the integral in the equality (2.21) takes the following form:

$$\int_{t_{k-1}}^{t_k}U\left(t_k-s,A\right)f\left(s\right)\,ds=\int_{0}^{\tau}U\left(\tau-s,A\right)f\left(t_{k-1}+s\right)\,ds.$$

If we decompose the function $f(t_{k-1}+s)$ into the Taylor series, and expand the semigroup $U(\tau-s,A)$ according to formula (2.15), we obtain:

$$U(\tau-s,A)\,f(t_{k-1}+s)=P_{3,k}(s)+\widetilde{R}_{4,k}(\tau,s)\,, \tag{2.23}$$

where

$$\begin{aligned}P_{3,k}(s)=&\Big(I-(\tau-s)A+\frac{(\tau-s)^2}{2}A^2-\frac{(\tau-s)^3}{6}A^3\Big)f(t_{k-1})\\&+s\Big(I-(\tau-s)A+\frac{(\tau-s)^2}{2}A^2\Big)f'(t_{k-1})\\&+\frac{s^2}{2}\left(I-(\tau-s)A\right)f''(t_{k-1})+\frac{s^3}{6}f'''(t_{k-1})\,,\end{aligned}$$

$$\begin{aligned}\widetilde{R}_{4,k}(\tau,s)=&\frac{1}{6}U(\tau-s,A)\int_0^s(s-\xi)^3f^{(IV)}(t_{k-1}+\xi)\,d\xi\\&+R_4(\tau-s,A)\,f(t_{k-1})+(\tau-s)AR_3(\tau-s,A)\,f'(t_{k-1})\\&+\frac{(\tau-s)^2}{2}A^2R_2(\tau-s,A)\,f''(t_{k-1})\\&+\frac{(\tau-s)^3}{6}A^3R_1(\tau-s,A)\,f'''(t_{k-1})\,.\end{aligned}$$

Hence according condition (b) and (d) of the Theorem 1.1 we obtain the following estimate:

$$\widetilde{R}_{4,k}(\tau,s)\le ce^{\omega_0\tau}\tau^4\sum_{i=0}^{4}\max_{s\in[t_{k-1},t_k]}\left\|f^{(i)}(s)\right\|_{A^{4-i}}. \tag{2.24}$$

From equality (2.21) with account of formula (2.22), we have:

$$\begin{aligned}R_{5,k}(\tau)=&\int_0^\tau U(\tau-s,A)\,f(t_{k-1}+s)\,ds\\&-\frac{\tau}{6}\Big(f(t_k)+4U\left(\frac{\tau}{2},A\right)f\left(t_{k-1/2}\right)+U(\tau,A)\,f(t_{k-1})\Big)\\=&\int_0^\tau P_{3,k}(s)\,ds+\int_0^\tau\widetilde{R}_{4,k}(\tau,s)\,ds\\&-\frac{\tau}{6}\Big(P_{3,k}(\tau)+4P_{3,k}\left(\frac{\tau}{2}\right)+P_{3,k}(0)\Big)\\&-\frac{\tau}{6}\widetilde{R}_{4,k}(\tau,0)+4\widetilde{R}_{4,k}\left(\tau,\frac{\tau}{2}\right)+\widetilde{R}_{4,k}(\tau,\tau)\,,\end{aligned} \tag{2.25}$$

Because of Simpson's formula is exact for the third order polynomial, for $R_{5,k}(\tau)$ we have:

$$R_{5,k}(\tau) = \int\limits_0^\tau \widetilde{R}_{4,k}(\tau, s)\, ds - \frac{\tau}{6}\left(\widetilde{R}_{4,k}(\tau, 0) + 4\widetilde{R}_{4,k}\left(\tau, \frac{\tau}{2}\right) + \widetilde{R}_{4,k}(\tau, \tau)\right).$$

hence according to inequality (2.22), we have:

$$\|R_{k,5}(\tau)\| \le ce^{\omega_0\tau}\tau^5 \sum_{i=0}^{4} \max_{s\in[t_{k-1},t_k]} \left\|f^{(i)}(s)\right\|_{A^{4-i}} \tag{2.26}$$

□

3. Proof of the theorem

Let us return to the proof of the Theorem 1.1.

Let us write formula (1.6) in the following form:

$$u(t_k) = U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A)\left(F_i^{(1)} + R_{5,k}(\tau)\right), \tag{3.1}$$

where

$$F_k^{(1)} = \frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + U(\tau, A)\, f(t_{k-1})\right). \tag{3.2}$$

Analogously let us present u_k as follows:

$$u_k = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau) F_i^{(2)}, \tag{3.3}$$

where

$$F_i^{(2)} = \frac{\tau}{6}\left(f(t_k) + 4V\left(\frac{\tau}{2}\right) f\left(t_{k-1/2}\right) + V(\tau)\, f(t_{k-1})\right). \tag{3.4}$$

From equalities (3.1) and (3.3) it follows:

$$\begin{aligned} u(t_k) - u_k = {} & \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \\ & + \sum_{i=0}^{k}\left[U^{k-i}(\tau, A)F_i^{(1)} - V^{k-i}(\tau)F_i^{(2)}\right] \\ & + \sum_{i=0}^{k} U^{k-i}(\tau, A)R_{k,5}(\tau) = \left[U^k(\tau, A) - V^k(\tau)\right]\varphi \end{aligned}$$

$$+\sum_{i=1}^{k}\left[\Big(U^{k-i}(\tau,A)-V^{k-i}(\tau)\Big)F_i^{(1)}+V^{k-i}(\tau)\big(F_i^{(1)}-F_i^{(2)}\big)\right]$$
$$+\sum_{i=0}^{k}U^{k-i}(\tau,A)R_{5,k}\,(\tau)\,. \quad (3.5)$$

From formulas (3.2) and (3.4) we have:

$$F_k^{(1)}-F_k^{(2)}=\frac{\tau}{6}\Big(4\left(U\left(\frac{\tau}{2},A\right)-V\left(\frac{\tau}{2}\right)\right)f\left(t_{k-1/2}\right)$$
$$+\left(U\left(\tau,A\right)-V\left(\tau\right)\right)f\left(t_{k-1}\right)\Big) \quad (3.6)$$

From here, according to inequality (2.17) and Lemma 2.1 we obtain the following estimate:

$$\left\|F_k^{(1)}-F_k^{(2)}\right\|\le ce^{\omega_0\tau}\tau^5\sup_{t\in[t_{k-1},t_k]}\|f(t)\|_{A^4}\,. \quad (3.7)$$

According to the Lemma 2.1 we have:

$$\left\|\sum_{i=1}^{k}\big(U^{k-i}(\tau,A)-V^{k-i}(\tau)\big)F_i^{(1)}\right\|\le ce^{\omega_0 t_k}t_k^2\tau^4\sup_{s,t\in[0,t_k]}\|U(s,A)f\left(t\right)\|_{A^5}\,. \quad (3.8)$$

From equality (3.5) according to inequalities (3.7), (3.8), (2.20) and the condition (b) of the Theorem 1.1 we obtain sought estimation.

Remark 3.1. The decomposition scheme constructed in this paper can be applied for solving heat equation, diffusion-reaction equation and other multidimensional evolution problems.

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