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Global Mandelbrot transformation for Maslov tunnel pseudodifferential equations

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Abstract. We construct the global in time Mandelbrot transformation relating the solution of the transport equation at nonsingular points for the tunnel in the sense of Maslov LPDE to the global generalized delta-shock wave type solution to the continuity equation in the discontinuous velocity field.

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1. Introduction

The goal of the present paper is to present a new approach to constructing singular (i.e., containing the Dirac δ-function as a summand) solutions to the continuity equation and to show how these results can be used to construct the global in time solution of the Cauchy problem for Kolmogorov–Feller-type equations.

In general, the relation between the solutions of the continuity equation and the system consisting of the Hamilton–Jacobi equation plus the transport equation is well known. The velocity field u is determined as the velocities of points on the projections of the trajectories of the Hamiltonian system corresponding to the Hamilton–Jacobi equation. In this velocity field, as it was mentioned by B. Mandelbrot, the squared solution of the transport equation satisfies the continuity equation

$$\rho_t + \langle \nabla, u\rho \rangle + a\rho = 0 \tag{1.1}$$

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with some additional term $a\rho$, which is defined below.

But this relation between the transport and continuity equations is known only in the domain where the action function is sufficiently smooth. We generalize this relation to the case in which the singular support of the velocity field is a stratified manifold transversal to the velocity field. This holds, for example, in the one-dimensional case under the condition that, for any $t \in [0, T]$, the singularity support is a discrete set without limit points.

2. Generalized solutions of the continuity equation

Here we follow the approach developed in [1], where the solution is understood in the sense of integral identity, which, in turn, follows from the fact that relation (1.1) is understood in the sense of $\mathcal{D}(\mathbb{R}^{n+1}_{x,t})$.

We specially note that the integral identities in [1] can be derived without using the construction of nonconservative product [2, 4] (or the measure solutions [5]), and the value of the velocity on the discontinuity lines (surfaces) is not given a priori but is calculated. Of course, in the case considered in [1], the integral identities exactly coincide in form with the identities derived using the construction of nonconservative product (measure solutions) under the above assumptions.

First, we consider an $n-1$-dimensional surface γ_t moving in $\mathbb{R}^n_x$, which is determined by the equation

$$\gamma_t = \{x; t = \psi(x)\},$$

where $\psi(x) \in C^1(\mathbb{R}^n)$, and $\nabla\psi \neq 0$ in the domain in $\mathbb{R}^n_x$ where we work.

This is equivalent to determining a surface by an equation of the form

$$S(x, t) = 0$$

($S(x, t) \in C^1$, $S(x, t) = 0$, $\nabla_{x,t} S|_{S=0} \neq 0$) under the condition that

$$\frac{\partial S}{\partial t} \neq 0.$$

But if $\frac{\partial S}{\partial t} = 0$, then we can make the change $x'_i = x_i - c_i t$, solve the problem with moving surface (it will be considered with appropriately chosen $c_1, c_2, \dots$), and then return to the original variables. Possible generalizations are considered later.

Next, we assume that $\zeta(x, t) \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^1_+)$. Then, by definition,

$$\langle \delta(t - \psi(x)), \zeta(x, t) \rangle = \int\limits_{\mathbb{R}^n} \zeta(x, \psi(x))\, dx,$$

where $\delta(z)$ is the Dirac delta function.

Let $\delta(t-\psi(x))$ be applied to the test function $\eta(x) \in C_0^\infty(\mathbb{R}^n)$, then

$$\langle \delta(t-\psi(x)), \eta(x)\rangle = \int\limits_{\gamma_t} \eta(x)\, d\omega_\psi,$$

where $d\omega$ is the Leray form [6] on the surface $\{t = \psi(x)\}$ such that $-d\psi d\omega_\psi = dx_1 \dots dx_n$.

One can show that (see [1,6])

$$\langle \delta(t-\psi(x)), \eta(x)\rangle = \int\limits_{\gamma_t} \frac{\eta(x)}{|\nabla\psi|}\, d\sigma.$$

First, we assume that the solution ρ to Eq. (1.1) has the form

$$\rho = R(x,t) + e(x)\delta(t-\psi(x)), \tag{2.1}$$

where $R(x,t)$ is a piecewise smooth function with possible discontinuity at $\{t=\psi(x)\}$:

$$R = R_0(x,t) + H(t-\psi)R_1(x,t),$$

$e(x) \in C(\mathbb{R}^n)$ and has a compact support, $\psi(x) \in C^2$ and $\nabla\psi \neq 0$ for $x \in \operatorname{supp} e(x)$, and $H(z)$ is the Heaviside function.

It is clear that the term

$$e(x)\delta'(t-\psi(x))$$

appears in (1.1) if we differentiate the $\delta(t-\psi)$ with respect to t. Hence it is necessary to have in (1.1)

$$\langle \nabla, \rho u\rangle = -e(x)\delta'(t-\psi) + \text{more smooth summands},$$

since $\nabla\delta(t-\psi) = -\nabla\psi\delta'(t-\psi)$. Then we must have

$$\rho u = \frac{e\nabla\psi}{|\nabla\psi|^2}\delta(t-\psi) + \text{more smooth summands}.$$

Now we formulate an integral identity, defining a generalized solution.

We denote $\Gamma_t = \{(x,t); t=\psi(x)\}$; this is an n-dimensional surface in $\mathbb{R}^n \times \mathbb{R}^1_+$. Let

$$u(x,t) = u_0(x,t) + H(t-\psi)u_1(x,t),$$

where ψ is the same function as previously, and $u_0, u_1 \in C(\mathbb{R}^n \times \mathbb{R}^1_+)$.

Let us consider Eq. (1.1) in the sense of distributions. For all $\zeta(x,t) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\zeta(x,0)=0$, we have

$$\Big\langle \frac{\partial p}{\partial t} + \langle \nabla, \rho u\rangle, \zeta \Big\rangle = -\langle \rho, \zeta_t\rangle - \langle \rho u, \nabla\zeta\rangle.$$

Substituting the singular terms for ρ and ρu calculated above, we come to the following definition.

Definition 2.1. *A function $\rho(x,t)$ determined by relation* (2.1) *is called a generalized δ-shock wave type solution to* (1.1) *on the surface $\{t = \psi(x)\}$ if the integral identity holds*

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta)\,dx\,dt + \int\limits_{\Gamma_t} \frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx = 0 \qquad (2.2)$$

for all test functions $\zeta(x,t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\zeta(x,0)=0$, $\frac{d}{dn_\perp} = \big(\frac{\nabla\psi}{|\nabla\psi|}, \nabla\big) + |\nabla\psi|\frac{\partial}{\partial t}$.

We have the relation

$$\int\limits_{\mathbb{R}^n} \frac{e}{|\nabla\psi|}\frac{d}{dn}\zeta(x,\psi)\,dx = \int\limits_{\Gamma_t} \frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx,$$

where

$$\frac{d}{dn_\perp} = \Big(\frac{\nabla\psi}{|\nabla\psi|}, \nabla\Big) + |\nabla\psi|\frac{\partial}{\partial t}.$$

We note that the vector $n_\perp$ is orthogonal to the vector $(\nabla\psi, -1)$, which is the normal on the surface Γ_t, i.e., $\frac{d}{dn_\perp}$ lies in the plane tangent to Γ_t.

We can give a geometric definition of the field $\frac{d}{dn_\perp}$. The trajectories of this vector field are curves lying on the surface Γ_t, and they are orthogonal to all sections of this surface produced by the planes $t = \text{const}$. Furthermore, it is clear that the expression $\frac{1}{|\nabla\psi|}$ is an absolute value of the normal velocity of a point on γ_t, i.e., on the cross-section of Γ_t by the plane $t = \text{const}$, and the expression $\frac{1}{|\nabla\psi|}\cdot\frac{\nabla\psi}{|\nabla\psi|} \overset{\text{def}}{=} \vec{V}_n$ is the vector of normal velocity of a point on γ_t. Thus, we have another representation:

$$\int\limits_{\Gamma_t} \frac{e}{|\nabla\psi|}\frac{d}{dn_\perp}\zeta(x,t)\,dx = \int\limits_{\Gamma_t} e\Big((\vec{V}_n, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\,dx,$$

where $V_n = \pi^*(v_n)$, v_n is the normal velocity of a point on γ_t, and π^* is induced by the projection mapping $\pi : \Gamma_t \to R^n_x$.

It follows from the last definition that the following relations must hold:

$$R_t + (\nabla, Ru) + aR\zeta = 0, \qquad (x,t) \notin \Gamma_t,$$

$$([R] - |\nabla\psi|[Ru_n]) + \Big(\frac{d}{dn}\Big)^* \frac{e}{|\nabla\psi|} = 0, (x,t) \in \Gamma_t,$$

The last relation can be rewritten in the form

$$\mathcal{K}E + \frac{d}{dn}E = [Ru_n]|\nabla\psi| - [R], \tag{2.3}$$

where $E = e/|\nabla\psi|$, the factor $\mathcal{K} = (\nabla, \frac{\nabla\psi}{|\nabla\psi|}) = \div\nu$ (ν is the normal on the surface $\{t = \psi(x)\}$) and, as is known, is the mean curvature of the cross-section of the surface Γ_t by the plane $t = \text{const}$, $\frac{d}{dn} = (\frac{\nabla\psi}{|\nabla\psi|}, \nabla)$.

Now we assume that there are two surfaces

$$\Gamma_t^{(i)} = \{(x,t); t = \psi_i(x)\}$$

in $\mathbb{R}^n \times \mathbb{R}^1_+$, $i = 1, 2$, whose intersection is a smooth surface

$$\hat{\gamma} = \{(x,t); (t = \psi_1) \cap (t = \psi_2)\}$$

belonging to the third surface $\Gamma_t^{(3)} = \{(x,t); t = \psi_3(x)\}$. Further, we assume that the surface $\Gamma_t^{(3)}$ is a continuation of the surfaces $\Gamma^{(i)}$ in the following sense. We let $n_\perp^{(i)}$ denote the curves on the surfaces $\Gamma_t^{(i)}$ and assume that each point $(\hat{x}, \hat{t})$ on the surface $\hat{\gamma}$ is assigned the graph consisting of the trajectories $n_\perp^{(1)}$ and $n_\perp^{(2)}$ entering $(\hat{x}, \hat{t})$ and the trajectory $n_\perp^{(3)}$ leaving this point (i.e., the trajectories $n_\perp^{(i)}$ fiber the surface $\Gamma^{(i)}$). We also assume that the surface $\Gamma_\cup = \Gamma^{(1)} \cup \Gamma^{(2)} \cup \Gamma^{(3)}$ consists of points belonging to these graphs. Next, we assume that $u(x,t)$ is a piecewise smooth vector field whose trajectories come to $\Gamma_\cup$.

Definition 2.2. *Let*

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{3} H(t - \psi_i) u_{1i}(x,t),$$

where ψ is the same function as previously, and $u_0, u_{1i} \in C(\mathbb{R}^n \times \mathbb{R}^1_+)$. The function $\rho(x,t)$ determined by relation

$$\rho(x,t) = R(x,t) + \sum_{i=1}^{3} e_i(x) \delta(t - \psi_i(x)),$$

where $R(x,t) \in C^1(\mathbb{R}^n \times \mathbb{R}^1_+) \setminus \{\bigcup \Gamma_t^{(i)}\}$, *is called a generalized* δ*-shock wave type solution to* (2.2) *on the graph* $\Gamma_\cup$ *if the integral identity*

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta)\, dx\, dt + \sum_{i=1}^{3} \int_{\Gamma_t^{(i)}} \frac{e_i}{|\nabla\psi_i|} \frac{d}{dn_\perp^{(i)}} \zeta(x,t)\, dx = 0 \quad (2.4)$$

holds for all test functions $\zeta(x,t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\zeta(x,0) = 0$, $\frac{d}{dn_\perp^{(i)}} = \left(\frac{\nabla\psi_i}{|\nabla\psi_i|}, \nabla\right) + |\nabla\psi_i| \frac{\partial}{\partial t}$.

Now we consider the case $\operatorname{codim} \Gamma_t > 1$.

First, we note that the second integral in (2.2) can be written as

$$\int_{\Gamma_t} \frac{e}{|\nabla\psi|} \frac{d}{dn_\perp} \zeta(x,t)\, dx = \int_{\Gamma_t} e\Big(\Big(\frac{\nabla\psi}{|\nabla\psi|^2}, \nabla\Big) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\, dx.$$

We note that if the surface Γ_t is determined by the equation $S(x,t) = 0$ rather than by a simpler equation presented at the beginning of this section, then

$$\vec{V}_n = -\frac{S_t}{|\nabla S|} \cdot \frac{\nabla S}{|\nabla S|} = -\frac{S_t}{|\nabla S|^2} \nabla S$$

and, of course, the new vector field $\frac{d}{dn_\perp} = (\vec{V}_n, \nabla) + \frac{\partial}{\partial t}$ remains tangent to Γ_t.

Therefore, in this more general case, using this new vector $\vec{V}_n$, we can again rewrite the integral identity from Definition 1.1 as

$$\int_0^\infty \int_{\mathbb{R}^n} \big(R\zeta_t + (uR, \nabla\zeta) + aR\zeta\big)\, dx dt + \int_{\Gamma_t} e\Big((\vec{V}_n, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\, dx = 0. \quad (2.5)$$

This form of integral identity can easily be generalized to the case in which Γ_t is a smooth surface in $\mathbb{R}^{n+1}$ of codimension > 1.

In this case, instead of $\vec{V}_n$, we can use a vector $\vec{v}$ such that it is transversal to Γ_t and the field $(\vec{v}, \nabla) + \frac{\partial}{\partial t}$ is tangent to Γ_t. We note that the vector $\vec{v}$ is uniquely determined by this condition, which can be treated as "the calculation of the velocity value on the discontinuity" from the viewpoint of [5] and [7].

Moreover, in this case, the expression for ρ does not contain the Heaviside function, and it is assumed that the trajectories of the field u are smooth, nonsingular outside Γ_t, and transversal to Γ_t at each point of Γ_t. In this case, the function ρ has the form

$$\rho = R(x,t) + e\delta(\Gamma_t),$$

where $R(x,t) \in C^1(\mathbb{R}^{n+1} \setminus \Gamma_t)$, $e \in C^1(\Gamma_t)$, and the delta function is determined as

$$\langle \delta(\Gamma_t), \zeta(x,t) \rangle = \int\limits_{\Gamma_t} \zeta\omega,$$

where ω is the Leray form on Γ_t. If $\Gamma_t = \{S_1(x,t) = 0 \cap \cdots \cap S_k(x,t) = 0\}$, $k \in [1,n]$, then ω is determined by the relation, see [6, p. 274],

$$dt\, dx_1 \cdots d\, x_n = dS_1 \cdots dS_k \omega.$$

In this case, we assume that the functions S_k are sufficiently smooth (for example, $C^2(\mathbb{R}^n \times \mathbb{R}^1_+)$) and their differentials on Γ_t are linearly independent.

Moreover, we can assume that the inequality

$$J = \frac{\mathcal{D}(S_1, \ldots, S_n)}{\mathcal{D}(t, x_1, \ldots, x_{n-1})} \neq 0$$

holds. This inequality is an analog of $S_t \neq 0$ at the beginning of this section and allows us to write ω in the form

$$\omega = J^{-1} dx_k \cdots dx_n.$$

The integral identity, an analog of (2.5), has the form

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^n} \big(R\zeta_t + (uR, \nabla\zeta) + aR\zeta\big)\, dxdt + \int\limits_{\Gamma_t} e\Big((v, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\omega = 0.$$

Integrating the last relation by parts, we obtain equations for determining the functions e and R similarly to (2.3).

Now we assume that the singular support of the velocity field is the stratified manifold $\bigcup \Gamma_{it}$ with smooth strata Γ_{it} of codimensions $n_i \geq 1$.

We also assume that the velocity field trajectories are transversal to $\bigcup \Gamma_t$ and are entering trajectories.

Then the general solution of Eq. (1.1) has the form

$$\rho = R(x,t) + \sum e_i \delta(\Gamma_{it}),$$

where $R(x,t)$ is a function smooth outside $\bigcup \Gamma_{it}$, e_i are functions defined on the strata Γ_{it}, and the sum is taken over all strata.

The integral identities determining such a generalized solution have the form

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta) + aR\zeta)\,dxdt \\ + \sum \int\limits_{\Gamma_{it}} e_i \Big[\Big((v_i, \nabla) + \frac{\partial}{\partial t}\Big)\zeta(x,t)\Big]\omega_i = 0. \quad (2.6)$$

This implies that, outside $\bigcup \Gamma_{it}$, the function R satisfies the continuity equation

$$R_t + \langle \nabla, uR \rangle + aR = 0,$$

and, on the strata Γ_{jt} for $n_j = 1$, equations of the form (2.3) hold, which contain the values of R brought to Γ_{jt} along the trajectories. For $n_l = n - k$, $k > 1$, on the strata Γ_{lt}, we have the equations

$$\frac{\partial}{\partial t} e_l \mu_l + (\nabla, v_l e_l \mu_l) = 0, \quad (2.7)$$

where μ_l is the density of the measure ω_l with respect to the measure on Γ_{lt} left-invariant with respect to the field $\frac{\partial}{\partial t} + \langle v_l, \nabla \rangle$. We note that it follows from the above that the function R is determined independently of the values of v_i on the strata under the condition that the field trajectories enter $\bigcup \Gamma_{it}$.

In conclusion, we consider the case where the coefficient a has a singular support on $\bigcup \Gamma_{it}$, i.e.,

$$a = f(u).$$

In this case, we set

$$a\rho = \check{a}\rho + \sum f(v_i) e_i \delta(\Gamma_{it}).$$

We note that such a choice of the definition of the term $a\rho$ in not unique in this case. But, first, it is consistent with the common concept of measure solutions and, second, it is of no importance for the construction of the solution outside $\bigcup \Gamma_{it}$ for the case in which the trajectories u enter $\bigcup \Gamma_{it}$.

In this case, identity (2.6) takes the form

$$\int_0^\infty \int_{\mathbb{R}^n} \big(R\zeta_t + (uR, \nabla\zeta) + f(u)R\zeta\big)\, dxdt + \sum \int_{\Gamma_{it}} e_i \Big[\Big((v_i, \nabla) + \frac{\partial}{\partial t} + f(v_i)\Big)\zeta(x,t)\Big]\omega_i = 0, \qquad (2.8)$$

and Eq. (2.8) can be rewritten in the form

$$\frac{\partial}{\partial t}(e_l\mu_l) + (\nabla, v_l e_l \mu_l) + f(v_l) = 0.$$

3. The Maslov tunnel asymptotics

We recall that the asymptotic solutions of the Cauchy problem for an equation with pure imaginary characteristics was first constructed by V. P. Maslov [8]. In the present paper, we consider only the following Cauchy problem

$$-h\frac{\partial u}{\partial t} + P\Big(\overset{2}{x}, -h\overset{1}{\frac{\partial}{\partial t}}\Big)u = 0, \qquad u|_{t=0} = e^{-S_0(x)/h}\varphi^0(x), \qquad (3.1)$$

where $P(x,\xi)$ is the (smooth) symbol of the Kolmogorov–Feller operator [9], $S_0(x) \geq 0$ is a smooth function, $\varphi^0(x) \in C_0^\infty$, $h \to +0$ is a small parameter characterizing the frequency and the amplitude of jumps of the corresponding random process.

It is clear [8] that, locally in t, the solution of problem (3.1) is constructed according to the scheme of the WKB method: the solution is constructed in the form

$$u = e^{S(x,t)}(\varphi_0(x,t) + h\varphi_1(x,t) + \cdots),$$

in this case, for the functions $S(x,t)$ and $\varphi_i(x,t)$ (we consider only the case $i = 0$) we obtain the following problems:

$$\frac{\partial S}{\partial t} + P\Big(x, \frac{\partial S}{\partial x}\Big) = 0, \qquad S|_{t=0} = S_0(x), \qquad (3.2)$$

$$\frac{\partial \varphi_0}{\partial t} + \Big(\nabla P\Big(x, \frac{\partial S}{\partial x}\Big), \nabla\varphi_0\Big) + \sum_{ij} \frac{\partial^2 P}{\partial x_i \partial x_j}\frac{\partial^2 S}{\partial x_i \partial x_j}\varphi_0 = 0, \qquad \varphi_0|_{t=0} = \varphi^0(x), \qquad (3.3)$$

As is known, the solution of problem (3.2) is constructed using the solutions of the Hamiltonian system

$$\dot{x} = \nabla_\xi P(x,p), \qquad x|_{t=0} = x_0, \tag{3.4}$$

$$\dot{x} = -\nabla_x P(x,p), \qquad p|_{t=0} = \nabla S_0(x_0).$$

This solution is smooth on the support of $\varphi_0(x,t)$ until the Jacobian $Dx/Dx_0 \neq 0$ for $x_0 \in \operatorname{supp} \varphi^0(x)$. We let g_H^t denote the translation mapping along the trajectories of the Hamiltonian system (3.4).

Recall that the plot

$$\Lambda_0^n = \{x = x_0, p = \nabla S_0(x_0)\}$$

is the initial Lagrangian manifold corresponding to Eq. (3.2), and $\Lambda_t^n = g_h^t \Lambda_0^n$ is the Lagrangian manifold corresponding to Eq. (3.2) at time t. Let $\pi : \Lambda_t^n \to \mathbb{R}_x^n$ be the projection of Λ_t^n on $\mathbb{R}_x^n$, which is assumed to be proper. The point $\alpha \in \Lambda_t^n$ is said to be essential if

$$\hat{S}(\alpha,t) = \min_{\beta \in \pi^{-1}(\alpha)} \hat{S}(\beta,t)$$

and nonessential otherwise. Here $\hat{S}$ is the action on Λ_t^n determined by the formula

$$\hat{S}(\alpha,t) = \int_0^t p\,dx - H\,dt,$$

where the integral is calculated along the trajectories of system (3.4) the projection of whose origin is $x_0 = \alpha$. As is known

$$S(x,t) = \hat{S}(\pi^{-1}x,t)$$

at nonessential points where the projection π is bijective.

The solution of problem (3.1) is given by the Maslov tunnel canonical operator.

To define this operator, following [8, 10] we introduce the set of essential points $\bigcup \gamma_{it} \subset \Lambda_t^n$. This set is closed because the projection π is proper.

Suppose that the open domains $U_j \subset \Lambda_t^n$ form a locally finite covering of the set $\bigcup \gamma_{it}$. If the set U_j consists of nonessential points, then we set

$$u_j = e^{-S_j(x,t)/h} \varphi_{0j}(x,t) \tag{3.5}$$

where

$$\varphi_{0j}(x,t) = \psi_{0j}(x,t) \Big(\det \frac{Dx_0}{Dx} \Big)^{1/2},$$

where $\psi_{0j}(x,t)$ is the solution of the equation

$$\frac{\partial \psi_{0j}}{\partial t} + (\nabla P(x, \nabla S_j), \nabla \psi_{0j}) + \frac{1}{2}\operatorname{tr}\frac{\partial^2 P}{\partial x \partial \xi}(x, \nabla S_j)\psi_{0j} = 0. \tag{3.6}$$

The solution u_j in the domain containing essential points (at which $d\pi$ is degenerate) is given in the following way: the canonical change of variables is performed so that the essential points become nonessential, then we determine a fragment of the solution in new coordinates by formula (3.5) and return to the old variables, applying the "quantum" inverse canonical transformation to the solution obtain in the new coordinates.

The Hamiltonian determining this canonical transformation has the form

$$H_\sigma = \frac{1}{2}\sum_{i=1}^{n} \sigma_k p_k^2,$$

where $\sigma_1, \dots, \sigma_n = \text{const} > 0$.

The canonical transformation to the new variables is given by the translation by the time -1 along the trajectories of the Hamiltonian H_σ. One can prove that the set of sets σ for which the change of variables takes a essential point into a nonessential is not empty.

Next, the solution near the essential point is determined by the relation

$$u_j = e^{\frac{1}{h}\hat{H}_\sigma}\tilde{u}_j, \tag{3.7}$$

where $\tilde{u}_j$ is given by formula (3.5) in the new variables and

$$\hat{H}_\sigma = \frac{1}{2}\sum_{k=1}^{n}\sigma_k\Big(-h\frac{\partial}{\partial x_k}\Big)^2.$$

On the intersections of singular and nonsingular charts, we must match S_j and ψ_{0j}. This can be done by applying the Laplace method to the integral (whose kernel is a fundamental solution for the operator $-h\frac{\partial}{\partial t} + \hat{H}_\sigma$) in the right-hand side of (3.7). In this case, since the solution is real, the Maslov index [8] well-known in hyperbolic problems does not appear. The complete representation of the solution of problem (3.1) is obtained by summing functions of the type (3.5) and (3.6) over all the domains U_j, for more detail, see [8, 10].

The asymptotics thus constructed is justified, i.e., the proximity between the exact and asymptotic solutions of the Cauchy problem (3.1) is proved [8, 9].

Precisely as in the preceding case where the solution of the continuity equation at nonessential points was independent of the values of the solution on the singularity support (of course, the inverse influence takes place), in the case of the canonical operator, the relation between the solutions at essential and nonessential point is also unilateral, namely, the essential points are "bypassed" using (3.7), but the values of the functions $\tilde{\psi}_{oj}$ contained in $\tilde{u}_j$ on the singularity support, do not determine the values at the regular points (but the converse is not true).

Now we note that the function $S(x,t)$ such that

$$S(x,t)|_{U_j} = S_j(\pi^{-1}(\alpha), t)$$

is globally determined and continuous at points of the domain $\pi(\bigcup \gamma_{it}) \subset \mathbb{R}^n_x$. We denote this set by $\bigcup \Gamma_{it}$ and assume that this is a stratified manifold with smooth strata Γ_{it} of different codimensions. We note that, for example, if the inequality $\nabla(S_i(x,t) - S_j(x,t)) \neq 0$ holds while we pass from one branch $\Lambda^n_t \cap \bigcup \gamma_{it}$ to another, then the set $\pi\{(\tilde{S}_i - \tilde{S}_j) = 0\}$ generates a smooth stratum of codimension 1. In the one-dimensional case, all strata are points or curves (under the above assumptions about the singularities are discrete).

Now we consider the equation for ψ^2_{0j}. We denote this function by ρ and then obtain

$$\frac{\partial \rho}{\partial t} + (\nabla, u\rho) + a\rho = 0, \tag{3.8}$$

where $u = \nabla_x iP(x, \nabla S)$ and $a = \operatorname{tr} \frac{\partial^2 P}{\partial x \partial \xi}(x, \nabla S)$.

If the condition

$$\operatorname{Hess} P(x,\xi) > 0$$

is satisfied, then it follows from the implicit function theorem that $\nabla S = F(x,u)$, where $F(x,u)$ is a smooth function and

$$a = f(x,u),$$

where $f(x,z)$ is again a smooth function.

Thus, we have proved the following theorem.

Theorem 3.1. *Suppose that the following conditions are satisfied for* $t \in [0,T]$.

(1) *There exists a smooth solution of the Hamiltonian system* (3.4).

(2) *The singularities of the velocity field* $u = \nabla P(x, \nabla S)$ *form a stratified manifold with smooth strata and* $\operatorname{Hess} P(x,\xi) > 0$.

(3) *There exists a generalized solution* ρ *of the Cauchy problem for Eq.* (3.8) *in the sense of the integral identity* (2.8).

Then at the points of $\Gamma \subset \Lambda_t^n$*, the asymptotic solution of the Cauchy problem* (3.1) *has the form*

$$u = \exp(-S(x,t)/h)\sqrt{\rho}.$$

This theorem is a global in time analog of the corresponding Mandelbrot statement.

References

[1] V. G. Danilov, *On singularities of continuity equations* // Nonlinear Analysis (2007), doi:10.1016/j.na/2006.12.044

[2] P. G. Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems* in Nonlinear Evolution Equations that Change Type, Springer, Berlin, 1990, pp. 126–138.

[3] G. Dal Maso, P. G. Le Floch, and F. Murat, *Definition and weak stability of nonconservative products* // J. Math. Pures Appl. **74** (1995), 483–548.

[4] A. I. Volpert, *The space BV and quasilinear equations* // Math. USSR Sb. **2** (1967), 225–267.

[5] Hanchun Yang, *Riemann problem for a class of coupled hyperbolic systems of conservation laws* // J. Diff. Equations **159** (1999), 447–484.

[6] I. M. Gelfand and G. E. Shilov, *Generalized Functions.* Academic Press, New York, 1964, Vol. 1, (translated from the Russian).

[7] Wanchung Sheng and Tong Zhang, *The Riemann problem for the transportation equation in gas dynamics* // Memories of AMS **137** (1999), N 64, 1–77.

[8] V. P. Maslov, *Asymptotic Methods and Perturbation Theory.* Nauka, Moscow, 1988.

[9] V. G. Danilov and S. M. Frolovitchev, *Exact asymptotics of the density of the transition probability for discontinuous Markov processes* // Math. Nachrichten **215** (2000), N 1, 55–90.

[10] V. P. Maslov and V. E. Nazaikinskii, *Tunnel canonical operator in thermodynamics* // Funktsional. Anal. i Prilozhen. **40** (2006) ,N 3, 12–29.

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