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Differential operator rings over 2-primal rings

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Abstract. Let R be a ring, and δ be a derivation of R. It is proved that R is a 2-primal Noetherian Q-algebra implies that the differential operator ring $R[x, \delta]$ is a 2-primal Noetherian.

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Introduction

A ring R always means an associative ring with identity. Q denotes the field of rational numbers. Spec(R) denotes the set of prime ideals of R. MinSpec(R) denotes the sets of minimal prime ideals of R. $P(R)$ and $N(R)$ denote the prime radical and the set of nilpotent elements of R respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

Before we discuss 2-primal rings, let us briefly recall the definitions of some Ore extensions concerning this paper.

Recall that a derivation of a ring R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. For example let $R = K[x]$, K is a field. Then the formal derivative d/dx is a derivation of R.

Differential operator ring $R[x, \delta]$ is the usual polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = xa + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x, \delta]$ by $D(R)$. If I is a δ-invariant (i.e. $\delta(I) \subseteq I$) ideal of R, then $I[x, \delta]$ is an ideal of $D(R)$. We denote $I[x, \delta]$ as usual by $D(I)$.

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Let σ be an endomorphism of a ring R. A σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. For example for any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a τ-derivation of R. Also let σ be an automorphism of a ring R and $\delta : R \to R$ any map. Let $\phi : R \to M_2(R)$ be a map defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix},$$

for all $r \in R$. Then δ is a σ-derivation of R.

Recall that the Ore extension $R[x, \sigma, \delta]$ is the usual polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take any $f(x) \in R[x, \sigma, \delta]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$. We denote $R[x, \sigma, \delta]$ by $O(R)$. If J is a σ-stable (i.e. $\sigma(J) = J$) ideal and δ-invariant (i.e. $\delta(J) \subseteq J$) ideal of R, then $J[x, \sigma, \delta]$ is an ideal of $O(R)$. We denote $J[x, \sigma, \delta]$ as usual by $O(J)$.

Now this article concerns the study of differential operator rings in terms of 2-primal rings. We recall that a ring R is 2-primal if and only if the set of nilpotent elements of R and the prime radical of R are same if and only if the prime radical is a completely semiprime ideal. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$, where $a \in R$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We note that a reduced ring is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [2, 4, 7, 9, 10].

2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [10], Greg Marks discusses the 2-primal property of $R[x, \sigma, \delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [9]. 2-primal near rings have been discussed by Argac and Groenewald in [2].

Let R be a ring, σ be an automorphism of R and δ be a σ-derivation of R. Recall that as defined in [4] a ring R is called a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ denotes the prime radical of R. Note that a ring with identity is not a δ-ring. The following result has been proved in Theorem 2.8 of [4] concerning δ-rings:

Let R be a δ-Noetherian Q-algebra such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in \mathrm{MinSpec}(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $R[x, \sigma, \delta]$ is 2-primal.

Now there arises a natural question:

Let R be a 2-primal ring. Is $R[x, \sigma, \delta]$ also a 2-primal ring? For the time being we are not able to answer this question, but towards this we prove the following result in this paper:

Let R be a 2-primal Noetherian Q-algebra. Then $R[x, \delta]$ is 2-primal Noetherian. This is proved in Theorem 1.2.

Before proving the above result, we find a relation between the minimal prime ideals of R and those of $R[x, \delta]$, where R is a Noetherian Q-algebra. This is proved in Theorem 1.1.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 3, 4, 8].

1. Differential operator ring $D(R) = R[x, \delta]$

Before we proceed further, we recall that Gabriel proved in Lemma 3.4 of [5] that if R is a Noetherian Q-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$, for all $P \in \text{MinSpec}(R)$. This result has been generalized in Theorem 2.2 of [4] for a σ-derivation δ of R and it has been proved that if R is a Noetherian Q-algebra. If σ is an automorphism of R and δ is a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$, then any $P \in \text{MinSpec}(O(R))$ with $\sigma(P) = P$ implies that $\delta(P) \subseteq P$.

The following Proposition follows immediately from Theorem 2.2 of [4], but we give a sketch of the proof in order to make the paper self contained.

Proposition 1.1. *Let R be a Noetherian Q-algebra. Let δ be a derivation of R. Then $\delta(P(R)) \subseteq P(R)$.*

Proof. Let $P_1 \in \text{MinSpec}(R)$. Let $T = R[[t]]$, the formal power series ring. Now it can be seen that $e^{t\delta}$ is an automorphism of T and $P_1T \in \text{MinSpec}(T)$. We also know that $(e^{t\delta})^k(P_1T) \in \text{MinSpec}(T)$ for all integers $k \geq 1$. Now T is Noetherian by Exercise 1ZA(c) of Goodearl and Warfield [6], and therefore Theorem 2.4 of Goodearl and Warfield [6] implies that MinSpec(T) is finite. So exists an integer an integer $n \geq 1$ such that $(e^{t\delta})^n(P_1T) = P_1T$; i.e. $(e^{nt\delta})(P_1T) = P_1T$. But R is a Q-algebra, therefore, $e^{t\delta}(P_1T) = P_1T$. Now for any $a \in P_1$, $a \in P_1T$ also, and so $e^{t\delta}(a) \in P_1T$; i.e. $a + t\delta(a) + (t^2/2!)\delta^2(a) + \cdots \in P_1T$, which implies that $\delta(a) \in P_1$. Therefore $\delta(P_1) \subseteq P_1$.

Now $P(R) \subseteq P$, for all $P \in \text{MinSpec}(R)$ implies that $\delta(P(R)) \subseteq \delta(P) \subseteq P$, for all $P \in \text{MinSpec}(R)$. Therefore

$$\delta(P(R)) \subseteq \cap_{P \in \text{MinSpec}(R)} P = P(R).$$

□

Proposition 1.2. *Let R be a Noetherian Q-algebra. Let δ be as usual. Then $D(N(R)) = N(D(R))$.*

Proof. It is easy to see that $D(N(R)) \subseteq N(D(R))$. We will show that $N(D(R)) \subseteq D(N(R))$.

Let $f = \sum_{i=0}^{m} x^i a_i \in N(D(R))$. Then $(f)(D(R)) \subseteq N(D(R))$, and $(f)(R) \subseteq N(D(R))$. Let $((f)(R))^k = 0$, $k > 0$. Then equating leading term to zero, we get $(x^m a_m R)^k = 0$. This implies on simplification that $x^{km}(a_m R)^k = 0$. Therefore $(a_m R)^k = 0 \subseteq P$, for all $P \in \mathrm{MinSpec}(R)$. So we have $a_m R \subseteq P$, for all $P \in \mathrm{MinSpec}(R)$. Therefore $a_m \in P(R) = N(R)$. Now $x^m a_m \in D(N(R)) \subseteq N(D(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(D(R))$, and with the same process, in a finite number of steps, it can be seen that $a_i \in P(R) = N(R)$, $0 \leq i \leq m-1$. Therefore $f \in D(N(R))$. Hence $N(D(R)) \subseteq D(N(R))$ and the result. □

Theorem 1.1. *Let R be a Noetherian Q-algebra and δ be a derivation of R. Then $P \in \mathrm{MinSpec}(D(R))$ if and only if $P = D(P \cap R)$ and $P \cap R \in \mathrm{MinSpec}(R)$.*

Proof. Let $P_1 \in \mathrm{MinSpec}(R)$. Then $\delta(P_1) \subseteq P_1$ by Proposition 1.1. Therefore by [11, 14.2.5 (*ii*)], $D(P_1) \in \mathrm{Spec}(D(R))$. Suppose $P_2 \subset D(P_1)$ is a minimal prime ideal of $D(R)$. Then

$$P_2 = D(P_2 \cap R) \subset D(P_1) \in \mathrm{MinSpec}(D(R)).$$

So $P_2 \cap R \subset P_1$ which is not possible.

Conversely suppose that $P \in \mathrm{MinSpec}(D(R))$. Then $P \cap R \in \mathrm{Spec}(R)$ by Lemma 2.21 of Goodearl and Warfield [6]. Let $P_1 \subset P \cap R$ be a minimal prime ideal of R. Then $D(P_1) \subset D(P \cap R)$ and as in first paragraph $D(P_1) \in \mathrm{Spec}(D(R))$, which is a contradiction. Hence $P \cap R \in \mathrm{MinSpec}(R)$. □

We are now in a position to prove the main result of this section in the form of the following Theorem.

Theorem 1.2. *Let R be a 2-primal Noetherian Q-algebra. Then $D(R)$ is 2-primal Noetherian.*

Proof. R is Noetherian implies $D(R)$ is Noetherian follows from Hilbert Basis Theorem, namely Theorem 1.12 of Goodearl and Warfield [6]. Now R is 2-primal implies $N(R) = P(R)$ and Proposition (1.1) implies that $\delta(N(R)) \subseteq N(R)$. Therefore $D(N(R)) = D(P(R))$. Now by Proposition 1.2 $D(N(R)) = N(D(R))$.

We now show that $D(P(R)) = P(D(R))$. It is easy to see that $D(P(R)) \subseteq P(D(R))$.

Now let

$$g = \sum_{i=0}^{t} x^i b_i \in P(D(R)).$$

Then $g \in P_i$, for all $P_i \in \text{MinSpec}(D(R))$. Now Theorem 1.1 implies that there exists $U_i \in \text{MinSpec}(R)$ such that $P_i = D(U_i)$. Now it can be seen that P_i are distinct implies that U_i are distinct. Therefore $g \in D(U_i)$. This implies that $b_i \in U_i$. Thus we have $b_i \in U_i$, for all $U_i \in \text{MinSpec}(R)$. Therefore $b_i \in P(R)$, which implies that $g \in D(P(R))$. So we have $P(D(R)) \subseteq D(P(R))$, and hence $D(P(R)) = P(D(R))$.

Thus we have

$$P(D(R)) = D(P(R)) = D(N(R)) = N(D(R)).$$

Hence D(R) is 2-primal. □

Question 1.1. *Let R be a 2-primal Noetherian Q-algebra. Is $O(R)$ 2-primal (even if $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$)?*

The main difficulty is that Proposition 1.2 and Theorem 1.1 do not hold.

A step towards the answer of the above question is the following Proposition and may give some idea:

Proposition 1.3. *Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation od R. Then:*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$, $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.*

2. *For any completely prime ideal U of $R[x, \sigma, \delta]$, $U \cap R$ is a completely prime ideal of R.*

Proof. See [4, Proposition 2.5]. □

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